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A q-logarithmic analogue of Euler's sine integral (*)

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ABSTRACT - We study a q-logarithmic analogue of Euler's sine integral proved in 1769. It is evaluated by the quantum dilogarithm function.

1. Introduction.

In [E] 1769, Euler calculated the famous integral

$$\int_0^{\frac{\pi}{2}} \log(\sin x)dx = -\frac{\pi}{2}\log 2. \tag{1.1}$$

This appears frequently in calculus texts as an example of somewhat tricky calculation. We remark that a refined formulation of (1.1) is given by the formula

$$\int_0^{\frac{\pi}{2}} \log(1 - e^{2ix})dx = -\frac{\pi^2 i}{8}. \tag{1.2}$$

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In fact, taking the real part of (1.2), we have (1.1) because

$$\begin{aligned}\operatorname{Re}\int_0^{\frac{\pi}{2}} \log(1-e^{2ix})dx &= \int_0^{\frac{\pi}{2}} \log|1-e^{2ix}|dx \\ &= \int_0^{\frac{\pi}{2}} \log(2\sin x)dx \\ &= \int_0^{\frac{\pi}{2}} \log(\sin x)dx + \frac{\pi}{2}\log 2\end{aligned}$$

by noticing that $\operatorname{Re}(-\frac{\pi^2 i}{8}) = 0$.

We formulate a slightly generalized version. We use Euler's dilogarithm function

$$\operatorname{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

and the double sine function

$$F(x) = e^x \prod_{n=1}^{\infty} \left\{ \left(\frac{1-\frac{x}{n}}{1+\frac{x}{n}} \right)^n e^{2x} \right\}$$

due to Hölder [H]. We define

$$F_1(x) = F(x)^{-2\pi i}(1-e^{2\pi i x})^{2\pi i x} e^{\pi^2(x^2-\frac{1}{6})}$$

for $\operatorname{Im}(x) \geq 0$. Then we have the

THEOREM 1.1. *For* $0 < a \leq 1$

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \log(1-ae^{2ix})dx &= -\frac{1}{2i}\{\operatorname{Li}_2(-a) - \operatorname{Li}_2(a)\} \\ &= \frac{1}{2i}\log\left(\frac{F_1\left(\frac{\log a}{2\pi i}+\frac{1}{2}\right)}{F_1\left(\frac{\log a}{2\pi i}\right)} \right).\end{aligned}$$

In the case of $a = 1$ in Theorem 1.1 we obtain (1.2) from the facts $F_1(0) = e^{-\frac{\pi^2}{6}}$ and $F_1(\frac{1}{2}) = e^{\frac{\pi^2}{12}}$. We can easily show these values are obtained from the definition of $F_1(x)$. Actually, the value of $F_1(0)$ is obvious from

$F(0)=1$. Also, using the result $F(\frac{1}{2})=\sqrt{2}$ of Hölder in [H], we have

$$F_1\left(\frac{1}{2}\right)=F\left(\frac{1}{2}\right)^{-2\pi i}2^{\pi i}e^{\frac{\pi^2}{12}}=e^{\frac{\pi^2}{12}}.$$

Now we present a q-analogue of Theorem 1.1 or (1.2). Let $q>1$. We put

$$\ell_q(x)=(q-1)\sum_{m=1}^{\infty}\frac{x-1}{x-1+q^m}$$

for $x\in\mathbb{C}$, which defines a meromorphic function in x. The function $\ell_q(x)$ is considered as a q-analogue of the usual logarithm

$$\log x=\sum_{m=1}^{\infty}\frac{(-1)^{n-1}(x-1)^n}{n}$$

expanded in $|x-1|<1$, as the following calculation shows:

$$\begin{aligned}\ell_q(x)&=(q-1)\sum_{m=1}^{\infty}\frac{(x-1)q^{-m}}{1+(x-1)q^{-m}}\\&=(q-1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{n-1}(x-1)^n q^{-nm}\\&=(q-1)\sum_{n=1}\frac{(-1)^{n-1}(x-1)^n}{q^n-1}=\sum_{n=1}\frac{(-1)^{n-1}(x-1)^n}{[n]_q},\end{aligned}$$

where $[n]_q=\frac{q^n-1}{q-1}$ and $\lim_{q\downarrow 1}[n]_q=n$.

We show that a q-analogue of (1.2) is expressed as

$$\int_0^{\frac{\pi}{2}}\ell_q(1-e^{2ix})dx=\frac{q-1}{2i}\log\left(\prod_{n=1}^{\infty}\frac{1+q^{-n}}{1-q^{-n}}\right). \tag{1.3}$$

To formulate the result neatly, we recall the quantum dilogarithm (see Kirillov [K]);

$$\mathrm{Li}_{2,q}(x)=\sum_{n=1}\frac{x^n}{n[n]_q}.$$

We also introduce

$$F_q(x)=\prod_{n=1}^{\infty}(1-q^{-n}e^{2\pi i x})^{q-1}.$$

Then we have the following theorem.

THEOREM 1.2. *For* $0 < a \leq 1$, *it holds that*

$$\int_0^{\frac{\pi}{2}} \ell_q(1 - ae^{2ix})dx = -\frac{1}{2i}\{\mathrm{Li}_{2,q}(-a) - \mathrm{Li}_{2,q}(a)\}$$

$$= \frac{1}{2i} \log \left(\frac{F_q\left(\frac{\log a}{2\pi i} + \frac{1}{2}\right)}{F_q\left(\frac{\log a}{2\pi i}\right)} \right).$$

If we put $a = 1$ in Theorem 1.2, the equation (1.3) follows immediately from the second expression. Moreover, we have the following limit formulas which allow us to obtain (1.2) as the limit $q \downarrow 1$ of (1.3).

PROPOSITION 1.3.

$$\lim_{q \downarrow 1} (q-1) \log \left(\prod_{n=1}^{\infty} (1 + q^{-n}) \right) = \frac{\pi^2}{12} \tag{1.4}$$

and

$$\lim_{q \downarrow 1} (q-1) \log \left(\prod_{n=1}^{\infty} (1 - q^{-n}) \right) = -\frac{\pi^2}{6}. \tag{1.5}$$

Furthermore, we prove the following

THEOREM 1.4. *Suppose* $\mathrm{Im}\, x \geq 0$. *Then we have*

$$\lim_{q \downarrow 1} F_q(x) = F_1(x).$$

This result indeed shows that "$\lim_{q \downarrow 1}$ (Theorem 1.2) = Theorem 1.1".
We add one remark. Since

$$\int_0^{\frac{\pi}{2}} \log(\sin x) dx = \frac{1}{4} \int_0^{2\pi} \log|\sin x| dx = \frac{1}{4} \mathrm{Re}\, \frac{1}{4} \int_0^{2\pi} \log(\sin x) dx,$$

it may be also reasonable to think the quantity

$$\frac{1}{4}\, \mathrm{Re}\, \frac{1}{4} \int_0^{2\pi} \ell_q(\sin x)\, dx$$

as an analogue of (1.1). From this viewpoint, we obtain the following

THEOREM 1.5. *For* $q > 2$

$$\frac{1}{4}\operatorname{Re}\frac{1}{4}\int_0^{2\pi}\ell_q(\sin x)\,dx = -\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{\binom{2n}{n}2^{-n}}{[n]_q}$$
$$= -\frac{(q-1)\pi}{2}\sum_{n=1}^{\infty}\left\{\frac{1}{\sqrt{1-2q^{-m}}}-1\right\}.$$

We note that, in the formula above, it is impossible to take the limit $q\downarrow 1$ directly.

2. Proof of Theorem 1.1.

The former part of the theorem is straightforward. In fact, we calculate

$$\int_0^{\frac{\pi}{2}}\log(1-ae^{2ix})dx = \int_0^{\frac{\pi}{2}}\left(-\sum_{n=1}^{\infty}\frac{a^n e^{2inx}}{n}\right)dx$$
$$= -\sum_{n=1}^{\infty}\frac{a^n}{n}\int_0^{\frac{\pi}{2}}e^{2inx}dx = -\sum_{n=1}^{\infty}\frac{a^n}{n}\left[\frac{e^{2inx}}{2in}\right]_0^{\frac{\pi}{2}}$$
$$= -\frac{1}{2i}\sum_{n=1}^{\infty}\frac{(-a)^n-a^n}{n^2}$$
$$= -\frac{1}{2i}\left\{\mathrm{Li}_2(-a)-\mathrm{Li}_2(a)\right\}.$$

To show the latter part of the theorem, it is sufficient to prove the following lemma.

LEMMA 2.1. *For* $\operatorname{Im} x \geq 0$

$$F_1(x) = \exp\left(-\mathrm{Li}_2(e^{2\pi ix})\right).$$

This is essentially proved in [KOW] or [KK], where the theory of multiple sine functions is developed in extending the results to the double sine functions due to Hölder [H] (for the period $(1,1)$) and Shintani [S] (for the period (ω_1,ω_2)). For reader's convenience, we sketch the proof. See also "Miscellaneous Examples" Chap. VII-20 of Whittaker-Watson [WW], which treats indeed the double sine function $F(x)$ of Hölder [H] (but without mentioning to this reference).

Proof of Lemma 2.1. We first show that

$$F(x) = \exp\left(\int_0^x \pi t \cot(\pi t)\, dt\right). \tag{2.1}$$

In fact, from the expression

$$\log F(x) = x + \sum_{n=1}^{\infty}\left\{n\left(\log\left(1-\frac{x}{n}\right) - \log\left(1+\frac{x}{n}\right)\right) + 2x\right\}$$

we obtain

$$\frac{F'(x)}{F(x)} = 1 + \sum_{n=1}^{\infty}\left\{n\left(\frac{1}{x-n} - \frac{1}{n+x}\right) + 2\right\}$$

$$= 1 + \sum_{n=1}^{\infty}\frac{2x^2}{x^2-n^2} = \pi x \cot(\pi x).$$

Hence, using $F(0) = 1$, we obtain (2.1).

It follows from (2.1) that

$$\log F(x) = \int_0^x \pi t \cot(\pi t)\, dt.$$

Suppose $\mathrm{Im}\,(x) \geq 0$. Then we have

$$\log F(x) = \int_0^x \pi i t \frac{e^{\pi i t} + e^{-\pi i t}}{e^{\pi i t} - e^{-\pi i t}} dt = -\int_0^x \pi i t \left(1 + 2\frac{e^{2\pi i t}}{1-e^{2\pi i t}}\right) dt$$

$$= -\pi i \frac{x^2}{2} - 2\pi i \sum_{n=1}^{\infty}\int_0^x t e^{2\pi i n t} dt$$

$$= -\frac{\pi i}{2}x^2 - 2\pi i \sum_{n=1}^{\infty}\left\{\frac{x e^{2\pi i n x}}{2\pi i n} - \frac{e^{2\pi i n x} - 1}{(2\pi i n)^2}\right\}$$

$$= -\frac{\pi i}{2}x^2 - x\sum_{n=1}^{\infty}\frac{e^{2\pi i n x}}{n} + \frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{e^{2\pi i n x} - 1}{n^2}$$

$$= -\frac{\pi i}{2}x^2 + x \log(1 - e^{2\pi i x}) + \frac{1}{2\pi i}\mathrm{Li}_2(e^{2\pi i x}) + \frac{\pi i}{12}.$$

Since $F_1(x) = F(x)^{-2\pi i}(1-e^{2\pi i x})^{2\pi i x} e^{\pi^2(x^2-\frac{1}{6})}$, this shows the claim in Lemma 2.1. $\square$

Thus Theorem 1.1 follows.

3. Proof of Theorem 1.2

The former part is exactly similar to the case of Theorem 1.1. Actually, we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \ell_q(1-ae^{2ix})dx &= \int_0^{\frac{\pi}{2}} \left(-\sum_{n=1}^{\infty}\frac{a^n e^{2inx}}{[n]_q}\right)dx \\ &= -\sum_{n=1}^{\infty}\frac{a^n}{[n]_q}\int_0^{\frac{\pi}{2}} e^{2inx}dx = -\frac{1}{2i}\sum_{n=1}^{\infty}\frac{(-a)^n - a^n}{[n]_q n} \\ &= -\frac{1}{2i}\{\mathrm{Li}_{2,q}(-a) - \mathrm{Li}_{2,q}(a)\}.\end{aligned}$$

To show the latter part, it suffices to use the following result.

LEMMA 3.1. *For* $\mathrm{Im}\,(x) \geq 0$

$$F_q(x) = \exp\left(-\mathrm{Li}_{2,q}(e^{2\pi ix})\right).$$

PROOF. This is easily seen as follows (see [KW] also):

$$\begin{aligned}\log F_q(x) =&(q-1)\sum_{n=1}^{\infty}\log\left(1-q^{-n}e^{2\pi ix}\right) \\ &= -(q-1)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{q^{-nm}e^{2\pi imx}}{m} = -(q-1)\sum_{m=1}^{\infty}\frac{e^{2\pi imx}}{m(q^m-1)} \\ &= -\sum_{m=1}^{\infty}\frac{e^{2\pi imx}}{m[m]_q} = -\mathrm{Li}_{2,q}(e^{2\pi ix}).\end{aligned}$$

Hence the claim follows. □

Thus, we obtain Theorem 1.2.

4. Proof of Theorem 1.4.

This is obvious from Lemma 2.1 and Lemma 3.1, because the formula

$$\lim_{q\downarrow 1}\mathrm{Li}_{2,q}(x) = \mathrm{Li}_2(x).$$

holds for $|x| < 1$.

5. Calculations of special values.

First, we notice that one can directly see that Hölder's result is equivalent to Euler's integral, that is,

$$F\left(\frac{1}{2}\right) = \sqrt{2} \iff (1.1)$$

as follows. From (1.2) we see that

$$\begin{aligned} F\left(\frac{1}{2}\right) &= \exp\left(\int_0^{\frac{1}{2}} \pi t \cot(\pi t)\, dt\right) \\ &= \exp\left(\left[t \log(\sin \pi t)\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \log(\sin \pi t)\, dt\right) \\ &= \exp\left(-\int_0^{\frac{1}{2}} \log(\sin \pi t)\, dt\right). \end{aligned}$$

Hence we observe that

$$\begin{aligned} F\left(\frac{1}{2}\right) = \sqrt{2} &\iff \int_0^{\frac{1}{2}} \log(\sin \pi t) dt = -\frac{1}{2}\log 2 \\ &\iff \int_0^{\frac{\pi}{2}} \log(\sin x)\, dx = -\frac{\pi}{2}\log 2. \end{aligned}$$

We now look at the case $a = 1$ of Theorem 1.2 and take the limit $q \downarrow 1$. Then we have (1.2) again by

$$\lim_{q\downarrow 1} (q-1) \log\left(\prod_{n=1}^{\infty} \frac{1+q^{-n}}{1-q^{-n}}\right) = \frac{\pi^2}{4}.$$

To see this limit formula, it is enough to show that (1.4) and (1.5) in Proposition 1.3.

PROOF OF PROPOSITION 1.3. Let us show these limit formulas by employing the modular form $\Delta(\tau)$ of Ramanujan defined by

$$\Delta(\tau) = e^{2\pi i \tau} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})^{24}$$

for $\operatorname{Im}\boldsymbol{\tau} > 0$. This has the functional equation, i.e. the automorphy of Δ (see, e.g., [We]);

$$\Delta\left(-\frac{1}{\boldsymbol{\tau}}\right) = \boldsymbol{\tau}^{12}\Delta(\boldsymbol{\tau}).$$

For $q > 1$, take $\boldsymbol{\tau} = \frac{\log(q^{-1})}{2\pi i}$. Then, since $\Delta(\boldsymbol{\tau}) = q^{-1}\prod_{n=1}^{\infty}(1-q^{-n})^{24}$, we have

$$\begin{aligned}
\log\left(\prod_{n=1}^{\infty}(1-q^{-n})\right) &= \frac{1}{24}\log\Delta(\boldsymbol{\tau}) + \frac{1}{24}\log q \\
&= \frac{1}{24}\log\left(\boldsymbol{\tau}^{-12}\Delta\left(-\frac{1}{\boldsymbol{\tau}}\right)\right) + \frac{1}{24}\log q \\
&= -\frac{1}{2}\log\boldsymbol{\tau} - \frac{1}{24}\frac{2\pi i}{\boldsymbol{\tau}} + \log\left(\prod_{n=1}^{\infty}(1-e^{-\frac{2\pi i n}{\boldsymbol{\tau}}})\right) + \frac{1}{24}\log q.
\end{aligned}$$

Hence it follows that

$$\begin{aligned}
&(q-1)\log\left(\prod_{n=1}^{\infty}(1-q^{-n})\right) \\
&= -\frac{\pi^2}{6}\frac{q-1}{\log q} - \frac{q-1}{2}\log\boldsymbol{\tau} + (q-1)\log\left(\prod_{n=1}^{\infty}(1-e^{-\frac{2\pi i n}{\boldsymbol{\tau}}})\right) + \frac{q-1}{24}\log q.
\end{aligned}$$

Taking $q\downarrow 1$, we have

$$\frac{q-1}{\log q} \to 1, \quad (q-1)\log\boldsymbol{\tau} \to 0, \quad e^{-\frac{2\pi i}{\boldsymbol{\tau}}} \to 0 \quad \text{and} \quad (q-1)\log q \to 0.$$

Therefore we obtain

$$\lim_{q\downarrow 1}(q-1)\log\left(\prod_{n=1}^{\infty}(1-q^{-n})\right) = -\frac{\pi^2}{6}.$$

(This process shows that one may have more detailed asymptotics.) Then, the relation

$$\begin{aligned}
&(q-1)\log\left(\prod_{n=1}^{\infty}(1+q^{-n})\right) = (q-1)\log\left(\prod_{n=1}^{\infty}\left(\frac{1-q^{-2n}}{1-q^{-n}}\right)\right) \\
&= \frac{1}{q+1}(q^2-1)\log\left(\prod_{n=1}^{\infty}(1-q^{-2n})\right) - (q-1)\log\left(\prod_{n=1}^{\infty}(1-q^{-n})\right)
\end{aligned}$$

yields

$$\lim_{q\downarrow 1}(q-1)\log\left(\prod_{n=1}^{\infty}(1+q^{-n})\right)=\frac{1}{2}\left(-\frac{\pi^2}{6}\right)-\left(-\frac{\pi^2}{6}\right)=\frac{\pi^2}{12}.$$

This completes the proof of Proposition 1.3. □

6. Proof of Theorem 1.5.

For $q>2$ we calculate

$$\int_0^{2\pi}\ell_q(\sin x)\,dx=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{[n]_q}\int_0^{2\pi}(\sin x-1)^n\,dx$$
$$=-\sum_{n=1}^{\infty}\frac{1}{[n]_q}\int_0^{2\pi}(1-\sin x)^n\,dx.$$

A simple calculation shows the following lemma.

LEMMA 6.1.

$$\int_0^{2\pi}(1-\sin x)^n\,dx=2\pi\binom{2n}{n}2^{-n}. \tag{6.1}$$

PROOF. We calculate as

$$\int_0^{2\pi}(1-\sin x)^n dx=\int_0^{2\pi}\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^{2n}dx=\int_0^{2\pi}\left(\sqrt{2}\sin\left(\frac{x}{2}-\frac{\pi}{4}\right)\right)^{2n}dx$$
$$=2^{n+1}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\sin^{2n}\theta\,d\theta=2^{n+2}\int_0^{\frac{\pi}{2}}\sin^{2n}\theta\,d\theta.$$

Therefore, by the well-known evaluation

$$\int_0^{\frac{\pi}{2}}\sin^{2n}\theta\,d\theta=\frac{\pi}{2}\binom{2n}{n}2^{-2n},$$

we get (6.1). □

From this lemma we get the former part of Theorem 1.4. We next show

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}2^{-n}}{[n]_q}=(q-1)\sum_{m=1}^{\infty}\left\{\frac{1}{\sqrt{1-2q^{-m}}}-1\right\}.$$

Recalling the binomial expansion

$$(1-x)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}\binom{2n}{n}2^{-2n}x^n \qquad (|x|<1),$$

we can calculate as

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\binom{2n}{n}2^{-n}}{[n]_q}&=(q-1)\sum_{n=1}^{\infty}\binom{2n}{n}2^{-n}(q^n-1)^{-1}\\&=(q-1)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\binom{2n}{n}2^{-n}q^{-nm}=(q-1)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\binom{2n}{n}2^{-2n}(2q^{-m})^n\\&=(q-1)\sum_{m=1}^{\infty}\left\{\frac{1}{\sqrt{1-2q^{-m}}}-1\right\}.\end{aligned}$$

Thus the theorem follows.

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