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Lamé operators with projective octahedral and icosahedral monodromies.

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ABSTRACT - We show that there exists a Lamé operator L_n with projective octahedral monodromy for each $n \in \frac{1}{2}(\mathbb{N}+\frac{1}{2}) \cup \frac{1}{3}(\mathbb{N}+\frac{1}{2})$, and with projective icosahedral monodromy for each $n \in \frac{1}{3}(\mathbb{N}+\frac{1}{2}) \cup \frac{1}{5}(\mathbb{N}+\frac{1}{2})$. To this end, we construct Grothendieck's dessins d'enfants corresponding to the Belyi morphisms which pull-back hypergeometric operators into Lamé operators L_n with the desired monodromies.

1. Introduction.

In this paper, we consider Lamé differential operators

$$L_n = \left(\frac{d}{dx}\right)^2 + \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-\lambda}\right) \cdot \frac{d}{dx} - \frac{n(n+1)x + B}{4x(x-1)(x-\lambda)}$$

whose solutions are algebraic over $\mathbb{C}(x)$. Here, λ is a complex number with $\lambda \neq 0, 1$, n (so-called degree parameter) is a rational number, and $B \in \mathbb{C}$ is the accessory parameter. The possible finite projective monodromies of L_n were studied by Baldassarri, Chiarellotto, and Dwork, and recently by Beukers, Dahmen, Liţcanu, van der Waall, and Zapponi. One of the most remarkable results is that there are at most finitely many equivalence classes of L_n for fixed $n \in \mathbb{Q}$ and fixed finite monodromy group. This was first done by Chiarellotto [C], and later shown by Liţcanu by using the notion of Grothendieck's dessins d'enfants [L1]. For details of the theory of Grothendieck's dessins d'enfants, see [S] and [SV]. Moreover, Chiarellotto and Liţcanu got the explicit formula for the number of equivalence classes with projective dihedral monodromy of order $2N$ for the case $n = 1$,

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which has been generalized more recently by Dahmen [D] for arbitrary $n \in \mathbb{Z}$. They translated the counting problem for the number of equivalence classes of Lamé operators into that for the number of the dessins compatible with the ramification data of Belyi morphisms which pull-back hypergeometric operators into the Lamé operators. This strategy is based on Klein's theorem, which claims that a second order Fuchsian differential operator with finite projective monodromy is a rational pull-back of a hypergeometric operator in the "*basic Schwarz list*".

To carry out this program, the method of Grothendieck's dessins d'enfants by Litcanu and Dahmen provides a powerful tool. Baldassarri determined the possible finite projective monodromy groups of L_n [B2], but recently, Liţcanu got the same results and the necessary conditions for n to have fixed possible finite projective monodromy group by using the notion of Grothendieck's dessins d'enfants [L2]. By [B2] and [L2], the possible finite projective monodromy groups are dihedral group $\mathcal{D}_{2N}$, octahedral group $\mathcal{S}_4$, or icosahedral group $\mathcal{A}_5$. More recently, Liţcanu [L2] proves the following theorem:

THEOREM 1.1 ([B2], [L2] Theorem 3.4). (1) *If the projective monodromy group of a Lamé operator L_n is dihedral of order at least* 6*, then* $n \in \mathbb{Z}$

(2) *If the projective monodromy group of a Lamé operator L_n is octahedral, then $n \in \frac{1}{2}(\mathbb{Z}+\frac{1}{2}) \cup \frac{1}{3}(\mathbb{Z}+\frac{1}{2})$.*

(3) *If the projective monodromy group of a Lamé operator L_n is icosahedral, then $n \in \frac{1}{3}(\mathbb{Z}+\frac{1}{2}) \cup \frac{1}{5}(\mathbb{Z}+\frac{1}{2})$.*

(4) *There is no Lamé operator with projective cyclic monodromy.*

(5) *There is no Lamé operator with projective tetrahedral monodromy.*

The proof of this theorem is based on the analysis of the Belyi morphism which pull-backs the hypergeometric operator into the Lamé operator, as well as the combinatorial data of the corresponding dessin. Conversely, the following problem arises:

PROBLEM. (1) *For each $n \in \mathbb{Z}$, does there exist a Lamé operator L_n with projective dihedral monodromy?*

(2) *For each $n \in \frac{1}{2}(\mathbb{Z}+\frac{1}{2}) \cup \frac{1}{3}(\mathbb{Z}+\frac{1}{2})$, does there exist a Lamé operator L_n with projective octahedral monodromy?*

(3) *For each $n \in \frac{1}{3}(\mathbb{Z}+\frac{1}{2}) \cup \frac{1}{5}(\mathbb{Z}+\frac{1}{2})$, does there exist a Lamé operator L_n with projective icosahedral monodromy?*

If we replace n by $-n-1$, it is easy to see that $L_n = L_{-n-1}$. Hence we can assume $n > -\frac{1}{2}$. As we saw, (1) is solved by Beukers and van der Waall [BW, Theorem 5.1] and Dahmen [D]. Beukers and van der Waall [BW, Theorem 6.1] and Baldassarri [B2, (3.e)] gave some examples of Lamé operators which have projective octahedral monodromy and icosahedral monodromy.

The aim of this article is to solve (2) and (3) of the problem above. Assuming $n > -\frac{1}{2}$, we have a few possible negative n in each case. Such "exceptional" cases will be dealt with case by case in Remarks 3.3 and 3.4, where we will see that these cases can be easily dismissed. (Notice that the case $n \leq 0$ is also dismissed in [vdW, Theorem 6.8.9, Corollary 6.7.5].) Thus we may assume $n \geq 0$. In this situation, the following theorem gives the existence of Lamé operators L_n with projective octahedral monodromy and projective icosahedral monodromy for each n as in Theorem 1.1 (2) and (3):

MAIN THEOREM. (1) *For each* $n \in \frac{1}{2}(\mathbb{N} + \frac{1}{2}) \cup \frac{1}{3}(\mathbb{N} + \frac{1}{2})$, *there exists a Lamé operator* L_n *with projective octahedral monodromy.*

(2) *For each* $n \in \frac{1}{3}(\mathbb{N} + \frac{1}{2}) \cup \frac{1}{5}(\mathbb{N} + \frac{1}{2})$, *there exists a Lamé operator* L_n *with projective icosahedral monodromy.*

We will prove this theorem by constructing explicitly the dessins compatible with the ramification data of the Belyi morphisms which pull-back hypergeometric operators with the same projective monodromy group into the Lamé operators L_n for each n. From this theorem, we can see that there exist infinitely Lamé operators with projective octahedral monodromy and infinitely many ones with projective icosahedral monodromy, fact that seems unknown. Note that this theorem does not answer the counting problem of the numbers of the equivalence classes of Lamé operators with projective octahedral and icosahedral monodromies.

2. Preliminaries.

Our first aim in this section is to reduce the existence of the Lamé operators with projective octahedral (resp. icosahedral) monodromy to the existence of the Belyi morphisms which pull-back the hypergeometric operators with projective octahedral (resp. icosahedral) monodromies into the Lamé operators. The second aim is to reduce it to the existence of the corresponding dessins.

2.1. – *Hypergeometric operators and Lamé operators.*

In this subsection, we review some results on hypergeometric operators, their rational pull-backs, and Lamé operators.

Let us first consider the linear differential operator on $\mathbb{P}^1$:

$$L = D^n + a_1(z) \cdot D^{n-1} + \cdots + a_{n-1}(z) \cdot D + a_n(z), \tag{1}$$

with $D^i = (\frac{d}{dz})^i$ and $a_j(z)$ rational functions in $\mathbb{C}(z)$ for $1 \leq i, j \leq n$. The linear operator (1) is said to be *Fuchsian* if any point on $\mathbb{P}^1$ is regular or regular singular.

Let $f \colon \mathbb{P}^1 \to \mathbb{P}^1$ be a holomorphic mapping from a projective line with an inhomogeneous coordinate y to another copy of the projective line with the inhomogeneous coordinate z, and let L be a linear differential operator as in (1). The *pull-back* f^*L of the operator L by the map f is the operator defined by

$$f^*L = D'^n + a_1(f(y)) \cdot D'^{n-1} + \cdots + a_{n-1}(f(y)) \cdot D' + a_n(f(y)),$$

where $D' = \frac{1}{f'(y)} \frac{d}{dy}$.

PROPOSITION 2.1. *Let L be a Fuchsian operator on $\mathbb{P}^1$, and $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism. Then f^*L is again Fuchsian.*

PROOF. See [vdW, Proposition 2.6.3]. □

Throughout this paper, we treat projective monodromies of the Fuchsian operators rather than (full) monodromies of them. Let us consider the natural projection P :

$$P : GL(n, \mathbb{C}) \longrightarrow PGL(n, \mathbb{C}).$$

The monodromy group G of the operator (1) is defined in $GL(n, \mathbb{C})$. Then the natural image of G by P is said to be the *projective monodromy* of the operator (1), and we denote it by PG, i.e., $PG = G \cdot Z/Z$ where $Z = \{\lambda \cdot I_n | \lambda \in \mathbb{C}^*\}$. Here, PG is a subgroup of $PGL(n, \mathbb{C})$ and determined up to conjugate.

Let us consider a second order Fuchsian operator with finite projective monodromy. If it has precisely three regular singular points, it is the so-called hypergeometric operator and it has the following normalized form

$$H_{\lambda,\mu,\nu} = \left(\frac{d}{dx}\right)^2 + \left\{\frac{1-\lambda^2}{4x^2} + \frac{1-\mu^2}{4(x-1)^2} + \frac{\lambda^2+\mu^2+\nu^2-1}{4x(x-1)}\right\},$$

where $\lambda + \mu + \upsilon > 1$. The regular singular points of $H_{\lambda,\mu,\upsilon}$ are $0, 1, \infty$, and their exponent differences are λ, μ, υ, respectively. The finite projective monodromy groups of $H_{\lambda,\mu,\upsilon}$ are classified as in the following "*basic Schwarz list*".

(λ, μ, υ)	projective monodromy of $H_{\lambda,\mu,\upsilon}$
$(1/n, 1, 1/n)$	$\mathcal{C}_n$: cyclic of order n
$(1/2, 1/n, 1/2)$	$\mathcal{D}_{2n}$: dihedral of order $2n$
$(1/2, 1/3, 1/3)$	$\mathcal{A}_4$: tetrahedral
$(1/2, 1/3, 1/4)$	$\mathcal{S}_4$: octahedral
$(1/2, 1/3, 1/5)$	$\mathcal{A}_5$: icosahedral

In general, second order Fuchsian operators with finite projective monodromy are characterized by the following theorem by Klein.

THEOREM 2.2 (Klein). *Let L be a second order Fuchsian operator with finite projective monodromy PG in normalized form on $\mathbb{P}^1$, i.e. $L = (\frac{d}{dx})^2 + Q(x)$, $Q(x) \in \mathbb{C}(x)$. Then there exists a morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ which ramifies at most over the set $\{0, 1, \infty\}$, and a unique hypergeometric operator H in the Schwarz list, having the same projective monodromy PG, such that $f^*H = L$. Moreover, the morphism f as above is unique up to Möbius transformations.*

PROOF. See [K] or [B1, Theorem 1.8]. $\square$

PROPOSITION 2.3. *Let L be a Fuchsian operator on $\mathbb{P}^1$ with projective monodromy PG_L, and let f^*L be a pull-back of L by a morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$, with projective monodromy PG_{f^*L}. Then PG_{f^*L} is conjugate in $PGL(2, \mathbb{C})$ to a subgroup of PG_L.*

PROOF. See [vdW, Corollary 2.6.10]. $\square$

A Lamé operator is a second order Fuchsian operator having four regular singular points on $\mathbb{P}^1$ with exponent differences $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$, and $|n + \frac{1}{2}|$ where n is a rational number. If its four regular singular points are $0, 1, \lambda$, and ∞, and their exponent differences are $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$, and $|n + \frac{1}{2}|$ respectively, then, after suitable transformation, we can assume it has the following

Riemann scheme :

$$\left\{\begin{matrix} x=0 & x=1 & x=\lambda & x=\infty \\ 0 & 0 & 0 & -\frac{n}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{n+1}{2} \end{matrix}\right\},$$

and the Lamé operator L_n has the following form:

$$L_n = \left(\frac{d}{dx}\right)^2 + \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-\lambda}\right) \cdot \frac{d}{dx} - \frac{n(n+1)x+B}{4x(x-1)(x-\lambda)},$$

where $B \in \mathbb{C}$ is the accessory parameter.

We are interested in the Lamé operators with finite projective monodromies. Such an operator is, due to Theorem 2.2, of the form f^*H for a hypergeometric operator sitting in the Schwarz list and a holomorphic map $f: \mathbb{P}^1 \to \mathbb{P}^1$ which ramifies at most over 0, 1, and ∞. Thus we are naturally led to the following notion:

Let C be an algebraic curve defined over $\mathbb{C}$. A morphism $f : C \to \mathbb{P}^1$ is said to be *Belyi morphism* if f has at most three critical values.

The Belyi morphisms that we are interested in are of the form $f : \mathbb{P}^1 \to \mathbb{P}^1$ satisfying the following condition (★):

CONDITION (★) : $f^*H_{\lambda,\mu,\upsilon}$ has four regular singular points $0, 1, \lambda$, and ∞ and their exponent differences are $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$, and $|n+\frac{1}{2}|$ respectively.

PROPOSITION 2.4. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a Belyi morphism satisfying the condition* (★)*, $H_{1/2,1/3,1/4}$* (*resp. $H_{1/2,1/3,1/5}$*) *the hypergeometric operator with projective octahedral* (*resp. icosahedral*) *monodromy. Let $n \in \frac{1}{2}(\mathbb{Z}+\frac{1}{2}) \cup \frac{1}{3}(\mathbb{Z}+\frac{1}{2})$* (*resp. $n \in \frac{1}{3}(\mathbb{Z}+\frac{1}{2}) \cup \frac{1}{5}(\mathbb{Z}+\frac{1}{2})$*)*. Then the second order Fuchsian operator $f^*H_{1/2,1/3,1/4}$* (*resp. $f^*H_{1/2,1/3,1/5}$*) *is a Lamé operator having projective octahedral* (*resp. icosahedral*) *monodromy.*

Conversely, any Lamé operator with projective octahedral (*resp. icosahedral*) *monodromy is of this form.*

PROOF. By Theorem 1.1, the projective monodromy of $f^*H_{1/2,1/3,1/4}$ is octahedral or icosahedral, and by Proposition 2.3, it must be conjugate to a subgroup of projective octahedral group. But projective icosahedral group cannot be conjugate to a subgroup of projective octahedral group, then the projective monodromy of $f^*H_{1/2,1/3,1/4}$ is octahedral. The icosahedral case is proved similarly.

The final assertion follows immediately from Theorem 2.2. □

By this, the construction of Lamé operator with projective octahedral monodromy amounts to the construction of the Belyi morphism satisfying the condition (★).

2.2. – *Belyi morphisms and Grothendieck's dessins d'enfants.*

This subsection gives some reviews about *Grothendieck's dessins d'enfants*. For more details, we refer to [S] and [SV]. Let us first recall Belyi's Theorem.

THEOREM 2.5 (Belyi's Theorem). *Let X be an algebraic curve over $\mathbb{C}$. Then X is defined over $\bar{\mathbb{Q}}$ if and only if there exists a morphism $\beta : X \to \mathbb{P}^1(\mathbb{C})$ which ramifies at most over $\{0, 1, \infty\}$.*

PROOF. Well-known; see [Be] or [S, Theorem I.2]. □

Let $\beta : X \to \mathbb{P}^1$ be a Belyi morphism. For a point $P \in X$, we denote by e_P the ramification index at P of β. The Belyi morphism β is said to be *clean* if $e_P = 2$ for any $P \in \beta^{-1}(1)$, and *preclean* if $e_P \leq 2$ for any $P \in \beta^{-1}(1)$. Consider a pair (X, β) consisting of a complex algebraic curve defined over $\bar{\mathbb{Q}}$ and a morphism $\beta : X \to \mathbb{P}^1$. The pair (X, β) is said to be a *Belyi pair* if the morphism β ramifies at most over $\{0, 1, \infty\}$. Two pairs (X, β) and (Y, α) are said to be *isomorphic* if there exists an isomorphism $\phi : X \to Y$ such that $\beta = \alpha \circ \phi$.

DEFINITION 2.6 (Dessins d'enfants). Let X be a compact Riemann surface, X_1 a connected 1-complex, X_0 the set of vertices of X_1, $[\imath]$ an isotopical class of inclusions $\imath : X_1 \hookrightarrow X$. The triple $D = (X_0 \subset X_1, [\imath])$ is said to be *Grothendieck's dessin d'enfant* on X if D satisfies the following conditions:

(1) The complement of X_0 in X_1 is a finite disjoint union of segments and each segment is homeomorphic to the interval $(0, 1)$.

(2) The complement of $\imath(X_1)$ in X is a finite disjoint union of open cells (simply connected regions).

(3) Each element of X_0 is equipped with the mark "•" or "$*$" and if two different elements of X_0 are connected by a segment, one is equipped with "•" and another with "$*$".

DEFINITION 2.7. Two Grothendieck's dessins $D = (X_0 \subset X_1, [\imath])$ on X

and $D' = (X'_0 \subset X'_1, [\imath'])$ on X' are said to be *equivalent* if there exists a homeomorphism $\phi : X \to X'$ such that $\phi|_{\imath(X_1)} : \imath(X_1) \to \imath'(X'_1)$ and $\phi|_{\imath(X_0)} : \imath(X_0) \to \imath'(X'_0)$ are homeomorphisms.

DEFINITION 2.8. A Grothendieck's dessin $D = (X_0 \subset X_1, [\imath])$ is said to be *preclean* if all vertices with the mark "$*$" have valencies ≤ 2. If all vertices with the mark "$*$" have valencies 2, D is said to be *clean*.

Let (X, β) be a Belyi pair. Then from the Belyi pair (X, β), we can construct a dessin $D = (\beta^{-1}(\{0,1\}) \subset \beta^{-1}([0,1]))$ by putting the mark "$\bullet$" on the vertices of $\beta^{-1}(0)$, and "$*$" on the vertices of $\beta^{-1}(1)$.

THEOREM 2.9 (Grothendieck Correspondence). *This correspondence gives a bijection between the set of isomorphism classes of preclean Belyi pairs and the set of equivalence classes of preclean dessins.*

PROOF. See [S, Theorem I.5]. □

The procedure for getting Belyi pairs from dessins is given in [S, Chapter I, §3]. By this correspondence, the most important thing is that the ramification multiplicities of points in $\beta^{-1}(0)$ (resp. $\beta^{-1}(1)$) are translated to the valencies of "$\bullet$" (resp. "$*$").

By Grothendieck Correspondence, we can construct a Lamé operator L_n with projective octahedral and icosahedral monodromy, if there exists a dessin d'enfant corresponding to a Belyi morphism which satisfies the condition (★). In the next section, we are going to construct the dessins such that the corresponding Belyi morphisms satisfy the condition (★) for each n.

DEFINITION 2.10. A Belyi morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ is said to be $*$-*morphism* if $\{0, 1, \infty\} \subseteq f^{-1}(\{0, 1, \infty\})$.

REMARK 2.11. Under the action of $PGL(2, \mathbb{C})$, any Belyi morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ is transformed to a $*$-morphism.

3. Constructions of the Dessins.

3.1. – *The case of projective octahedral monodromy.*

We start this subsection by preparing some notations. We denote a second order Fuchsian differential operator on $\mathbb{P}^1$ by L and its exponent difference at $P \in \mathbb{P}^1$ by $\Delta_{P,L}$. Set $\Delta_L = \sum_{P \in \mathbb{P}^1} (\Delta_{P,L} - 1)$.

We need the following useful lemma.

LEMMA 3.1 ([BD] Lemma 1.5). *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism, and L a Fuchsian second order differential operator. Then*

$$\deg(f) = \frac{\Delta_{f^*L} + 2}{\Delta_L + 2}.$$

PROOF. See [BD, Lemma 5.1]. □

Let L_n denote a Lame operator with projective octahedral monodromy. By Theorem 1.1, we have $n \in \frac{1}{2}(\mathbb{N} + \frac{1}{2}) \cup \frac{1}{3}(\mathbb{N} + \frac{1}{2})$. Due to Proposition 2.4, there is a $*$-morphism such that $L_n = f^* H_{1/2,1/3,1/4}$. We want to construct such a $*$-morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$.

In the octahedral case, by Lemma 3.1,

$$\deg(f) = 12n,$$

and Riemann-Hurwitz formula implies

$$\# f^{-1}(\{0, 1, \infty\}) = 12n + 2.$$

Then we can assume $f^{-1}(\{0, 1, \infty\}) = \{0, 1, \lambda, \infty, a_1, \cdots a_{12n-2}\}$ where, $a_1, \cdots a_{12n-2}$ denote distinct points different from $0, 1, \lambda, \infty$, and thus, possible ramification data of such an f is given as follows:

	0	1	λ	∞	$a_1, \cdots a_{12n-2}$	deg
0	$0, 1$	$0, 1$	$0, 1$	$0, 2n+1$	$0, 2$	$12n$
1	0	0	0	$0, 3n+\frac{3}{2}$	$0, 3$	$12n$
∞	$0, 2$	$0, 2$	$0, 2$	$0, 4n+2$	$0, 4$	$12n$

Here, we explain how to read this table. For $P \in \{0, 1, \infty\}$ (an entry of the first column) and $Q \in \{0, 1, \infty, a_1, \cdots a_{12n-2}\}$ (an entry of the first row), the possible ramification index of f at P is written in the corresponding entry (i.e., (Q, P)-th entry); the number 0 occurs when $f(Q) \neq P$. These values $e_{Q,P}$ are calculated by the formula

$$\Delta_{Q, L_n} = e_{Q,f} \cdot \Delta_{P,H}$$

where $H = H_{1/2,1/3,1/4}$. Moreover, these values must satisfy the following compatibility conditions:

(1) The summation of every row is equal to $\deg(f)$.

(2) Every column contains only one non-zero number.

Let us ask, conversely, if we can construct the $*$-morphism f, or what amounts to the same, the corresponding dessin, starting from a table as above which satisfies the above compatibility conditions for each $n \in \frac{1}{2}(\mathbb{N}+\frac{1}{2}) \cup \frac{1}{3}(\mathbb{N}+\frac{1}{2})$.

(1) The case for $n \in \frac{1}{2}(\mathbb{N}+\frac{1}{2})$

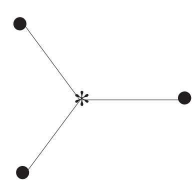
	0	1	λ	∞	$a_1, \cdots a_{12n-2}$	deg
0	1	1	1	0	$(6n-\frac{3}{2})$ pts with *mult.* $=2$	$12n$
1	0	0	0	0	$4n$ pts with *mult.* $=3$	$12n$
∞	0	0	0	$4n+2$	$(2n-\frac{1}{2})$ pts with *mult.* $=4$	$12n$

When we have $n \in \frac{1}{2}(\mathbb{N}+\frac{1}{2})$, then $N = 2n - \frac{1}{2}$ is a natural number, and it is the number of loops of valency 4. Written in terms of the new parameter N, the table becomes the following:

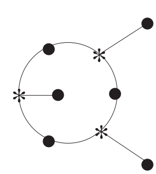
	0	1	λ	∞	$a_1, \cdots a_{12n-2}$	deg
0	1	1	1	0	$3N$ pts with *mult.* $=2$	$6N+3$
1	0	0	0	0	$(2N+1)$ pts with *mult.* $=3$	$6N+3$
∞	0	0	0	$2N+3$	N pts with *mult.* $=4$	$6N+3$

So it suffices to construct the dessins compatible with the table for all $N \in \mathbb{N}$. Now, we construct dessins.

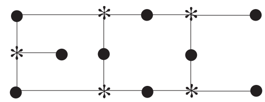
For $N=0$,



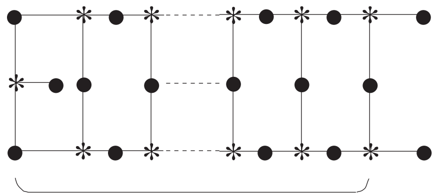
For $N=1$,



For $N = 2$,



For $N = k \geq 2$,



It is easy to check that these are the dessins we want, and thus we could draw dessins inductively for all $N \in \mathbb{N}$.

REMARK 3.2. For each $N \geq 1$, our dessin drawn above is one of those which are compatible with the table. There may exist other dessins compatible with the table.

(2) The case for $n \in \frac{1}{3}(\mathbb{N} + \frac{1}{2})$

	0	1	λ	∞	$a_1, \cdots a_{12n-2}$	deg
0	1	1	0	0	$(6n-1)$ pts with $mult. = 2$	$12n$
1	0	0	0	$3n + \frac{3}{2}$	$(3n - \frac{1}{2})$ pts with $mult. = 3$	$12n$
∞	0	0	2	0	$(3n - \frac{1}{2})$ pts with $mult. = 4$	$12n$

When we have $n \in \frac{1}{3}(\mathbb{N} + \frac{1}{2})$, then $N = 3n - \frac{1}{2}$ is a natural number, and it is the number of loops of valency 4. Then the table becomes the following:

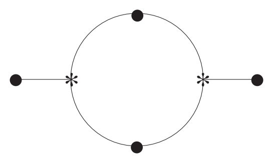
	0	1	λ	∞	$a_1, \cdots a_{12n-2}$	deg
0	1	1	0	0	$2N$ pts with $mult. = 2$	$4N + 2$
1	0	0	0	$N + 2$	N pts with $mult. = 3$	$4N + 2$
∞	0	0	2	0	N pts with $mult. = 4$	$4N + 2$

As in the previous case, it suffices to construct the dessins compatible with the table for all $N \in \mathbb{N}$.

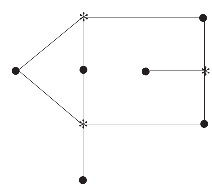
Now, we construct dessins. (For each $N \geq 1$, our dessin is one of those which are compatible with the table.)

For $N = 0$,

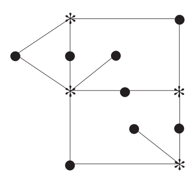
For $N = 1$,



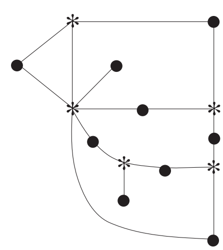
For $N = 2$,



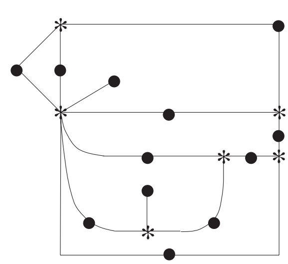
For $N = 3$,



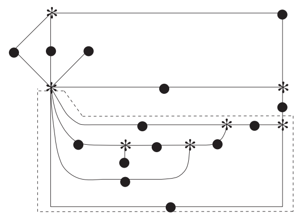
For $N = 4$,



For $N = 5$,

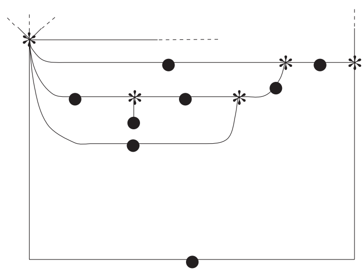


For $N = 6$,

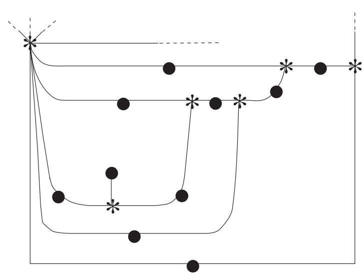


For $N \geq 6$, we can construct dessins inductively according to the following operations. We operate the lower-half part of the dessin. (We draw only the part which is enclosed by the dotted line in the dessin of $N = 6$.)

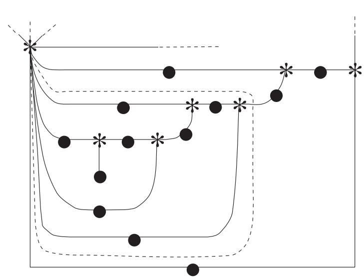
If the dessin with N loops of valency 4 is



then the dessin with $N + 1$ loops of valency 4 is obtained by



and the dessin with $N + 2$ loops of valency 4 is obtained by



We can see that the part which is enclosed by the dotted line in the above dessin repeatedly appears when N is even. Also we can easily see that dessins inductively constructed by this operation are compatible with the table above. In fact, in each step of this operation, the number of "$\bullet$" with valency 2 increase by two, that of "$*$" with valency 3 increase by one, that of edges increase by four, and that of loops having a_i in the fiber over ∞ with valency 4 inside increase by one.

Remark 3.3. As we saw in the introduction, we can assume degree parameter $n > -1/2$. If there exists Lamé operator with projective octahedral monodromy for $n = -1/4$ (resp. $n = -1/6$), there would exist a Belyi morphism which pulls-back $H_{1/2,1/3,1/4}$ into $L_{-1/4}$ (resp. $L_{-1/6}$). But this morphism has to have negative degree, and hence the corresponding dessin does not exist. Therefore, the Lamé operator with projective octahedral monodromy for $n = -1/4$ and $n = -1/6$ does not exist.

3.2. – *The case of projective icosahedral monodromy.*

As in the previous subsection, we want $*$ - morphisms $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that $L_n = f^* H_{1/2,1/3,1/5}$ for $n \in \frac{1}{3}(\mathbb{N} + \frac{1}{2}) \cup \frac{1}{5}(\mathbb{N} + \frac{1}{2})$.

Lemma 3.1 and Riemann-Hurwitz formula implies

$$\# f^{-1}(\{0, 1, \infty\}) = 30n + 2.$$

So we can assume $f^{-1}(\{0, 1, \infty\}) = \{0, 1, \lambda, \infty, a_1, \cdots a_{30n-2}\}$, and the possible ramification data of f is according to the following table.

	0	1	λ	∞	$a_1, \cdots a_{12n-2}$	deg
0	1	1	1	$0,\ 2n+1$	$0,\ 2$	$30n$
1	0	0	0	$0,\ 3n+\frac{3}{2}$	$0,\ 3$	$30n$
∞	0	0	0	$0,\ 5n+\frac{5}{2}$	$0,\ 5$	$30n$

In this subsection, we construct dessins compatible with the ramification data above for each $n \in \frac{1}{3}(\mathbb{N} + \frac{1}{2}) \cup \frac{1}{5}(\mathbb{N} + \frac{1}{2})$.

(3) The case for $n \in \frac{1}{3}(\mathbb{N}+\frac{1}{2})$

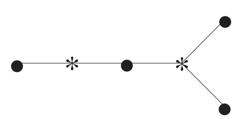
	0	1	λ	∞	$a_1, \cdots a_{12n-2}$	deg
0	1	1	1	0	$(15n-\frac{3}{2})$ pts with *mult.* $=2$	$30n$
1	0	0	0	$3n+\frac{3}{2}$	$(9n-\frac{1}{2})$ pts with *mult.* $=3$	$30n$
∞	0	0	0	0	$6n$ pts with *mult.* $=5$	$30n$

When we have $n \in \frac{1}{3}(\mathbb{N}+\frac{1}{2})$, then $N = 3n - \frac{1}{2}$ is a natural number, and it is the half of the number of the inner loops, which are all of valency 5. Then the table becomes the following.

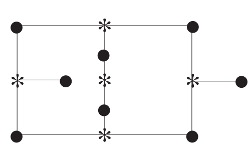
	0	1	λ	∞	$a_1, \cdots a_{12n-2}$	deg
0	1	1	1	0	$(5N+1)$ pts with *mult.* $=2$	$10N+5$
1	0	0	0	$N+2$	$(3N+1)$ pts with *mult.* $=3$	$10N+5$
∞	0	0	0	0	$(2N+1)$ pts with *mult.* $=5$	$10N+5$

So it suffices to construct the dessins compatible with the table for all $N \in \mathbb{N}$. Now, we construct dessins. (For each $N \geq 1$, our dessin is one of those which are compatible with the table.)

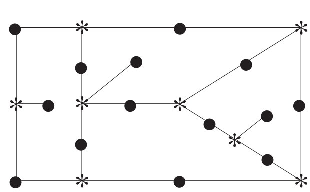
For $N=0$,



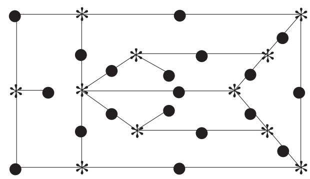
For $N=1$,



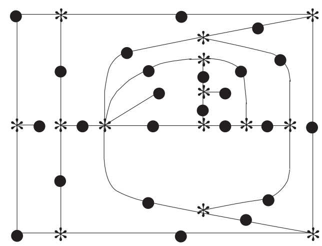
For $N=2$,



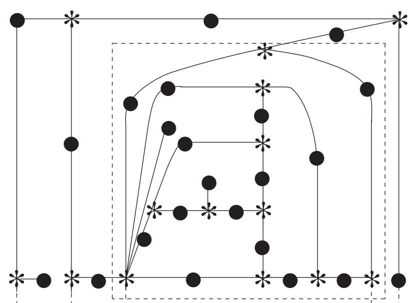
For $N = 3$,



For $N = 4$,

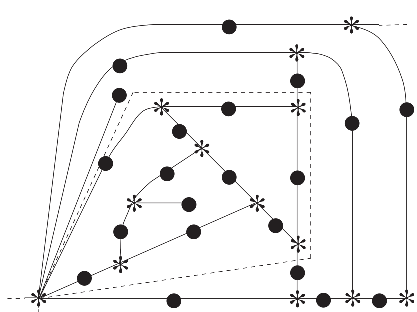


For $N = 5$,



(We draw only the upper-half part, the lower part looks like the same as $N = 4$.)

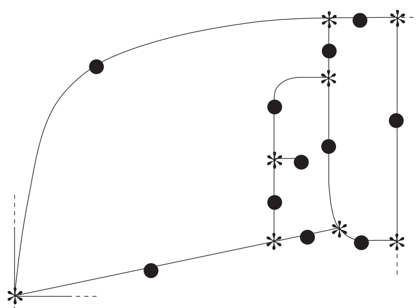
For $N = 6$,



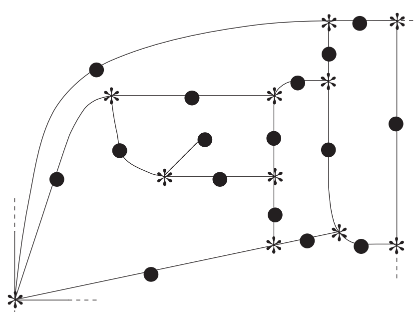
(We draw only the main part which is enclosed by the dotted line in the dessin of $N = 5$, the other parts are same as $N = 5$.)

For $N \geq 6$, we can construct dessins inductively according to the following operations. We operate the upper half part of the dessin. (We draw only the main part that is enclosed by the dotted line in the dessin of $N = 6$.)

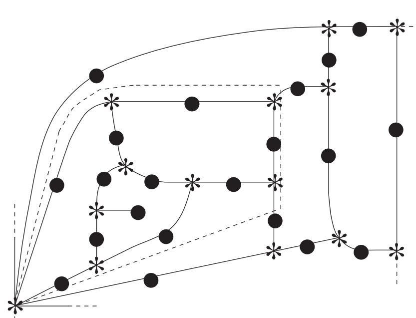
If the dessin with $N = 2m$ inner loops is



then the dessin with $2m + 1$ inner loops is obtained by



and the dessin with $2m + 2$ inner loops is obtained by



We can easily see that the part which is enclosed by the dotted line in the above dessin repeatedly appears when N is even and that the dessins inductively constructed by this operation are compatible with the table.

(4) The case for $n \in \frac{1}{5}(\mathbb{N} + \frac{1}{2})$

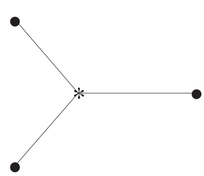
	0	1	λ	∞	$a_1, \cdots a_{12n-2}$	deg
0	1	1	1	0	$(15n - \frac{3}{2})$ pts with $mult. = 2$	$30n$
1	0	0	0	0	$10n$ pts with $mult. = 3$	$30n$
∞	0	0	0	$5n + \frac{5}{2}$	$(5n - \frac{1}{2})$ pts with $mult. = 5$	$30n$

When we have $n \in \frac{1}{5}(\mathbb{N} + \frac{1}{2})$, then $N = 5n - \frac{1}{2}$ is a natural number, and it is the number of loops of valency 5. Then the table becomes the following:

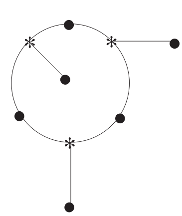
	0	1	λ	∞	$a_1, \cdots a_{12n-2}$	deg
0	1	1	1	0	$3N$ pts with $mult. = 2$	$6N + 3$
1	0	0	0	0	$(2N + 1)$ pts with $mult. = 3$	$6N + 3$
∞	0	0	0	$N + 3$	N pts with $mult. = 5$	$6N + 3$

So it suffices to construct the dessins compatible with the table for all $N \in \mathbb{N}$. Now, we construct dessins. (For each $N \geq 1$, our dessin is one of those which are compatible with the table.)

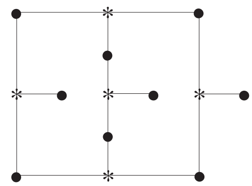
For $N = 0$,



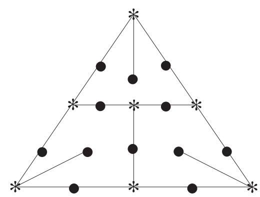
For $N = 1$,



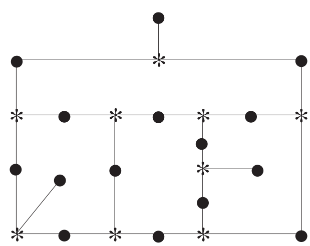
For $N = 2$,



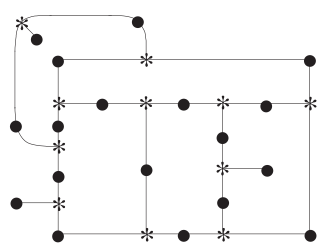
For $N = 3$,



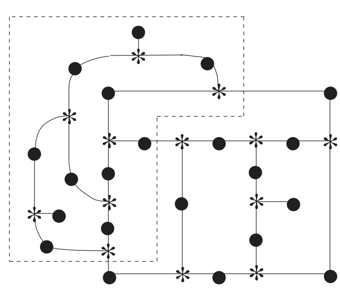
For $N = 4$,



For $N = 5$,

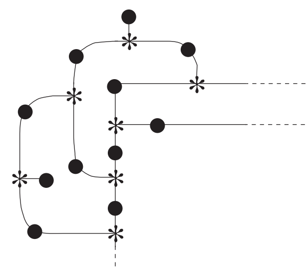


For $N = 6$,

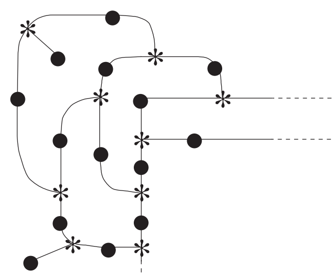


For $N \geq 6$, we can construct dessins inductively according to the following operations. We operate the left side of the dessin. (We draw only the main part which is enclosed by the dotted line in the dessin of $N = 6$.)

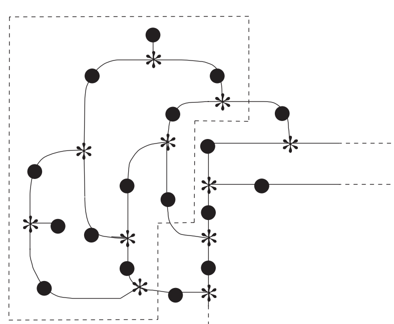
If the dessin with $N = 2k$ ($k \geq 3$) loops of valency 5 is



then the dessin with $N + 1 = 2k + 1$ loops of valency 5 is obtained by



and the dessin with $N + 2 = 2k + 2$ loops of valency 5 is obtained by



We can easily see that the part which is enclosed by the dotted line in the above dessin repeatedly appears when N is even and that the dessins inductively constructed by this operation are compatible with the table.

REMARK 3.4. The "exceptional" values of n in the icosahedral case are $n = -1/6$ and $n = -1/10$. Similarly to the case of octahedral monodromy, we can show that these cases do not occur.

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