Lamé operators with projective octahedral and icosahedral monodromies.

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Abstract - We show that there exists a Lamé operator $L_n$ with projective octahedral monodromy for each $n \in \frac{1}{2}(N + \frac{1}{2}) \cup \frac{1}{2}(N + \frac{1}{2})$, and with projective icosahedral monodromy for each $n \in \frac{1}{2}(N + \frac{1}{2}) \cup \frac{1}{2}(N + \frac{1}{2})$. To this end, we construct Grothendieck’s dessins d’enfants corresponding to the Belyi morphisms which pull-back hypergeometric operators into Lamé operators $L_n$ with the desired monodromies.

1. Introduction.

In this paper, we consider Lamé differential operators

$$L_n = \left( \frac{d}{dx} \right)^2 + \frac{1}{2} \left( \frac{1}{x} + \frac{1}{x - 1} + \frac{1}{x - \lambda} \right) \cdot \frac{d}{dx} - \frac{n(n + 1)x + B}{4x(x - 1)(x - \lambda)}$$

whose solutions are algebraic over $\mathbb{C}(x)$. Here, $\lambda$ is a complex number with $\lambda \neq 0, 1$, $n$ (so-called degree parameter) is a rational number, and $B \in \mathbb{C}$ is the accessory parameter. The possible finite projective monodromies of $L_n$ were studied by Baldassarri, Chiarellotto, and Dwork, and recently by Beukers, Dahmen, Lițcanu, van der Waall, and Zapponi. One of the most remarkable results is that there are at most finitely many equivalence classes of $L_n$ for fixed $n \in \mathbb{Q}$ and fixed finite monodromy group. This was first done by Chiarellotto [C], and later shown by Lițcanu by using the notion of Grothendieck’s dessins d’enfants [L1]. For details of the theory of Grothendieck’s dessins d’enfants, see [S] and [SV]. Moreover, Chiarellotto and Lițcanu got the explicit formula for the number of equivalence classes with projective dihedral monodromy of order $2N$ for the case $n = 1$.

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which has been generalized more recently by Dahmen [D] for arbitrary \( n \in \mathbb{Z} \). They translated the counting problem for the number of equivalence classes of Lamé operators into that for the number of the dessins compatible with the ramification data of Belyi morphisms which pull-back hypergeometric operators into the Lamé operators. This strategy is based on Klein’s theorem, which claims that a second order Fuchsian differential operator with finite projective monodromy is a rational pull-back of a hypergeometric operator in the “basic Schwarz list”.

To carry out this program, the method of Grothendieck’s dessins d’enfants by Litcanu and Dahmen provides a powerful tool. Baldassarri determined the possible finite projective monodromy groups of \( L_n \) [B2], but recently, Litcanu got the same results and the necessary conditions for \( n \) to have fixed possible finite projective monodromy group by using the notion of Grothendieck’s dessins d’enfants [L2]. By [B2] and [L2], the possible finite projective monodromy groups are dihedral group \( D_{2n} \), octahedral group \( S_4 \), or icosahedral group \( A_5 \). More recently, Litcanu [L2] proves the following theorem:

**Theorem 1.1 ([B2], [L2] Theorem 3.4).** (1) If the projective monodromy group of a Lamé operator \( L_n \) is dihedral of order at least 6, then \( n \in \mathbb{Z} \).

(2) If the projective monodromy group of a Lamé operator \( L_n \) is octahedral, then \( n \in \frac{1}{2}(\mathbb{Z} + \frac{1}{2}) \cup \frac{1}{3}(\mathbb{Z} + \frac{1}{2}) \).

(3) If the projective monodromy group of a Lamé operator \( L_n \) is icosahedral, then \( n \in \frac{1}{3}(\mathbb{Z} + \frac{1}{2}) \cup \frac{1}{5}(\mathbb{Z} + \frac{1}{2}) \).

(4) There is no Lamé operator with projective cyclic monodromy.

(5) There is no Lamé operator with projective tetrahedral monodromy.

The proof of this theorem is based on the analysis of the Belyi morphism which pull-backs the hypergeometric operator into the Lamé operator, as well as the combinatorial data of the corresponding dessin. Conversely, the following problem arises:

**Problem.** (1) For each \( n \in \mathbb{Z} \), does there exist a Lamé operator \( L_n \) with projective dihedral monodromy?

(2) For each \( n \in \frac{1}{2}(\mathbb{Z} + \frac{1}{2}) \cup \frac{1}{3}(\mathbb{Z} + \frac{1}{2}) \), does there exist a Lamé operator \( L_n \) with projective octahedral monodromy?

(3) For each \( n \in \frac{1}{3}(\mathbb{Z} + \frac{1}{2}) \cup \frac{1}{5}(\mathbb{Z} + \frac{1}{2}) \), does there exist a Lamé operator \( L_n \) with projective icosahedral monodromy?
If we replace \( n \) by \(-n - 1\), it is easy to see that \( L_n = L_{-n-1} \). Hence we can assume \( n > -\frac{1}{2} \). As we saw, (1) is solved by Beukers and van der Waall [BW, Theorem 5.1] and Dahmen [D]. Beukers and van der Waall [BW, Theorem 6.1] and Baldassarri [B2, (3.e)] gave some examples of Lamé operators which have projective octahedral monodromy and icosahedral monodromy.

The aim of this article is to solve (2) and (3) of the problem above. Assuming \( n > -\frac{1}{2} \) we have a few possible negative \( n \) in each case. Such “exceptional” cases will be dealt with case by case in Remarks 3.3 and 3.4, where we will see that these cases can be easily dismissed. (Notice that the case \( n \leq 0 \) is also dismissed in [vdW, Theorem 6.8.9, Corollary 6.7.5].) Thus we may assume \( n \geq 0 \). In this situation, the following theorem gives the existence of Lamé operators \( L_n \) with projective octahedral monodromy and projective icosahedral monodromy for each \( n \) as in Theorem 1.1 (2) and (3):

**Main Theorem.** (1) For each \( n \in \frac{1}{2}(N + \frac{1}{2}) \cup \frac{1}{3}(N + \frac{1}{2}) \), there exists a Lamé operator \( L_n \) with projective octahedral monodromy.

(2) For each \( n \in \frac{1}{3}(N + \frac{1}{2}) \cup \frac{1}{5}(N + \frac{1}{2}) \), there exists a Lamé operator \( L_n \) with projective icosahedral monodromy.

We will prove this theorem by constructing explicitly the dessins compatible with the ramification data of the Belyi morphisms which pull-back hypergeometric operators with the same projective monodromy group into the Lamé operators \( L_n \) for each \( n \). From this theorem, we can see that there exist infinitely Lamé operators with projective octahedral monodromy and infinitely many ones with projective icosahedral monodromy, fact that seems unknown. Note that this theorem does not answer the counting problem of the numbers of the equivalence classes of Lamé operators with projective octahedral and icosahedral monodromies.

2. Preliminaries.

Our first aim in this section is to reduce the existence of the Lamé operators with projective octahedral (resp. icosahedral) monodromy to the existence of the Belyi morphisms which pull-back the hypergeometric operators with projective octahedral (resp. icosahedral) monodromies into the Lamé operators. The second aim is to reduce it to the existence of the corresponding dessins.
2.1. \textit{Hypergeometric operators and Lamé operators.}

In this subsection, we review some results on hypergeometric operators, their rational pull-backs, and Lamé operators.

Let us first consider the linear differential operator on $\mathbb{P}^1$:

\begin{equation}
L = D^n + a_1(z) \cdot D^{n-1} + \cdots + a_{n-1}(z) \cdot D + a_n(z),
\end{equation}

with $D^i = \left( \frac{d}{dz} \right)^i$ and $a_j(z)$ rational functions in $\mathbb{C}(z)$ for $1 \leq i, j \leq n$. The linear operator (1) is said to be \textit{Fuchsian} if any point on $\mathbb{P}^1$ is regular or regular singular.

Let $f: \mathbb{P}^1 \to \mathbb{P}^1$ be a holomorphic mapping from a projective line with an inhomogeneous coordinate $y$ to another copy of the projective line with the inhomogeneous coordinate $z$, and let $L$ be a linear differential operator as in (1). The \textit{pull-back} $f^* L$ of the operator $L$ by the map $f$ is the operator defined by

\[ f^* L = D^n + a_1(f(y)) \cdot D^{n-1} + \cdots + a_{n-1}(f(y)) \cdot D' + a_n(f(y)), \]

where $D' = \frac{1}{f'(y)} \frac{d}{dy}$.

\textbf{Proposition 2.1.} Let $L$ be a Fuchsian operator on $\mathbb{P}^1$, and $f: \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism. Then $f^* L$ is again Fuchsian.

\textbf{Proof.} See [vdW, Proposition 2.6.3]. \hfill \Box

Throughout this paper, we treat projective monodromies of the Fuchsian operators rather than (full) monodromies of them. Let us consider the natural projection $P$:

\[ P : GL(n, \mathbb{C}) \to PGL(n, \mathbb{C}). \]

The monodromy group $G$ of the operator (1) is defined in $GL(n, \mathbb{C})$. Then the natural image of $G$ by $P$ is said to be the \textit{projective monodromy} of the operator (1), and we denote it by $PG$, i.e., $PG = G \cdot Z/Z$ where $Z = \{ \lambda \cdot I_n | \lambda \in \mathbb{C}^* \}$. Here, $PG$ is a subgroup of $PGL(n, \mathbb{C})$ and determined up to conjugate.

Let us consider a second order Fuchsian operator with finite projective monodromy. If it has precisely three regular singular points, it is the so-called hypergeometric operator and it has the following normalized form

\[ H_{\mu, \nu, \nu} = \left( \frac{d}{dx} \right)^2 + \left\{ \frac{1 - \lambda^2}{4x^2} + \frac{1 - \mu^2}{4(x - 1)^2} + \frac{\lambda^2 + \mu^2 + \nu^2 - 1}{4x(x - 1)} \right\}, \]
where $\lambda + \mu + v > 1$. The regular singular points of $H_{\lambda,\mu,v}$ are $0, 1, \infty$, and their exponent differences are $\lambda, \mu, v$, respectively. The finite projective monodromy groups of $H_{\lambda,\mu,v}$ are classified as in the following “basic Schwarz list”.

\[
\begin{array}{|c|c|}
\hline
(\lambda, \mu, v) & \text{projective monodromy of } H_{\lambda,\mu,v} \\
\hline
(1/n, 1, 1/n) & C_n : \text{cyclic of order } n \\
(1/2, 1/n, 1/2) & D_{2n} : \text{dihedral of order } 2n \\
(1/2, 1/3, 1/3) & A_4 : \text{tetrahedral} \\
(1/2, 1/3, 1/4) & S_4 : \text{octahedral} \\
(1/2, 1/3, 1/5) & A_5 : \text{icosahedral} \\
\hline
\end{array}
\]

In general, second order Fuchsian operators with finite projective monodromy are characterized by the following theorem by Klein.

**Theorem 2.2 (Klein).** Let $L$ be a second order Fuchsian operator with finite projective monodromy $PG$ in normalized form on $\mathbb{P}^1$, i.e. $L = \left( \frac{d}{dx} \right)^2 + Q(x)$, $Q(x) \in \mathbb{C}(x)$. Then there exists a morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ which ramifies at most over the set $\{0, 1, \infty\}$, and a unique hypergeometric operator $H$ in the Schwarz list, having the same projective monodromy $PG$, such that $f^*H = L$. Moreover, the morphism $f$ as above is unique up to Möbius transformations.

**Proof.** See [K] or [B1, Theorem 1.8]. \(\square\)

**Proposition 2.3.** Let $L$ be a Fuchsian operator on $\mathbb{P}^1$ with projective monodromy $PG_L$, and let $f^*L$ be a pull-back of $L$ by a morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$, with projective monodromy $PG_{f^*L}$. Then $PG_{f^*L}$ is conjugate in $PGL(2, \mathbb{C})$ to a subgroup of $PG_L$.

**Proof.** See [vdW, Corollary 2.6.10]. \(\square\)

A Lamé operator is a second order Fuchsian operator having four regular singular points on $\mathbb{P}^1$ with exponent differences $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$, and $|n + \frac{1}{2}|$ where $n$ is a rational number. If its four regular singular points are $0, 1, \lambda$, and $\infty$, and their exponent differences are $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$, and $|n + \frac{1}{2}|$ respectively, then, after suitable transformation, we can assume it has the following
Riemann scheme:

\[
\begin{align*}
\begin{array}{cccc}
  x = 0 & x = 1 & x = \lambda & x = \infty \\
  0 & 0 & 0 & -\frac{n}{2} \\
  \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{n+1}{2}
\end{array}
\end{align*}
\]

and the Lamé operator \( L_n \) has the following form:

\[
L_n = \left( \frac{d}{dx} \right)^2 + \frac{1}{2} \left( \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-\lambda} \right) \cdot \frac{d}{dx} - \frac{n(n+1)x+B}{4x(x-1)(x-\lambda)},
\]

where \( B \in \mathbb{C} \) is the accessory parameter.

We are interested in the Lamé operators with finite projective monodromies. Such an operator is, due to Theorem 2.2, of the form \( f^*H \) for a hypergeometric operator sitting in the Schwarz list and a holomorphic map \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) which ramifies at most over 0, 1, and \( \infty \). Thus we are naturally led to the following notion:

Let \( C \) be an algebraic curve defined over \( \mathbb{C} \). A morphism \( f : C \to \mathbb{P}^1 \) is said to be \textit{Belyi morphism} if \( f \) has at most three critical values.

The Belyi morphisms that we are interested in are of the form \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) satisfying the following condition (\( \bigstar \)):

\textbf{CONDITION (\( \bigstar \))}: \( f^*H_{2,2,2} \) has four regular singular points 0, 1, \( \lambda \), and \( \infty \) and their exponent differences are \( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \), and \( |n + \frac{1}{2}| \) respectively.

\textbf{Proposition 2.4.} Let \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) be a Belyi morphism satisfying the condition (\( \bigstar \)), \( H_{1/2,1/3,1/4} \) (resp. \( H_{1/2,1/3,1/5} \)) the hypergeometric operator with projective octahedral (resp. icosahedral) monodromy. Let \( n \in \frac{1}{2}(Z + \frac{1}{2}) \cup \frac{1}{3}(Z + \frac{1}{2}) \) (resp. \( n \in \frac{1}{3}(Z + \frac{1}{2}) \cup \frac{1}{5}(Z + \frac{1}{2}) \)). Then the second order Fuchsian operator \( f^*H_{1/2,1/3,1/4} \) (resp. \( f^*H_{1/2,1/3,1/5} \)) is a Lamé operator having projective octahedral (resp. icosahedral) monodromy.

Conversely, any Lamé operator with projective octahedral (resp. icosahedral) monodromy is of this form.

\textbf{Proof.} By Theorem 1.1, the projective monodromy of \( f^*H_{1/2,1/3,1/4} \) is octahedral or icosahedral, and by Proposition 2.3, it must be conjugate to a subgroup of projective octahedral group. But projective icosahedral group cannot be conjugate to a subgroup of projective octahedral group, then the projective monodromy of \( f^*H_{1/2,1/3,1/4} \) is octahedral. The icosahedral case is proved similarly.

The final assertion follows immediately from Theorem 2.2. \( \square \)
By this, the construction of Lamé operator with projective octahedral monodromy amounts to the construction of the Belyi morphism satisfying the condition (★).

2.2. – Belyi morphisms and Grothendieck’s dessins d’enfants.

This subsection gives some reviews about Grothendieck’s dessins d’enfants. For more details, we refer to [S] and [SV]. Let us first recall Belyi’s Theorem.

**Theorem 2.5** (Belyi’s Theorem). Let \( X \) be an algebraic curve over \( \mathbb{C} \). Then \( X \) is defined over \( \overline{\mathbb{Q}} \) if and only if there exists a morphism \( \beta : X \to \mathbb{P}^1(\mathbb{C}) \) which ramifies at most over \( \{0, 1, \infty\} \).

**Proof.** Well-known; see [Be] or [S, Theorem I.2]. \( \square \)

Let \( \beta : X \to \mathbb{P}^1 \) be a Belyi morphism. For a point \( P \in X \), we denote by \( e_P \) the ramification index at \( P \) of \( \beta \). The Belyi morphism \( \beta \) is said to be clean if \( e_P = 2 \) for any \( P \in \beta^{-1}(1) \), and preclean if \( e_P \leq 2 \) for any \( P \in \beta^{-1}(1) \). Consider a pair \((X, \beta)\) consisting of a complex algebraic curve defined over \( \overline{\mathbb{Q}} \) and a morphism \( \beta : X \to \mathbb{P}^1 \). The pair \((X, \beta)\) is said to be a Belyi pair if the morphism \( \beta \) ramifies at most over \( \{0, 1, \infty\} \). Two pairs \((X, \beta)\) and \((Y, \alpha)\) are said to be isomorphic if there exists an isomorphism \( \phi : X \to Y \) such that \( \beta = \alpha \circ \phi \).

**Definition 2.6** (Dessins d’enfants). Let \( X \) be a compact Riemann surface, \( X_1 \) a connected 1-complex, \( X_0 \) the set of vertices of \( X_1 \), \([i]\) an isotopical class of inclusions \( i : X_1 \hookrightarrow X \). The triple \( D = (X_0 \subset X_1, [i]) \) is said to be Grothendieck’s dessin d’enfant on \( X \) if \( D \) satisfies the following conditions:

1. The complement of \( X_0 \) in \( X_1 \) is a finite disjoint union of segments and each segment is homeomorphic to the interval \((0, 1)\).
2. The complement of \( i(X_1) \) in \( X \) is a finite disjoint union of open cells (simply connected regions).
3. Each element of \( X_0 \) is equipped with the mark “●” or “★” and if two different elements of \( X_0 \) are connected by a segment, one is equipped with “●” and another with “★”.

**Definition 2.7.** Two Grothendieck’s dessins \( D = (X_0 \subset X_1, [i]) \) on \( X \)
and \( D' = (X'_0 \subset X'_1, [t']) \) on \( X' \) are said to be equivalent if there exists a homeomorphism \( \phi : X \rightarrow X' \) such that \( \phi|_{i(X_1)} : i(X_1) \rightarrow i'(X'_1) \) and \( \phi|_{i(X_0)} : i(X_0) \rightarrow i'(X'_0) \) are homeomorphisms.

**Definition 2.8.** A Grothendieck’s dessin \( D = (X_0 \subset X_1, [i]) \) is said to be preclean if all vertices with the mark “*” have valencies \( \leq 2 \). If all vertices with the mark “*” have valencies \( 2 \), \( D \) is said to be clean.

Let \( (X, \beta) \) be a Belyi pair. Then from the Belyi pair \( (X, \beta) \), we can construct a dessin \( D = (\beta^{-1}(\{0, 1\}) \subset \beta^{-1}([0, 1])) \) by putting the mark “•” on the vertices of \( \beta^{-1}(0) \), and “*” on the vertices of \( \beta^{-1}(1) \).

**Theorem 2.9** (Grothendieck Correspondence). This correspondence gives a bijection between the set of isomorphism classes of preclean Belyi pairs and the set of equivalence classes of preclean dessins.

**Proof.** See [S, Theorem I.5].

The procedure for getting Belyi pairs from dessins is given in [S, Chapter I, §3]. By this correspondence, the most important thing is that the ramification multiplicities of points in \( \beta^{-1}(0) \) (resp. \( \beta^{-1}(1) \)) are translated to the valencies of “•” (resp. “*”).

By Grothendieck Correspondence, we can construct a Lamé operator \( L_n \) with projective octahedral and icosahedral monodromy, if there exists a dessin d’enfant corresponding to a Belyi morphism which satisfies the condition (★). In the next section, we are going to construct the dessins such that the corresponding Belyi morphisms satisfy the condition (★) for each \( n \).

**Definition 2.10.** A Belyi morphism \( f : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) is said to be \(*\)-morphism if \( \{0, 1, \infty\} \subseteq f^{-1}(\{0, 1, \infty\}) \).

**Remark 2.11.** Under the action of \( \text{PGL}(2, \mathbb{C}) \), any Belyi morphism \( f : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) is transformed to a \(*\)-morphism.

3. **Constructions of the Dessins.**

3.1. – The case of projective octahedral monodromy.

We start this subsection by preparing some notations. We denote a second order Fuchsian differential operator on \( \mathbb{P}^1 \) by \( L \) and its exponent difference at \( P \in \mathbb{P}^1 \) by \( A_{P, L} \). Set \( A_L = \sum_{P \in \mathbb{P}^1} (A_{P, L} - 1) \).
We need the following useful lemma.

**Lemma 3.1 ([BD] Lemma 1.5).** Let \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) be a morphism, and \( L \) a Fuchsian second order differential operator. Then

\[
\deg(f) = \frac{\Delta_f - L + 2}{\Delta_L + 2}.
\]

**Proof.** See [BD, Lemma 5.1]. \( \square \)

Let \( L_n \) denote a Lame operator with projective octahedral monodromy. By Theorem 1.1, we have \( n \in \frac{1}{2}(N + \frac{1}{2}) \cup \frac{1}{3}(N + \frac{1}{2}) \). Due to Proposition 2.4, there is a \( * \)-morphism such that \( L_n = f^*H_{1/2,1/3,1/4} \). We want to construct such a \( * \)-morphism \( f : \mathbb{P}^1 \to \mathbb{P}^1 \).

In the octahedral case, by Lemma 3.1,

\[
\deg(f) = 12n,
\]

and Riemann-Hurwitz formula implies

\[
\#f^{-1}(\{0, 1, \infty\}) = 12n + 2.
\]

Then we can assume \( f^{-1}(\{0, 1, \infty\}) = \{0, 1, \lambda, \infty, a_1, \ldots, a_{12n-2}\} \) where, \( a_1, \ldots, a_{12n-2} \) denote distinct points different from \( 0, 1, \lambda, \infty \), and thus, possible ramification data of such an \( f \) is given as follows:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>( \lambda )</th>
<th>( \infty )</th>
<th>( a_1, \ldots, a_{12n-2} )</th>
<th>deg</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0, 1</td>
<td>0, 1</td>
<td>0, 1</td>
<td>0, 2n + 1</td>
<td>0, 2</td>
<td>12n</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0, 3n + \frac{3}{2}</td>
<td>0, 3</td>
<td>12n</td>
</tr>
<tr>
<td>( \infty )</td>
<td>0, 2</td>
<td>0, 2</td>
<td>0, 2</td>
<td>0, 4n + 2</td>
<td>0, 4</td>
<td>12n</td>
</tr>
</tbody>
</table>

Here, we explain how to read this table. For \( P \in \{0, 1, \infty\} \) (an entry of the first column) and \( Q \in \{0, 1, \infty, a_1, \ldots, a_{12n-2}\} \) (an entry of the first row), the possible ramification index of \( f \) at \( P \) is written in the corresponding entry (i.e., \((Q, P)\)-th entry); the number 0 occurs when \( f(Q) \neq P \). These values \( e_{Q,P} \) are calculated by the formula

\[
\Delta_{Q,L_n} = e_{Q,f} \cdot \Delta_{P,H}
\]

where \( H = H_{1/2,1/3,1/4} \). Moreover, these values must satisfy the following compatibility conditions:

1. The summation of every row is equal to \( \deg(f) \).
(2) Every column contains only one non-zero number.

Let us ask, conversely, if we can construct the *-morphism $f$, or what amounts to the same, the corresponding dessin, starting from a table as above which satisfies the above compatibility conditions for each $n \in \frac{1}{2}(N + \frac{1}{2}) \cup \frac{1}{3}(N + \frac{1}{2})$.

(1) The case for $n \in \frac{1}{2}(N + \frac{1}{2})$

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>(\lambda)</th>
<th>(\infty)</th>
<th>(a_1, \ldots, a_{12n-2})</th>
<th>deg</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>((6n - \frac{3}{2})) pts with mult. = 2</td>
<td>12n</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(4n) pts with mult. = 3</td>
<td>12n</td>
</tr>
<tr>
<td>(\infty)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4n + 2</td>
<td>((2n - \frac{1}{2})) pts with mult. = 4</td>
<td>12n</td>
</tr>
</tbody>
</table>

When we have $n \in \frac{1}{2}(N + \frac{1}{2})$, then $N = 2n - \frac{1}{2}$ is a natural number, and it is the number of loops of valency 4. Written in terms of the new parameter $N$, the table becomes the following:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>(\lambda)</th>
<th>(\infty)</th>
<th>(a_1, \ldots, a_{12n-2})</th>
<th>deg</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>(3N) pts with mult. = 2</td>
<td>6N + 3</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>((2N + 1)) pts with mult. = 3</td>
<td>6N + 3</td>
</tr>
<tr>
<td>(\infty)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2N + 3</td>
<td>(N) pts with mult. = 4</td>
<td>6N + 3</td>
</tr>
</tbody>
</table>

So it suffices to construct the dessins compatible with the table for all $N \in \mathbb{N}$. Now, we construct dessins.

For $N = 0$,

```
  *---*---*
      |
      *
```

For $N = 1$,

```
  *---*---*
      |
      *---*
```
For $N = 2$,

![Diagram](123)

For $N = k \geq 2$,

![Diagram](456)

k-loops

It is easy to check that these are the dessins we want, and thus we could draw dessins inductively for all $N \in \mathbb{N}$.

**Remark 3.2.** For each $N \geq 1$, our dessin drawn above is one of those which are compatible with the table. There may exist other dessins compatible with the table.

**(2) The case for $n \in \frac{1}{3}(N + \frac{1}{2})$**

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>$\hat{\lambda}$</th>
<th>$\infty$</th>
<th>$a_1, \cdots a_{12n-2}$</th>
<th>deg</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$(6n - 1)$ pts with $\text{mult.} = 2$</td>
<td>$12n$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$3n + \frac{3}{2}$</td>
<td>$(3n - \frac{1}{2})$ pts with $\text{mult.} = 3$</td>
<td>$12n$</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>$(3n - \frac{1}{2})$ pts with $\text{mult.} = 4$</td>
<td>$12n$</td>
</tr>
</tbody>
</table>

When we have $n \in \frac{1}{3}(N + \frac{1}{2})$, then $N = 3n - \frac{1}{2}$ is a natural number, and it is the number of loops of valency 4. Then the table becomes the following:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>$\hat{\lambda}$</th>
<th>$\infty$</th>
<th>$a_1, \cdots a_{12n-2}$</th>
<th>deg</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$2N$ pts with $\text{mult.} = 2$</td>
<td>$4N + 2$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$N + 2$</td>
<td>$N$ pts with $\text{mult.} = 3$</td>
<td>$4N + 2$</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>$N$ pts with $\text{mult.} = 4$</td>
<td>$4N + 2$</td>
</tr>
</tbody>
</table>

As in the previous case, it suffices to construct the dessins compatible with the table for all $N \in \mathbb{N}$. 

Now, we construct dessins. (For each $N \geq 1$, our dessin is one of those which are compatible with the table.)

For $N = 0$,

For $N = 1$,

For $N = 2$,

For $N = 3$,

For $N = 4$,

For $N = 5$, 
For $N = 6$, 

For $N \geq 6$, we can construct dessins inductively according to the following operations. We operate the lower-half part of the dessin. (We draw only the part which is enclosed by the dotted line in the dessin of $N = 6$.)

If the dessin with $N$ loops of valency 4 is

then the dessin with $N + 1$ loops of valency 4 is obtained by

and the dessin with $N + 2$ loops of valency 4 is obtained by
We can see that the part which is enclosed by the dotted line in the above dessin repeatedly appears when \( N \) is even. Also we can easily see that dessins inductively constructed by this operation are compatible with the table above. In fact, in each step of this operation, the number of “•” with valency 2 increase by two, that of “*” with valency 3 increase by one, that of edges increase by four, and that of loops having \( a_i \) in the fiber over \( \infty \) with valency 4 inside increase by one.

**Remark 3.3.** As we saw in the introduction, we can assume degree parameter \( n > -1/2 \). If there exists Lamé operator with projective octahedral monodromy for \( n = -1/4 \) (resp. \( n = -1/6 \)), there would exist a Belyi morphism which pulls-back \( H_{1/2,1/3,1/5} \) into \( L_{-1/4} \) (resp. \( L_{-1/6} \)). But this morphism has to have negative degree, and hence the corresponding dessin does not exist. Therefore, the Lamé operator with projective octahedral monodromy for \( n = -1/4 \) and \( n = -1/6 \) does not exist.

### 3.2. The case of projective icosahedral monodromy.

As in the previous subsection, we want * - morphisms \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) such that \( L_n = f^*H_{1/2,1/3,1/5} \) for \( n \in \frac{1}{3}(N + \frac{1}{2}) \cup \frac{1}{5}(N + \frac{1}{2}) \).

Lemma 3.1 and Riemann-Hurwitz formula implies

\[
\#f^{-1}(\{0, 1, \infty\}) = 30n + 2.
\]

So we can assume \( f^{-1}(\{0, 1, \infty\}) = \{0, 1, \lambda, \infty, a_1, \ldots, a_{30m-2}\} \), and the possible ramification data of \( f \) is according to the following table.

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( \infty )</th>
<th>( a_1, \ldots, a_{30m-2} )</th>
<th>( \text{deg} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0, 2n + 1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0, 3n + ( \frac{3}{2} )</td>
</tr>
<tr>
<td>\infty</td>
<td>0</td>
<td>0</td>
<td>0, 5n + ( \frac{5}{2} )</td>
</tr>
</tbody>
</table>

In this subsection, we construct dessins compatible with the ramification data above for each \( n \in \frac{1}{3}(N + \frac{1}{2}) \cup \frac{1}{5}(N + \frac{1}{2}) \).
(3) The case for \( n \in \frac{1}{3}(N + \frac{1}{2}) \)

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>( \lambda )</th>
<th>( \infty )</th>
<th>( a_1, \ldots a_{12n-2} )</th>
<th>deg</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>( (15n - \frac{3}{2}) ) pts with mult. = 2</td>
<td>30n</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( 3n + \frac{3}{2} )</td>
<td>( (9n - \frac{1}{2}) ) pts with mult. = 3</td>
<td>30n</td>
</tr>
<tr>
<td>( \infty )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( 6n ) pts with mult. = 5</td>
<td>30n</td>
</tr>
</tbody>
</table>

When we have \( n \in \frac{1}{3}(N + \frac{1}{2}) \), then \( N = 3n - \frac{1}{2} \) is a natural number, and it is the half of the number of the inner loops, which are all of valency 5. Then the table becomes the following.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>( \lambda )</th>
<th>( \infty )</th>
<th>( a_1, \ldots a_{12n-2} )</th>
<th>deg</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>( (5N + 1) ) pts with mult. = 2</td>
<td>10N + 5</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( N + 2 )</td>
<td>( (3N + 1) ) pts with mult. = 3</td>
<td>10N + 5</td>
</tr>
<tr>
<td>( \infty )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( (2N + 1) ) pts with mult. = 5</td>
<td>10N + 5</td>
</tr>
</tbody>
</table>

So it suffices to construct the dessins compatible with the table for all \( N \in \mathbb{N} \). Now, we construct dessins. (For each \( N \geq 1 \), our dessin is one of those which are compatible with the table.)

For \( N = 0 \),

For \( N = 1 \),

For \( N = 2 \),
For $N = 3$, 

For $N = 4$, 

For $N = 5$, 

(We draw only the upper-half part, the lower part looks like the same as $N = 4$.) 

For $N = 6$, 
Lamé operators with projective octahedral etc.

(We draw only the main part which is enclosed by the dotted line in the dessin of \( N = 5 \), the other parts are same as \( N = 5 \).

For \( N \geq 6 \), we can construct dessins inductively according to the following operations. We operate the upper half part of the dessin. (We draw only the main part that is enclosed by the dotted line in the dessin of \( N = 6 \).

If the dessin with \( N = 2m \) inner loops is

then the dessin with \( 2m + 1 \) inner loops is obtained by

and the dessin with \( 2m + 2 \) inner loops is obtained by

We can easily see that the part which is enclosed by the dotted line in
the above dessin repeatedly appears when $N$ is even and that the dessins inductively constructed by this operation are compatible with the table.

(4) The case for $n \in \frac{1}{5}(N + \frac{1}{2})$

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>$\lambda$</th>
<th>$\infty$</th>
<th>$a_1, \cdots a_{12n-2}$</th>
<th>deg</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$(15n - \frac{3}{2})$ pts with $mult. = 2$</td>
<td>$30n$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$10n$ pts with $mult. = 3$</td>
<td>$30n$</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$5n + \frac{5}{2}$</td>
<td>$(5n - \frac{1}{2})$ pts with $mult. = 5$</td>
<td>$30n$</td>
</tr>
</tbody>
</table>

When we have $n \in \frac{1}{5}(N + \frac{1}{2})$, then $N = 5n - \frac{1}{2}$ is a natural number, and it is the number of loops of valency 5. Then the table becomes the following:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>$\lambda$</th>
<th>$\infty$</th>
<th>$a_1, \cdots a_{12n-2}$</th>
<th>deg</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$3N$ pts with $mult. = 2$</td>
<td>$6N + 3$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$(2N + 1)$ pts with $mult. = 3$</td>
<td>$6N + 3$</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$N + 3$</td>
<td>$N$ pts with $mult. = 5$</td>
<td>$6N + 3$</td>
</tr>
</tbody>
</table>

So it suffices to construct the dessins compatible with the table for all $N \in \mathbb{N}$. Now, we construct dessins. (For each $N \geq 1$, our dessin is one of those which are compatible with the table.)

For $N = 0$,

For $N = 1$,  

For $N = 0$,

For $N = 1$,
For $N = 2$,

For $N = 3$,

For $N = 4$,

For $N = 5$,

For $N = 6$,
For $N \geq 6$, we can construct dessins inductively according to the following operations. We operate the left side of the dessin. (We draw only the main part which is enclosed by the dotted line in the dessin of $N = 6$.)

If the dessin with $N = 2k$ ($k \geq 3$) loops of valency 5 is

then the dessin with $N + 1 = 2k + 1$ loops of valency 5 is obtained by

and the dessin with $N + 2 = 2k + 2$ loops of valency 5 is obtained by

We can easily see that the part which is enclosed by the dotted line in the above dessin repeatedly appears when $N$ is even and that the dessins inductively constructed by this operation are compatible with the table.
Remark 3.4. The “exceptional” values of \( n \) in the icosahedral case are \( n = -1/6 \) and \( n = -1/10 \). Similarly to the case of octahedral monodromy, we can show that these cases do not occur.

Acknowledgements. I would like to thank Răzvan Liţcanu for showing me the new version of [L2]. I am deeply grateful to my adviser, Professor Fumiharu Kato for his advice and support. Thanks are also due to Masao Aoki and Koki Itoh for their useful comments. I would like to express my gratitude to the referee for his careful reading of the text and numerous useful comments.

REFERENCES


Manoscritto pervenuto in redazione il 13 gennaio 2005, modificato l’11 maggio 2005