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# An asymptotic higher order very ampleness theorem for blowings-up of projective spaces at general points.

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ABSTRACT - In this paper we consider $k$-very ampleness for linear systems on blowings-up of projective spaces at general points. Using a recent theorem by J. Alexander and A. Hirschowitz we prove a theorem that is sharp from an asymptotic point of view.

## Introduction and statement of the theorem.

Let $X^n \subset P^N$ be a smooth $n$-dimensional non-degenerated irreducible projective variety over an algebraically closed field K. This embedding can be described by means of a line bundle $L$ on $X$ and a $K$-subvectorspace $V \subset H^0(X, L)$. Here $L$ is the restriction of $O_{P^N}(1)$ to $X$ and $V$ corresponds to $H^0(P^N, O_{P^N}(1))$. We say the embedding is $k$-very ample if and only if for each $Z \subset X$, 0-dimensional subscheme of length $k+1$, the natural map $e_Z : V \rightarrow H^0(Z, O_Z \otimes L)$ is surjective. In case $V = H^0(X, L)$ we say $L$ is $k$-vary ample.

Let $\mathrm{Sec}_k(X) \subset P^N$ be the closure of the union of the linear spans $\langle P_0, \ldots, P_k \rangle$ with $P_0, \ldots, P_k$ different points on $X$. Since $\dim(\mathrm{Sec}_k(X)) \leqslant (k+1)\,n + k$, one can expect $k$-very ampleness if $N \geqslant (k+1)(n+1)-1$.

From this consideration it is natural to expect the following to be true.

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CONJECTURE. Let $\pi : X \to P^n$ be the blowing-up of $P^n$ at $r$ general points $P_1, \dots, P_r$. Let $E_i = \pi^{-1}(P_i)$ and let $k$ be a positive integer. There exists a function $d(k)$ such that for $d \geqslant d(k)$ and

$$\binom{d+n}{n} - r\binom{k+n-1}{n} \geqslant (k+1)(n+1)$$

the line bundle $L = \pi^*(O_{P^n}(d)) \otimes O_X\left(-k\sum_{i=1}^{r} E_i\right)$ is $k$-very ample.

This conjecture is true for $k=1$ and $d(k)=3$ (see [6]). For $n=2$ it is proved that $L$ is $k$-very ample for $d \geqslant 4k+1$ in case $r \leqslant \frac{(d+3)^2}{(k+1)^2} - 4$ (see [7]). This corresponds to an upper bound on $r$ which is a polynomial in $d$ with highest degree term equal to $\frac{d^2}{(k+1)^2}$. In case $n=2$, the conjecture should give an upper bound on $r$ which is a polynomial in $d$ with highest degree term $\frac{d^2}{k(k+1)}$. For arbitrary $n$ the conjecture should give an upper bound on $r$ which is a polynomial in $d$ with highest degree term $\frac{d^n/n!}{\binom{k+n-1}{n}}$.

In this short note we show that a recent theorem of J. Alexander and A. Hirschowitz implies the following.

THEOREM. There exists a function $d(k)$ such that in case $d \geqslant d(k)$ and $r \leqslant \binom{d-k+n}{n} \Big/ \binom{k+n-1}{n}$ then $L = \pi^*(O_{P^n}(d)) \otimes O_X\left(-k\sum_{i=1}^{r} E_i\right)$ is $k$-very ample on the blowing-up $X$ of $P^n$ at $r$ general points.

Note that the theorem gives an upper bound on $r$ which is a polynomial in $d$ with highest degree term as in the conjecture.

Although this theorem improves the upper bound on $r$ in case $n=2$ given in [7] for large values of $k$, it shoud be noted that the lower bound on $d$ in [7] is very good.

*Proof of the theorem.*

An application of Theorem 1.1 of J. Alexander and A. Hirschowitz (see [1]) gives the following. There exists a function $d_0(k)$ such that, for $d' \geqslant d_0(k)$ and for $P_1, \dots, P_r$ general points on $P^n$ the natural map $H^0(P^n, O_{P^n}(d')) \to H^0(Z, O_Z(d'))$ has maximal rank where $Z$ is the zero-dimensional subscheme of $P^n$ with ideal $\prod_{i=1}^{r} M^k_{P^n, P_i}$.

The length of $Z$ as above is equal to $r\binom{k+n-1}{n}$. Since $h^0(P^n, O_{P^n}(d')) = \binom{d'+n}{n}$, we conclude that the rational map is surjective in case $r\binom{k+n-1}{n} \leqslant \binom{d'+n}{n}$, hence its cokernel is 0 and, since $h^1(P^n, O_{P^n}(d')) = 0$ this implies $h^1(P^n, I_Z(d')) = 0$. From Proposition 2.2 in [2] this implies $L = \pi^*(O_{P^n}(d'+k)) \otimes O_X\left(-k\sum_{i=1}^{r} E_i\right)$ is $k$-very ample. In other words, $L = \pi^*(O_{P^n}(d)) \otimes O_X\left(-k\sum_{i=1}^{r} E_i\right)$ is $k$-very ample in case $d \geqslant d_0(k) + k := d(k)$ and

$$r\binom{k+n-1}{n} \leqslant \binom{d-k+n}{n}, \quad \text{i.e.} \quad r \leqslant \binom{d-k+n}{n} \bigg/ \binom{k+n-1}{n}.$$

REMARKS. The theorem of J. Alexander and A. Hirschowitz not only concerns projective spaces. Using the Riemann-Roch Theorem and Theorem 2.1 in [2], together with Serre's vanishing theorem, one can state and prove an asymptotic $k$-very ampleness theorem for blowings-up at general points in a more general context.

In case $n = 2$ and $k < 12$ one can find $k$-very ampleness results with reasonable bounds on $d$ using results from [4] and [5].

For arbitrary $n$ and $k = 2$ one finds 2-very ampleness results with reasonable bounds on $d$ using the results of J. Alexander and A. Hischowitz on the associated fat points. See also [3] and the references in that paper.

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