

## On the Global Stability of Contact Discontinuity for Compressible Navier-Stokes Equations.

FEIMIN HUANG (\*) - HUIJIANG ZHAO (\*\*)

ABSTRACT - The asymptotic behavior of the solutions toward the contact discontinuity for the one-dimensional compressible Navier-Stokes equations with a free boundary is investigated. It is shown that the viscous contact discontinuity introduced in [3] is asymptotic stable with arbitrarily *large* initial perturbation if the adiabatic exponent  $\gamma$  is near 1. The case the asymptotic state is given by a combination of viscous contact discontinuity and the rarefaction wave is further investigated. Both the strength of rarefaction wave and the initial perturbation can be arbitrarily *large*.

### 1. Introduction.

We study in present paper the large time behavior of the solutions for the one-dimensional compressible Navier-Stokes (NS) equations. It is known that there have been a lot of works on this subject. Most of these results are concerned with the rarefaction wave and viscous shock wave. We refer to [4, 6-9, 11-15, 17-19] and references therein. It is noted that Nishihara, Yang and Zhao recently established in [17,

(\*) Indirizzo dell'A.: Institute of Applied Mathematics, AMSS, Academia Sinica, Beijing 100080, China. E-mail: fhuang@mail.amt.ac.cn

(\*\*) Indirizzi dell'A.: Laboratory of Mathematical Physics, Wuhan Institute of Physics and Mathematics, Academia Sinica, Wuhan 430071, China. E-mail: hhjjzhao@hotmail.com

School of Political Science and Economics and Graduate School of Science and Engineering, Waseda University, Tokyo 169-8050, Japan.

[18] the global stability of strong rarefaction wave for the  $3 \times 3$  equations. However few result is known for the contact discontinuity except [3] due to various difficulties. As an elementary hyperbolic wave, the asymptotic stability of contact discontinuity should be investigated. Liu and Xin [10, 20] first studied in 1995 the nonlinear stability of contact discontinuity for an artificial viscosity system. It was shown in [10, 20] that the contact discontinuity can not be the asymptotic state and a linear diffusive wave called viscous contact wave (or viscous contact discontinuity), which approximates the contact discontinuity on any finite time interval, instead dominates the large time behavior of the solutions. Motivated by [10, 20], Huang, Matsumura and Shi [3] investigated the contact discontinuity case for a physical system—compressible NS equations, with a free boundary. Unlike the artificial viscous system, the contact discontinuity for compressible NS equations is approximated by a nonlinear diffusive wave. The nonlinear stability of the viscous contact wave was established if the initial perturbation is small(see [3]). This means the nonlinear stability of [3] is *local*. It is worthy to point out that the stability of [10, 20] is also *local*. Thus, a natural question is whether the viscous contact wave constructed in [3] is still stable or not under arbitrarily *large* perturbation. Our main purpose of this paper is to give a positive answer to this question when the adiabatic exponent  $\gamma$  is near 1. Furthermore, the case the asymptotic state is given by a superposition of the viscous contact discontinuity and the rarefaction wave is also treated. In this situation, both the strength of rarefaction wave and the initial perturbation can be arbitrarily *large*.

We now formulate our main results. The 1- $d$  compressible NS equations reads in *Lagrangian* coordinates:

$$(1.1) \quad \begin{cases} v_t - u_x = 0, \\ u_t + p_x = \mu \left( \frac{u_x}{v} \right)_x, \\ \left( e + \frac{u^2}{2} \right)_t + (pu)_x = \left( \kappa \frac{\theta_x}{v} + \mu \frac{uu_x}{v} \right)_x, \end{cases}$$

where  $u(x, t)$  is the velocity,  $v(x, t) > 0$  the specific volume,  $\theta(x, t)$  the absolute temperature,  $\mu > 0$  the viscosity constant and  $\kappa > 0$  the coeffi-

cient of heat conduction. Here we consider the perfect gas so that the pressure

$$(1.2) \quad p = \frac{R\theta}{v} = Av^{-\gamma} e^{\frac{\gamma-1}{R}s},$$

and the internal energy  $e = \frac{R}{\gamma-1}\theta + \text{const.}$ , where  $s$  is the entropy,  $\gamma > 1$  is the adiabatic exponent and  $A, R$  are positive constants. By (1.2)<sub>2</sub>, the entropy  $s$  can also be regarded as a function of  $v$  and  $\theta$ . Our initial and boundary conditions are

$$(1.3) \quad \begin{cases} \theta|_{x=0} = \theta_-, \\ \left( p(v, \theta) - \mu \frac{u_x}{v} \right)(0, t) = p_0, \quad t > 0, \\ (v, u, \theta)(x, 0) = (v_0, u_0, \theta_0)(x) \rightarrow (v_+, u_+, \theta_+) \text{ as } x \rightarrow +\infty, \end{cases}$$

where (1.3)<sub>2</sub> means the gas is attached at the boundary  $x = 0$  to the atmosphere with pressure  $p_0$  (see [19]),  $v_+ > 0, u_+, \theta_+ > 0$  are constants and  $\theta_0(0) = \theta_-$  holds as a compatibility condition.

It is well known that the contact discontinuity is a unique Riemann solution

$$(1.4) \quad (\bar{V}, \bar{U}, \bar{\Theta})(x, t) = \begin{cases} (v_-, u_-, \theta_-), & x < 0, \\ (v_+, u_+, \theta_+), & x > 0, \end{cases}$$

of the following Riemann problem

$$(1.5) \quad \begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ \left( e + \frac{u^2}{2} \right)_t + (pu)_x = 0, \\ (v, u, \theta)(x, 0) = (v_-, u_-, \theta_-), \quad x < 0, \\ (v, u, \theta)(x, 0) = (v_+, u_+, \theta_+), \quad x > 0, \end{cases}$$

if  $u_- = u_+$  and  $p_- = \frac{R\theta_-}{v_-} = p_+ = \frac{R\theta_+}{v_+}$ . In view of [3], the contact discontinuity  $(\bar{V}, \bar{U}, \bar{\Theta})$  in the half space  $x > 0$  can be approximated by a viscous contact wave  $(V, U, \Theta)$  satisfying

$$\|(V - \bar{V}, U - \bar{U}, \Theta - \bar{\Theta})\|_{L^p(0, +\infty)} = O(\kappa)(1+t)^{1/2p}, \quad \text{all } p \geq 1.$$

Here

$$(1.6) \quad V(x, t) = \frac{R}{p_+} \Theta(x, t), \quad U(x, t) = u_+ + \frac{\kappa(\gamma - 1)}{\gamma R} \frac{\Theta(x, t)_x}{\Theta(x, t)},$$

and  $\Theta(x, t) = \Theta(\xi)$ ,  $\xi = \frac{x}{\sqrt{1+t}}$  is the unique self similarity solution of

$$(1.7) \quad \Theta_t = a \left( \frac{\Theta_x}{\Theta} \right)_x, \quad \Theta(0, t) = \theta_-, \quad \Theta(+\infty, t) = \theta_+, \quad a = \frac{\kappa p_+(\gamma - 1)}{\gamma R^2} > 0.$$

From (1.7),  $\Theta(\xi)$  satisfies

$$(1.8) \quad -\frac{1}{2} \xi \Theta' = a \left( \frac{\Theta'}{\Theta} \right)', \quad \Theta(0) = \theta_-, \quad \Theta(+\infty) = \theta_+, \quad ' = \frac{d}{d\xi}.$$

By the same lines as in [2, 3], we have

$$(1.9) \quad \begin{cases} C_1(\gamma - 1)^{-1/2} |\theta_+ - \theta_-| \leq |\Theta'(0)| \leq C_2(\gamma - 1)^{-1/2} |\theta_+ - \theta_-|, \\ \left| \left( |\Theta_{xx}|, \frac{1}{\sqrt{\gamma - 1}} |\Theta_x|, \frac{1}{\gamma - 1} |\Theta - \theta_+| \right) \right| \leq C_3 e^{-\frac{C_4 \xi^2}{\gamma - 1}}, \text{ as } \xi \rightarrow \infty, \end{cases}$$

where  $C_i, i = 1, 2, 3, 4$  are positive constants depending on  $\theta_{\pm}$ . By (1.9), the lemma 1.1 of [3] reads in the following style.

LEMMA 1.1. *If  $|\theta_+ - \theta_-| \leq M(\gamma - 1)$ , then*

$$(1.10) \quad \int_0^\infty \Theta_x^4 dx \leq C(\gamma - 1)^{5/2} (1+t)^{-3/2}, \quad \int_0^\infty \Theta_{xx}^2 dx \leq C(\gamma - 1)^{1/2} (1+t)^{-3/2},$$

$$(1.11) \quad \int_0^\infty \Theta_{xxx}^2 dx \leq C(\gamma - 1)^{-1/2} (1+t)^{-5/2}, \quad \int_0^\infty x(\Theta_x^2 + |\Theta_{xx}|) dx \leq C(\gamma - 1),$$

where the constant  $C$  only depends on  $M$ .

The main aim of present paper is to show the global stability of the viscous contact wave and the superposition of a viscous contact wave and a rarefaction wave. The precise statement of our first result is

THEOREM 1.2. *Assume that  $p_0 = p_+$  and  $|\theta_+ - \theta_-| \leq M(\gamma - 1)$  holds for some constant  $M$ . Assume that  $(V, U, \Theta)$  is the viscous con-*

tact wave constructed in (1.6) and (1.7) and the initial data satisfies

$$(1.12) \quad \begin{cases} 0 < M_0^{-1} \leq v_0(x), \theta_0(x) \leq M_0, & u_0(x) - U(x, 0) \in H^1(0, \infty), \\ (v_0(x) - V(x, 0), s_0(x) - S(x, 0)) \in H_0^1(0, \infty) \end{cases}$$

for some positive constant  $M_0$ , where  $S = \frac{R}{\gamma - 1} \left( \ln \frac{R\Theta}{A} + (\gamma - 1) \ln V \right)$  and  $s_0(x) = s(x, 0)$ . Then there exists a positive constant  $\delta_0 > 0$  such that if  $\gamma < 1 + \delta_0$ , the problem (1.1)-(1.3) has a unique global solution  $(v, u, \theta)(x, t)$  satisfying

$$(1.13) \quad (v - V, u - U, \theta - \Theta)(x, t) \in C(0, +\infty; H^1(0, +\infty)),$$

$$(1.14) \quad (v - V)_x(x, t) \in L^2(0, +\infty; L^2(0, +\infty)),$$

$$(1.15) \quad (u - U, \theta - \Theta)_x(x, t) \in L^2(0, +\infty; H^1(0, +\infty)),$$

and

$$(1.16) \quad \sup_{x \geq 0} |(v - V, u - U, \theta - \Theta)(x, t)| \rightarrow 0, \quad \text{as } t \rightarrow +\infty.$$

REMARK 1.3. The difference  $|\theta_+ - \theta_-|$  is naturally bounded by  $(\gamma - 1)$  multiplying some constant  $M$  from the physical point of view.

When  $p_0 \neq p_+$ , there are two subcases: the superposition of a viscous contact wave and a 3-rarefaction wave or that of a viscous contact wave and a 3-viscous shock wave. Here we only consider the previous one. In this situation, by the basic theory of hyperbolic system of conservation laws, there exists a unique point  $(v_m, u_m, \theta_m)$  such that  $p_0 = p_m = \frac{R\theta_m}{v_m}$  and  $(v_m, u_m, \theta_m)$  belongs to the 3-rarefaction wave curve  $R(v_+, u_+, \theta_+)$  in the phase plane, where

$$(1.17) \quad R(v_+, u_+, \theta_+) = \left\{ (v, u, \theta) \mid s = s_+, u = u_+ - \int_{v_+}^v \lambda(\eta, s_+) d\eta, v > v_+ \right\},$$

$$(1.18) \quad \lambda(v, s) = \sqrt{A\gamma v^{-\gamma-1} e^{\frac{\gamma-1}{R}s}}.$$

The precise statement of our second result is

**THEOREM 1.4.** *Assume that there exists a unique point  $(v_m, u_m, \theta_m)$  such that  $p_0 = p_m$  and  $(v_m, u_m, \theta_m) \in R(v_+, u_+, \theta_+)$  and  $|\theta_+ - \theta_m| + |\theta_m - \theta_-| \leq M(\gamma - 1)$  holds for some constant  $M$ . Assume that  $(V^{cd}, U^{cd}, \Theta^{cd})$  is the viscous contact wave constructed in (1.6) and (1.7), where  $(v_+, u_+, \theta_+)$  is replaced by  $(v_m, u_m, \theta_m)$  and  $(V^r, U^r, \Theta^r)$  is the smooth rarefaction wave constructed in (3.4) satisfying  $S^r = s(V^r, \Theta^r) = s(v_+, \theta_+) = s_+$ . Let*

$$(1.19) \quad V = V^{cd} + V^r - v_m, \quad U = U^{cd} + U^r - u_m, \quad S = S^{cd}.$$

*Assume that the initial data satisfies (1.12). Then there exists a positive constant  $\delta_0 > 1$  such that if  $\gamma < 1 + \delta_0$ , the problem (1.1)-(1.3) has a unique global solution  $(v, u, \theta)(x, t)$  satisfying (1.13)-(1.15). Furthermore,*

$$(1.20) \quad \sup_{x \geq 0} |(v - V^{cd} - v^r + v_m, u - U^{cd} - u^r + u_m, s - S^{cd})(x, t)| \rightarrow 0,$$

*where  $(v^r, u^r, \theta^r)$ ,  $(s^r = s(v^r, \theta^r) = s_+)$  is the 3-rarefaction wave uniquely determined by (3.1).*

Our plan of this paper is as follows. In sect. 2, the single contact discontinuity case is investigated. In sect. 3, the case including the rarefaction wave is treated.

**NOTATIONS.** Throughout this paper, several positive generic constants are denoted by  $c, C$  without confusions. For function spaces,  $H^l(\Omega)$  denotes the  $l$ -th order Sobolev space with its norm

$$(1.21) \quad \|f\|_l = \left( \sum_{j=0}^l \|\partial_x^j f\|^2 \right)^{1/2}, \quad \text{when } \|\cdot\| := \|\cdot\|_{L^2(\Omega)}.$$

The domain  $\Omega$  will be often abbreviated without confusions.

## 2. Global stability of viscous contact wave.

**PROOF OF THEOREM 1.2.** We put the perturbation  $(\phi, \psi, \zeta, \varphi)(x, t)$  by

$$(2.1) \quad \begin{cases} v(x, t) = V(x, t) + \phi(x, t), & u(x, t) = U(x, t) + \psi(x, t), \\ \theta(x, t) = \Theta(x, t) + \zeta(x, t), & s(x, t) = S(x, t) + \varphi(x, t), \end{cases}$$

where  $(V, U, \Theta)(x, t)$  is the viscous contact wave constructed in (1.6)

and (1.7). By (1.6) and (1.7), we have

$$(2.2) \quad \begin{cases} V_t - U_x = 0, \\ U_t + \left(\frac{R\Theta}{V}\right)_x = \mu \left(\frac{U_x}{V}\right)_x + F, \\ \frac{R}{\gamma-1} \Theta_t + p_+ U_x = \kappa \left(\frac{\Theta_x}{V}\right)_x + \mu \frac{U_x^2}{V} + G, \end{cases}$$

where

$$(2.3) \quad F = U_t - \mu \left(\frac{U_x}{V}\right)_x = \frac{\kappa(\gamma-1)}{R\gamma} \left[ (\ln \Theta)_{xt} - \mu \left(\frac{p_+}{R\Theta} (\ln \Theta)_{xx}\right)_x \right],$$

$$(2.4) \quad G = -\mu \frac{U_x^2}{V} = -\frac{\mu p_+}{R\Theta} \left(\frac{\kappa(\gamma-1)}{R\gamma} (\ln \Theta)_{xx}\right)^2.$$

From lemma 1.1, we have

$$(2.5) \quad \|F\|_{L^1} \leq C(\gamma-1)(1+t)^{-1}, \quad \|G\|_{L^1} \leq C(\gamma-1)^{5/2}(1+t)^{-3/2}.$$

Substituting (2.2) into (1.1) and (1.3) yields

$$(2.6) \quad \begin{cases} \phi_t - \psi_x = 0, \\ \psi_t + \left(\frac{R(\Theta + \zeta)}{V + \phi} - \frac{R\Theta}{V}\right)_x = \mu \left(\frac{U_x + \psi_x}{V + \phi} - \frac{U_x}{V}\right)_x - F, \\ \frac{R}{\gamma-1} \zeta_t + \frac{R(\Theta + \zeta)}{V + \phi} (U_x + \psi_x) - p_+ U_x, \\ = \kappa \left(\frac{\Theta_x + \zeta_x}{V + \phi}\right)_x - \kappa \left(\frac{\Theta_x}{V}\right)_x + \mu \frac{(U_x + \psi_x)^2}{V + \phi} - \mu \frac{U_x^2}{V} - G, \\ \left(\frac{R\theta}{V + \phi} - \mu \frac{U_x + \psi_x}{V + \phi}\right) \Big|_{x=0} = p_+, \\ \zeta(0, t) = 0, \\ (\phi, \psi, \zeta)(x, 0) = (v_0 - V, u_0 - U, \theta - \Theta)(x, 0). \end{cases}$$

We shall prove theorem 1.2 by the local existence and the a priori estimate. We look for the solution  $(\phi, \psi, \zeta)$  in the solution space  $X(0, +\infty)$ ,

where

$$(2.7) \quad X(0, T) = \left\{ (\phi, \psi, \zeta) : (\phi, \zeta) \in C(0, T; H_0^1), \psi \in C(0, T; H^1), \frac{1}{2} M_1^{-1} \leq v \leq 2M_1, \frac{1}{4} M_0^{-1} \leq \theta \leq 4M_0, \phi_x \in L^2(0, T; L^2), (\psi, \zeta)_x \in L^2(0, T; H^1) \right\}$$

for some  $0 < T \leq +\infty$ , where the constant  $M_1$  will be determined later. Since the local existence has already been established in [3], we only consider the a priori estimate here. We have

PROPOSITION 2.1. (*A Priori Estimate*). Assume that the conditions of theorem 1.2 hold, then there exists a positive constant  $\delta_0$  such that if  $\gamma < 1 + \delta_0$  and  $(\phi, \psi, \zeta) \in X(0, T)$  is a solution of (2.6) for some positive  $T$ , then the followings hold

$$(2.8) \quad M_1^{-1} \leq v(x, t) \leq M_1, \quad \frac{1}{2} M_0^{-1} \leq \theta(x, t) \leq 2M_0,$$

$$(2.9) \quad \|(\phi, \psi, \zeta)\|_1^2 + \int_0^t \{ \|\phi_x\|^2 + \|(\psi_x, \zeta_x)\|_1^2 \} d\tau \leq C(M_0)(1 + \|(\phi_0, \psi_0, \varphi_0)\|_1^2),$$

where the constant  $C(M_0)$  depends on  $M, M_0$  and the initial data, but does not depend on  $M_1$ .

Proposition 2.1 is proved by a series of lemmas.

LEMMA 2.2. It follows that

$$(2.10) \quad \left\| \left( \left( \sqrt{\Phi \left( \frac{v}{V} \right)}, \psi, \frac{1}{\sqrt{\delta}} \zeta \right) (t) \right) \right\|^2 + \int_0^t \left\| \left( \frac{\psi_x}{v^{1/2}}, \frac{\zeta_x}{v^{1/2}} \right) (\tau) \right\|^2 d\tau \leq C(M_0) \left\{ \|(\phi_0, \psi_0, \sqrt{\delta} \varphi_0)\|_1^2 + C(M_1) \left[ \delta + \delta \int_0^t \left\| \left( \frac{\phi_x}{v^{3/2}}, \frac{\zeta_x}{v^{1/2}} \right) \right\|^2 d\tau + \int_0^t \delta^2 (1 + \tau)^{-8/5} \left\| \left( \psi, \frac{1}{\sqrt{\delta}} \zeta \right) \right\|^2 (\tau) d\tau \right] \right\},$$

where  $\delta = \gamma - 1 < 1$ .



PROOF. Due to [3], we have

$$(2.11) \quad \left( \frac{1}{2} \psi^2 + R\Theta\Phi \left( \frac{v}{V} \right) + \frac{R}{\delta} \Theta\Phi \left( \frac{\theta}{\Theta} \right) \right)_t + \frac{\kappa\Theta}{v\theta^2} \zeta_x^2 + \frac{\mu\Theta\psi_x^2}{v\theta} + H_x + Q = -F\psi - \frac{\zeta G}{\theta},$$

where

$$(2.12) \quad \Phi(\sigma) = \sigma - 1 - \ln \sigma, \quad \Psi(\sigma) = \sigma^{-1} - 1 + \ln \sigma,$$

$$(2.13) \quad H = R \left[ \left( \frac{\Theta + \zeta}{v} - \frac{\Theta}{V} \right) \psi \right] - \mu \left[ \left( \frac{U_x + \psi_x}{v} - \frac{U_x}{V} \right) \psi \right] - \kappa \frac{\zeta}{\theta} \left( \frac{\theta_x}{v} - \frac{\Theta_x}{V} \right),$$

$$(2.14) \quad Q = p_+ \Psi \left( \frac{v}{V} \right) U_x + \frac{p_+}{\delta} \Psi \left( \frac{\theta}{\Theta} \right) U_x + \mu \psi_x U_x \left( \frac{1}{v} - \frac{1}{V} \right) - \frac{\zeta}{\theta} (p_+ - p) U_x - \kappa \frac{\Theta_x}{\theta^2 v} \zeta \zeta_x - \kappa \frac{\zeta_x \phi \Theta \Theta_x}{\theta^2 v V} + \kappa \frac{\zeta \phi}{\theta^2 v V} \Theta_x^2 - \frac{\mu \zeta U_x^2}{\theta} \left( \frac{1}{v} - \frac{1}{V} \right) - \frac{2\mu \zeta}{v\theta} \psi_x U_x.$$

By the formula of  $\Phi(\sigma)$ , we know that  $\Phi(1) = \Phi'(1) = 0$ ,  $\Psi(1) = \Psi'(1) = 0$  and  $\Phi(\sigma)$  is a strictly convex function. This yields that  $\Phi(\sigma) > 0$  and

$$(2.15) \quad \begin{cases} C_1(M_0) C_1(M_1) \phi^2 \leq \Phi \left( \frac{v}{V} \right) \leq C_2(M_0) C_2(M_1) \phi^2, \\ \left| \Psi \left( \frac{\theta}{\Theta} \right) \right| \leq C(M_0) C(M_1) \zeta^2, \end{cases}$$

for some constants  $C_i(M_0), C_i(M_1), i = 1, 2$ . Since

$$(2.16) \quad |Q| \leq \frac{\kappa\Theta}{4v\theta^2} \zeta_x^2 + \frac{\mu\Theta}{4v\theta} \psi_x^2 + C(M_0) C(M_1) \left[ |U_x| \left( \phi^2 + \frac{1}{\delta} \zeta^2 \right) + (\phi^2 + \zeta^2) \Theta_x^2 \right],$$

and

$$(2.17) \quad \int_0^\infty |F\psi| + \left| \frac{G\xi}{\theta} \right| dx \leq \frac{\mu}{4^3 M_0^2} \int_0^\infty \frac{\psi_x^2}{v} dx + \frac{\kappa}{4^4 M_0^3} \int_0^\infty \frac{\xi_x^2}{v} dx + \\ + C(M_0) C(M_1) \left[ \delta(1+t)^{-6/5} + \delta^2(1+t)^{-8/5} \left\| \left( \psi, \frac{1}{\sqrt{\delta}} \xi \right) \right\|^2 \right]$$

due to (2.5), integrating (2.11) over  $R_+ \times (0, t)$  gives

$$(2.18) \quad \left\| \left( \sqrt{\Phi\left(\frac{v}{V}\right)}, \psi, \frac{1}{\sqrt{\delta}} \xi \right) \right\|^2 + \int_0^t \int_0^\infty \frac{\psi_x^2}{v} + \frac{\xi_x^2}{v} dx dt - \int_0^t H(0, t) dt \leq \\ \leq C(M_0) C(M_1) \left\{ \int_0^t \int_0^\infty (\xi^2 + \phi^2)(|\Theta_{xx}| + \Theta_x^2) dx dt + \delta + \right. \\ \left. + \int_0^t \delta^2(1+\tau)^{-8/5} \left\| \left( \psi, \frac{1}{\sqrt{\delta}} \xi \right) \right\|^2 d\tau \right\} + C(M_0) \left\| \left( \phi_0, \psi_0, \frac{1}{\sqrt{\delta}} \xi_0 \right) \right\|^2.$$

On the other hand, the boundary condition (1.2) exactly gives the value of  $\phi(0, t)$ . From [3] and the fact that  $\phi_0(x) \in H_0^1(0, \infty)$ , we have

$$(2.19) \quad \phi(0, t) = \phi_0(0) e^{-\frac{p_+}{\mu}t} = 0,$$

which yields that  $H(0, t) = 0$ . Note that

$$(2.20) \quad |\xi(x, t)| \leq x^{1/2} \|\xi_x\|, \quad |\phi(x, t)| \leq x^{1/2} \|\phi_x\|,$$

due to [16], applying Lemma 1.1 and the boundary condition (2.19), we have

$$(2.21) \quad \int_0^t \int_0^\infty (\xi^2 + \phi^2)(|\Theta_{xx}| + \Theta_x^2) dx dt \leq \\ \leq C(M_0) C(M_1) \delta \int_0^t \left\| \left( \frac{\phi_x}{v^{3/2}}, \frac{\xi_x}{v^{1/2}} \right) \right\|^2 dt.$$

By (1.2) and (2.1), we have

$$(2.22) \quad \|\xi_0\|_1 \leq C\delta \|(\phi_0, \varphi_0)\|_1.$$

Thus, combining (2.18)-(2.22) yields lemma 2.2.

LEMMA 2.3. *There exists a small positive constant  $\delta_0$  such that if  $\delta \leq \delta_0$ , then*

$$(2.23) \quad \left\| \left( \phi, \psi, \frac{1}{\sqrt{\delta}} \zeta \right) (t) \right\|^2 + \|\phi_x\|^2 + \int_0^t \{ \|\phi_x, \psi_x, \zeta_x(\tau)\|^2 \} d\tau \leq \\ \leq C(M_0) \{ 1 + \|(\phi_0, \psi_0, \varphi_0)\|_1^2 \}.$$

PROOF. Following [14], we introduce a new variable  $\tilde{v} = \frac{v}{V}$ . Then (2.6)<sub>2</sub> can be rewritten by the new variable as

$$(2.24) \quad \left( \mu \left( \frac{\tilde{v}_x}{\tilde{v}} \right) - \psi \right)_t - p_x = F.$$

Multiplying (2.24) by  $\frac{\tilde{v}_x}{\tilde{v}}$ , we have

$$(2.25) \quad \left( \frac{\mu}{2} \left( \frac{\tilde{v}_x}{\tilde{v}} \right)^2 - \psi \frac{\tilde{v}_x}{\tilde{v}} \right)_t + \left( \psi \frac{\tilde{v}_t}{\tilde{v}} \right)_x + \frac{R\theta}{v} \left( \frac{\tilde{v}_x}{\tilde{v}} \right)^2 - \frac{R}{v} \zeta_x \frac{\tilde{v}_x}{\tilde{v}} + \\ + \frac{R\theta}{v} \left( \frac{1}{\Theta} - \frac{1}{\theta} \right) \Theta_x \frac{\tilde{v}_x}{\tilde{v}} = \frac{\psi_x^2}{v} + \psi_x U_x \left( \frac{1}{v} - \frac{1}{V} \right) + F \frac{\tilde{v}_x}{\tilde{v}}.$$

The Cauchy inequality yields that

$$(2.26) \quad \left| \frac{R}{v} \zeta_x \frac{\tilde{v}_x}{\tilde{v}} \right| + \left| \psi_x U_x \left( \frac{1}{v} - \frac{1}{V} \right) \right| \leq \frac{R\theta}{4v} \left( \frac{\tilde{v}_x}{\tilde{v}} \right)^2 + \\ + C(M_0) \left( \frac{\zeta_x^2}{v} + \frac{\psi_x^2}{v} + C(M_1) \phi^2 U_x^2 \right),$$

$$(2.27) \quad \left| \frac{R\theta}{v} \left( \frac{1}{\Theta} - \frac{1}{\theta} \right) \Theta_x \frac{\tilde{v}_x}{\tilde{v}} \right| + \left| F \frac{\tilde{v}_x}{\tilde{v}} \right| \leq \frac{R\theta}{4v} \left( \frac{\tilde{v}_x}{\tilde{v}} \right)^2 + \\ + C(M_0) C(M_1) (\zeta^2 \Theta_x^2 + |F|^2),$$

and

$$(2.28) \quad c_1 \frac{\phi_x^2}{v^2} - C(M_0) C(M_1) \phi^2 \Theta_x^2 \leq \left( \frac{\tilde{v}_x}{\tilde{v}} \right)^2 \leq c_3 \frac{\phi_x^2}{v^2} + \\ + C(M_0) C(M_1) \phi^2 \Theta_x^2.$$

Similar to (2.19), we also have  $\left(\frac{\tilde{v}_t}{\tilde{v}}\right)(0, t) = \left(\frac{\phi_t}{v}\right)(0, t) = 0$ . Note that the right hand sides of (2.26)-(2.28) have already been investigated in Lemma 2.2. Integrating (2.25) over  $R_+ \times (0, t)$  and using Lemma 2.2, (2.15), (2.26)-(2.28) and the fact that

$$\left| \psi \frac{\tilde{v}_x}{\tilde{v}} \right| \leq \frac{\mu}{4} \left(\frac{\tilde{v}_x}{\tilde{v}}\right)^2 + C\psi^2,$$

we get

$$\begin{aligned} (2.29) \quad & \left\| \left( \sqrt{\Phi\left(\frac{v}{V}\right)}, \psi, \frac{1}{\sqrt{\delta}} \xi \right) (t) \right\|^2 + \\ & + \left\| \frac{\tilde{v}_x}{\tilde{v}} \right\|^2 + \int_0^t \left\| \left( \frac{\phi_x}{v^{3/2}}, \frac{\psi_x}{v^{1/2}}, \frac{\xi_x}{v^{1/2}} \right) (\tau) \right\|^2 d\tau \leq \\ & \leq C(M_0) \left\{ \|\phi_0, \psi_0, \sqrt{\delta}\varphi_0\|_1^2 + C(M_1) \left[ \delta + \delta \int_0^t \left\| \left( \frac{\phi_x}{v^{3/2}}, \frac{\xi_x}{v^{1/2}} \right) (\tau) \right\|^2 d\tau + \right. \right. \\ & \quad \left. \left. + \int_0^t \delta^2 (1 + \tau)^{-8/5} \left\| \left( \psi, \frac{1}{\sqrt{\delta}} \xi \right) \right\|^2 (\tau) d\tau \right] \right\}, \end{aligned}$$

We now choose a small positive constant  $\delta_0 < 1$  so that  $C(M_0) \cdot C(M_1) \delta_0 < \frac{1}{2}$ . Then for any  $\delta < \delta_0$ , the Gronwall's inequality yields

$$\begin{aligned} (2.30) \quad & \left\| \left( \sqrt{\Phi\left(\frac{v}{V}\right)}, \psi, \frac{1}{\sqrt{\delta}} \xi \right) (t) \right\|^2 + \\ & + \left\| \frac{\tilde{v}_x}{\tilde{v}} \right\|^2 + \int_0^t \left\| \left( \frac{\phi_x}{v^{3/2}}, \frac{\psi_x}{v^{1/2}}, \frac{\xi_x}{v^{1/2}} \right) (\tau) \right\|^2 d\tau \leq \\ & \leq C(M_0) \{ 1 + \|\phi_0, \psi_0, \sqrt{\delta}\varphi_0\|_1^2 \}. \end{aligned}$$

We note here that the constant  $C(M_0)$  does not depend on  $M_1$ .

We now use the method of [14] and (2.30) to show (2.8)<sub>1</sub>. To this end, let

$$(2.31) \quad \eta(\tilde{v}) = \int_1^{\tilde{v}} \frac{\sqrt{\Phi(\sigma)}}{\sigma} d\sigma, \quad \Phi(\sigma) = \sigma - 1 - \ln \sigma.$$

Since

$$(2.32) \quad \eta(\tilde{v}) \rightarrow \begin{cases} -\infty, & \tilde{v} \rightarrow 0+, \\ +\infty, & \tilde{v} \rightarrow +\infty, \end{cases}$$

and

$$(2.33) \quad \begin{aligned} |\eta(\tilde{v}(x, t))| &= \\ &= \left| \int_{+\infty}^x \frac{\partial}{\partial y} \eta(\tilde{v}(y, t)) dy \right| \leq \int_{R_+} \left( \Phi\left(\frac{v}{V}\right) + \left(\frac{\tilde{v}_x}{\tilde{v}}\right)^2 \right) (x, t) dx, \end{aligned}$$

the inequality (2.30) yields that there exists a positive constant  $M_2$  which only depends on  $M, M_0$  and the initial data such that

$$(2.34) \quad M_2^{-1} \leq v(x, t) \leq M_2.$$

Let  $M_1 = M_2$ , then (2.8)<sub>1</sub> is proved. After we obtain the a priori estimate (2.34), it is easy to imply (2.23) from (2.15) and (2.30).

LEMMA 2.4. *It follows that*

$$(2.35) \quad \begin{aligned} \|(\psi_x, \xi_x)(t)\|^2 + \\ + \int_0^t \{ \|(\psi_{xx}, \xi_{xx})(\tau)\|^2 \} d\tau \leq C(M_0)(1 + \|(\phi_0, \psi_0, \xi_0)\|_1^2). \end{aligned}$$

PROOF. We first estimate the term  $\psi_x$ . Multiplying (2.6)<sub>2</sub> by  $-\psi_{xx}$ , we have

$$(2.36) \quad \begin{aligned} \left( \frac{1}{2} \psi_x^2 \right)_t - (\psi_t \psi_x)_x + \mu \frac{\psi_{xx}^2}{v} - p_x \psi_{xx} + \\ + \mu \left( \frac{U_x}{v} - \frac{U_x}{V} \right)_x \psi_{xx} + \mu \psi_x \left( \frac{1}{v} \right)_x \psi_{xx} = F \psi_{xx}. \end{aligned}$$

Since  $p_+$  is constant, we have

$$(2.37) \quad |p_x \psi_{xx}| \leq \frac{\mu}{4v} \psi_{xx}^2 + C(M_0)(\phi^2 + \zeta^2) \Theta_x^2 + C(M_0)(\phi_x^2 + \zeta_x^2).$$

On the other hand,

$$(2.38) \quad \left| \mu \left( \frac{U_x}{v} - \frac{U_x}{V} \right)_x \psi_{xx} \right| + \left| \mu \psi_x \left( \frac{1}{v} \right)_x \psi_{xx} \right| \leq \\ \leq \frac{\mu}{4v} \psi_{xx}^2 + C(M_0) \delta^2 (\Theta_x^4 + \Theta_{xx}^2 + \Theta_{xxx}^2) + C(M_0)(\phi_x^2 + \psi_x^2) + \\ + C(M_0) |\phi_x| |\psi_x| |\psi_{xx}|.$$

We compute by lemma 2.3

$$(2.39) \quad \int_0^\infty |\phi_x| |\psi_x| |\psi_{xx}| dx \leq \sup \{ |\psi_x| \} \|\phi_x\| \|\psi_{xx}\| \\ \leq \sqrt{2} \|\psi_{xx}\|^{3/2} \|\psi_x\|^{1/2} \|\phi_x\| \\ \leq \mu \left\| \frac{\psi_{xx}}{2v^{1/2}} \right\|^2 + C(M_0) \|\psi_x\|^2.$$

Since  $\psi_x(0, t) = \phi_t(0, t) = 0$ , integrating (2.36) over  $R_+ \times (0, t)$  and using (2.37)-(2.39) and Lemmas 1.1 and 2.3 imply

$$(2.40) \quad \|\psi_x(t)\|^2 + \int_0^t \|\psi_{xx}(\tau)\|^2 d\tau \leq C(M_0)(1 + \|(\phi_0, \psi_0, \zeta_0)\|_1^2).$$

We now estimate  $\zeta_x$ . Multiplying (2.5)<sub>3</sub> by  $-\zeta_{xx}$ , we have

$$(2.41) \quad \left( \frac{R}{2\delta} \zeta_x^2 \right)_t - \left( \frac{R}{\delta} \zeta_t \zeta_x \right)_x + \kappa \frac{\zeta_{xx}^2}{v} + \mu \left( \frac{u_x^2}{v} - \frac{U_x^2}{V} \right) \zeta_{xx} + \tilde{Q} = G \zeta_{xx},$$

where

$$(2.42) \quad \tilde{Q} = -(p-p_+) U_x \zeta_{xx} + \kappa \zeta_x \left( \frac{1}{v} \right)_x \zeta_{xx} - p \psi_x \zeta_{xx} + \kappa \left( \frac{\Theta_x}{v} - \frac{\Theta_x}{V} \right)_x \zeta_{xx}.$$

By the same method for the estimate on  $\psi_x$ , we obtain

$$(2.43) \quad \left\| \frac{\zeta_x}{\delta^{1/2}}(t) \right\|^2 + \int_0^t \|\zeta_{xx}(\tau)\|^2 d\tau \leq C(M_0)(1 + \|(\phi_0, \psi_0, \zeta_0)\|_1^2).$$

We omit the details here. Combining (2.40) and (2.43) yields lemma 2.4.

By lemmas 2.3 and 2.4, we have

$$(2.44) \quad \|\zeta(x, t)\|_{L^\infty} \leq C\|\zeta\|_1 \leq C(M_0) \delta^{1/2}(1 + \|(\phi_0, \psi_0, \zeta_0)\|_1).$$

Choosing  $\delta$  suitably small yields

$$(2.45) \quad \frac{1}{2}M_0^{-1} \leq \theta(x, t) = \Theta(x, t) + \zeta(x, t) \leq 2M_0.$$

Thus proposition 2.1 is obtained from (2.34), (2.45) and lemmas 2.3 and 2.4. Theorem 1.2 is easy from the local existence and proposition 2.1.

### 3. Superposition of viscous contact wave and rarefaction wave.

PROOF OF THEOREM 1.4. In this case, there exists a unique point  $(v_m, u_m, \theta_m) \in R(V_+, u_+, \theta_+)$  such that  $p_0 = p_m$  and the superposition of the viscous contact wave connecting  $(v_-, u_m, \theta_-)$  with  $(v_m, u_m, \theta_m)$  and the 3-rarefaction wave connecting  $(v_m, u_m, \theta_m)$  with  $(v_+, u_+, \theta_+)$ , where  $v_- = \frac{R\theta_-}{p_0}$ . The 3-rarefaction wave  $(v^r, u^r, \theta^r)\left(\frac{x}{t}\right)$  connecting  $(v_m, u_m, \theta_m)$  and  $(v_+, u_+, \theta_+)$  is the weak solution of the Riemann problem

$$(3.1) \quad \begin{cases} v_t - u_x = 0, \\ u_t + p(v, \theta)_x = 0, \\ \left( e(v, \theta) + \frac{u^2}{2} \right)_t + (p(v, \theta) u)_x = 0, \\ (v_0, u_0, \theta_0)(x) = \begin{cases} (v_m, u_m, \theta_m), & x < 0, \\ (v_+, u_+, \theta_+), & x > 0. \end{cases} \end{cases}$$

Since the rarefaction wave is a weak solution, it is necessary to construct a smooth approximate rarefaction wave  $(V^r, U^r, \Theta^r)(x, t)$  of

$(v^r, u^r, \theta^r)\left(\frac{x}{t}\right)$  in  $\mathbb{R}_+ \times (0, +\infty)$ . To this end, we apply the idea of [4] to construct the smooth rarefaction wave. The advantage of this kind smooth wave is that the boundary effect can be exactly eliminated. We first construct the solution  $w(x, t)$  of the following problem

$$(3.2) \quad \begin{cases} w_t + ww_x = 0, & (x, t) \in \mathbb{R} \times (0, +\infty), \\ w|_{t=0} = \begin{cases} w_-, & x < 0, \\ w_- + \tilde{w}K_q \int_0^{\varepsilon x} z^q e^{-z} dz, & x \geq 0, \end{cases} \end{cases}$$

where  $w_{\pm} = \lambda(v_{\pm}, s_+)$ ,  $s_+ = s(v_+, \theta_+)$ ,  $\tilde{w} = w_+ - w_-$ ,  $K_q$  is a constant such that  $K_q \int_0^{+\infty} z^q e^{-z} dz = 1$  for large constant  $q \geq 8$  and  $\varepsilon$  is a small positive constant. We have the following properties of  $w(x, t)$  due to [4].

LEMMA 3.1 [4]. *Let  $0 < w_- < w_+$ , then the problem (3.2) has a unique smooth solution  $w(x, t)$  satisfying*

i)  $w_- \leq w(x, t) < w_+$ ,  $w_x \geq 0$ , for  $x \geq 0, t \geq 0$ .

ii) For any  $p(1 \leq p \leq +\infty)$ , there exists a constant  $C_{p,q}$  such that for  $t \geq 0$ ,

$$\begin{aligned} \|w_x(\cdot, t)\|_{L^p} &\leq C_{p,q} \min(\tilde{w}\varepsilon^{1-1/p}, \tilde{w}^{1/p}t^{-1+1/p}), \\ \|w_{xx}(\cdot, t)\|_{L^p} &\leq C_{p,q} \min(\tilde{w}\varepsilon^{2-1/p}, \tilde{w}^{1/q}\varepsilon^{1-1/p+1/q}t^{-1+1/q}). \end{aligned}$$

iii) When  $x \leq w_-t$ ,  $w(x, t) - w_- = w_x(x, t) = w_{xx}(x, t) = 0$ .

iv)  $\limsup_{t \rightarrow +\infty, x \in \mathbb{R}} |w(x, t) - w^R(x, t)| = 0$ .

Here  $w^R(x, t)$  is the Riemann solution of the scalar equation (3.2) with the initial data  $w_0(x) = w_-$ , if  $x < 0$  and  $w_0(x) = w_+$ , if  $x > 0$ .

The smooth approximation  $(\tilde{V}^r, \tilde{U}^r, \tilde{\Theta}^r)$  to  $(v^r, u^r, \theta^r)(x, t)$  is given by

$$(3.3) \quad \begin{cases} \tilde{S}^r(x, t) = s_+, \\ \lambda(\tilde{V}^r(x, t), s_+) = w(x, t), \\ \tilde{U}^r(x, t) = u_+ - \int_{v_+}^{\tilde{V}^r(x, t)} \lambda(\eta, s_+) d\eta. \end{cases}$$



Setting

$$(3.4) \quad (V^r, U^r, \Theta^r)(x, t) := (\tilde{V}^r, \tilde{U}^r, \tilde{\Theta}^r)(x, t + t_0) \Big|_{x \geq 0},$$

where the constant  $t_0 > 1$  will be given later. Then we have

$$(3.5) \quad \begin{cases} V_t^r - U_x^r = 0, \\ U_t^r + p(V^r, \Theta^r)_x = 0, \\ \left( e(V^r, \Theta^r) + \frac{1}{2}(U^r)^2 \right)_t + (p(V^r, \Theta^r) U^r)_x = 0, \\ (V^r, U^r, \Theta^r) \Big|_{x=0} = (v_m, u_m, \theta_m), \\ (V^r, U^r, \Theta^r) \Big|_{t=0} = (V_0^r, U_0^r, \Theta_0^r)(x) = (V^r, U^r, \Theta^r)(x, 0). \end{cases}$$

Due to Lemma 3.1,  $(V^r, U^r, \Theta^r)$  has the following properties.

LEMMA 3.2. *The smooth rarefaction wave  $(V^r, U^r, \Theta^r)(x, t)$  satisfies, if  $q \geq p$ ,*

i)  $U_x^r(x, t) \geq 0, |U_x^r| \leq C\varepsilon$ , for  $t \geq 0, x \geq 0$ .

ii) For any  $p(1 \leq p \leq +\infty)$ , there exists a constant  $C_{p,q}$  such that

$$\begin{aligned} \|(V_x^r, U_x^r, \Theta_x^r)\|_{L^p(x \geq 0)} &\leq C_{p,q} \min \{ \varepsilon^{1-1/p}, (t_0 + t)^{-1+1/p} \}, \\ \|(V_{xx}^r, U_{xx}^r, \Theta_{xx}^r)\|_{L^p(x \geq 0)} &\leq C_{p,q} \min \{ \varepsilon^{2-1/p}, (t_0 + t)^{-1+1/q} \}, \quad t \geq 0. \end{aligned}$$

iii)  $(V^r, U^r, \Theta^r) \Big|_{x \leq \lambda(v_m, s_+)t} = (v_m, u_m, \theta_m)$ ,

$$(V_x^r, U_x^r, \Theta_x^r, V_{xx}^r, U_{xx}^r, \Theta_{xx}^r) \Big|_{x \leq \lambda(v_m, s_+)t} = 0.$$

iv)  $\limsup_{t \rightarrow +\infty, x \in \{x \geq 0\}} \left| (V^r, U^r, \Theta^r)(x, t) - (v^r, u^r, \theta^r) \left( \frac{x}{t} \right) \right| = 0$ .

Let  $(V^{cd}, U^{cd}, \Theta^{cd})$  be the viscous contact wave constructed in (1.6) and (1.7), where  $(v_+, u_+, \theta_+)$  is replaced by  $(v_m, u_m, \theta_m)$ . Then  $(V^{cd}, U^{cd}, \Theta^{cd})$  satisfies

$$(3.6) \quad \begin{cases} V_t^{cd} - U_x^{cd} = 0, \\ U_t^{cd} + \left( \frac{R\Theta^{cd}}{V^{cd}} \right)_x = \mu \left( \frac{U_x^{cd}}{V^{cd}} \right)_x + F^{cd}, \\ \frac{R}{\gamma - 1} \Theta_t^{cd} + p_0 U_x^{cd} = \kappa \left( \frac{\Theta_x^{cd}}{V^{cd}} \right)_x + \mu \frac{(U_x^{cd})^2}{V^{cd}} + G^{cd}, \end{cases}$$

where

$$(3.7) \quad F^{cd} = U_t^{cd} - \mu \left( \frac{U_x^{cd}}{V^{cd}} \right)_x, \quad G^{cd} = -\mu \frac{(U_x^{cd})^2}{V^{cd}}.$$

Let

$$(3.8) \quad \begin{pmatrix} V \\ U \\ S \end{pmatrix} (x, t) = \begin{pmatrix} V^{cd}(x, t) + V^r(x, t) - v_m \\ U^{cd}(x, t) + U^r(x, t) - u_m \\ S^{cd}(x, t) \end{pmatrix},$$

and

$$(3.9) \quad (\phi, \psi, \theta, \varphi)(x, t) = (v - V, u - U, \theta - \Theta, s - S)(x, t),$$

where  $\Theta(x, t) = \frac{A}{R} V^{-\gamma+1} e^{\frac{\gamma-1}{R} S}$  satisfies  $\Theta(0, t) = \theta_-$ . Then the system

(1.1) is rewritten as

$$(3.10) \quad \begin{cases} \phi_t - \psi_x = 0, \\ \psi_t + [p(v, \theta) - p(V, \Theta)]_x = \mu \left( \frac{u_x}{v} \right)_x - \mu \left( \frac{U_x}{V} \right)_x + F, \\ \frac{R}{\gamma - 1} \zeta_t + p\psi_x + (p - p(V, \Theta)) U_x = \\ \qquad \qquad \qquad = \left( \frac{\kappa\theta_x}{v} \right)_x + \frac{\mu u_x^2}{v} - \left( \frac{\kappa\Theta_x}{V} \right)_x - \frac{\mu U_x^2}{V} + G, \\ \left( \frac{R\theta_-}{V + \phi} - \mu \frac{U_x + \psi_x}{V + \phi} \right) \Big|_{x=0} = p_0, \\ \zeta(0, t) = 0, \\ (\phi, \psi, \zeta) |_{t=0} = (v - V, u - U, \theta - U)(x, 0) =: (\phi_0, \psi_0, \zeta_0)(x), \end{cases}$$

where

$$(3.11) \quad F = -[p(V, \Theta) - p(V^r, \Theta^r)]_x + \left[ \mu \left( \frac{U_x}{V} \right)_x - U_t^{cd} \right] =: -F_1 + F_2,$$

$$(3.12) \quad G = -[p(V, \Theta) U_x - p_0 U_x^{cd} - p(V^r, \Theta^r) U_x^r] + \\ + \left[ \left( \frac{\kappa\Theta_x}{V} \right)_x + \frac{\mu U_x^2}{V} - \left( \frac{\kappa\Theta_x^{cd}}{V^{cd}} \right)_x \right] =: -G_1 + G_2.$$

To prove theorem 1.4, it is sufficient to show the same a priori estimate as proposition 2.1. We shall follow the same idea of § 2 to achieve our goal. Similar to (2.11), we have

$$(3.13) \quad \left( \frac{1}{2} \psi^2 + R\Theta\Phi\left(\frac{v}{V}\right) + \frac{R}{\delta} \Theta\Phi\left(\frac{\theta}{\Theta}\right) \right)_t + \frac{\kappa\Theta}{v\theta^2} \zeta_x^2 + \frac{\mu\Theta\psi_x^2}{v\theta} + \\ + H_x + Q_1 U_x^r + Q_2 = F\psi + \frac{\zeta G}{\theta},$$

where  $\Phi$ ,  $\Psi$  and  $H$  are defined in §2 and

$$(3.14) \quad \left\{ \begin{array}{l} Q_1 = p(V, S) \left( \frac{p}{p(V, S)} - 1 + \gamma(\tilde{v} - 1) - \frac{\delta}{R}(s - S) \right), \\ Q_2 = Q_1 U_x^{cd} + Vp(V, S) \left( -\frac{\delta}{R} \Phi\left(\frac{v}{V}\right) + \Psi\left(\frac{\theta}{\Theta}\right) \right) S_t^{cd} + \\ \quad + \mu\psi_x U_x \left( \frac{1}{v} - \frac{1}{V} \right) - \kappa \frac{\Theta_x}{\theta^2 v} \zeta \zeta_x - \kappa \frac{\zeta_x \phi \Theta \Theta_x}{\theta^2 v V} + \\ \quad + \kappa \frac{\zeta \phi}{\theta^2 v V} \Theta_x^2 - \frac{\mu \zeta U_x^2}{\theta} \left( \frac{1}{v} - \frac{1}{V} \right) - \frac{2\mu \zeta}{v\theta} \psi_x U_x. \end{array} \right.$$

Note that  $p(v, s) = Av^{-\gamma} e^{\frac{\gamma-1}{R}s}$  is convex to  $v$  and  $s$ . Thus we have  $Q_1 > 0$  and  $Q_1 U_x^r \geq 0$  due to lemma 3.2. By the definition, we have

$$(3.15) \quad S_t^{cd} = \frac{R\gamma}{\delta V^{cd}} V_t^{cd} = \frac{R\gamma}{\delta V^{cd}} U_x^{cd},$$

and

$$(3.16) \quad R\Theta_x = p(V, S) \left( -\delta + \frac{\gamma V}{V^{cd}} \right) V_x^{cd} - \delta p(V, S) V_x^r.$$

We obtain

$$(3.17) \quad |Q_2| \leq \frac{\kappa\Theta}{4v\theta^2} \zeta_x^2 + \frac{\mu\Theta}{4v\theta} \psi_x^2 + \\ + C(M_0)C(M_1)\{(\phi^2 + \zeta^2)(|\Theta_{xx}^{cd}| + |\Theta_x^{cd}|^2) + (|U_x^r|^2 + \delta^2|V_x^r|^2)(\zeta^2 + \phi^2)\}.$$

Here do not need to estimate the terms involving the contact wave in (3.17) because we have already done in the previous section. By lemma

3.2, we have

$$\begin{aligned}
 (3.18) \quad & \int_0^\infty (|U_x^r|^2 + |V_x^r|^2)(\zeta^2 + \phi^2) \leq \\
 & \leq C(M_0) C(M_1)(t_0 + t)^{-1} (\|\phi\| \|\phi_x\| + \|\zeta\| \|\zeta_x\|) \leq \frac{\mu}{4^4 M_0^3} \left\| \frac{\zeta_x}{v^{1/2}} \right\|^2 + \\
 & + C(M_0) C(M_1) t_0^{-1/4} \left\{ \left\| \frac{\phi_x}{v^{3/2}} \right\| + (1+t)^{-3/2} \|\zeta\|^2 + (1+t)^{-3/2} \left\| \sqrt{\Phi \left( \frac{v}{V} \right)} \right\|^2 \right\}.
 \end{aligned}$$

We now estimate the terms of right hand side of (3.13). Since

$$\begin{aligned}
 (3.19) \quad |F_1| & \leq C(M_0)(|V^r - v_m| + |\Theta^r - \theta_m|) |\Theta_x^{cd}| + \\
 & + C(M_0)(|V^{cd} - v_m| + |\Theta^{cd} - \theta_m|) (|\Theta_x^r|),
 \end{aligned}$$

and  $\Theta_x^r = V^r - v_m = \Theta^r - \theta_m = 0$  for any  $x < \lambda(v_m, s_+)t$  due to lemma 3.2, lemma 1.1 gives

$$(3.20) \quad \|F_1\|_{L^1} \leq C(M_0) \delta e^{-Ct^2/\delta(1+t)},$$

when  $t$  is large. On the other hand,

$$\begin{aligned}
 (3.21) \quad |F_2| & \leq C(M_0) \cdot \\
 & \cdot (|U_t^{cd}| + |U_{xx}^{cd}| + |U_x^{cd} V_x^{cd}| + |U_{xx}^r| + |U_x^r| |V_x^{cd}| + |U_x^{cd} V_x^r|).
 \end{aligned}$$

Similar to (3.20), we have from lemmas 1.1 and 3.2,

$$\begin{aligned}
 (3.22) \quad & \int_0^\infty (|U_{xx}^r| + |U_x^r| |V_x^{cd}| + |U_x^{cd} V_x^r|) dx \leq \\
 & \leq C(M_0)((1+t)^{-1+1/q} + \delta e^{-Ct^2/\delta(1+t)}),
 \end{aligned}$$

with some constant  $q \geq 8$ . Thus we have

$$\begin{aligned}
 (3.23) \quad & \int_0^\infty |F\psi| dx \leq \frac{\mu}{4^3 M_0^2} \int_0^\infty \frac{\psi_x^2}{v} dx + C(M_0) C(M_1) \{ \delta(1+t)^{-6/5} + \\
 & + t_0^{-1/16} (1+t)^{-17/16} + \delta e^{-Ct^2/\delta(1+t)} + [t_0^{-1/8} (1+t)^{-9/8} + \delta^2(1+t)^{-8/5} + \\
 & + \delta^2 e^{-Ct^2/\delta(1+t)}] \|\psi\|^2 \}.
 \end{aligned}$$

In the same way, we obtain

$$(3.24) \quad \int_0^\infty \left| G \frac{\xi}{\theta} \right| dx \leq \frac{\mu}{4^4 M_0^3} \int_0^\infty \frac{\xi_x^2}{v} dx + C(M_0) C(M_1) \left\{ \delta(1+t)^{-6/5} + \right. \\ \left. + t_0^{-1/16}(1+t)^{-17/16} + \delta e^{-Ct^2/\delta(1+t)} + [t_0^{-1/8}(1+t)^{-9/8} + \delta^2(1+t)^{-8/5} + \right. \\ \left. + \delta^2 e^{-Ct^2/\delta(1+t)} \right] \left\| \frac{\xi}{\sqrt{\delta}} \right\|^2 \Big\}.$$

The details are omitted. Thus integrating (3.13) over  $R_+ \times (0, t)$ , combining (2.18), (3.14)-(3.24) and using the fact that  $H(0, t) = 0$  yields

$$(3.25) \quad \left\| \left( \sqrt{\Phi \left( \frac{v}{V} \right)}, \psi, \frac{1}{\sqrt{\delta}} \xi \right) (t) \right\|^2 + \int_0^t \left\{ \left\| \left( \frac{\psi_x}{v^{1/2}}, \frac{\xi_x}{v^{1/2}} \right) (\tau) \right\|^2 \right\} d\tau \leq \\ \leq C(M_0) \left\{ \|(\phi_0, \psi_0, \sqrt{\delta}\varphi_0)\|_1^2 + C(M_1) [\delta + t_0^{-1/16} + \right. \\ \left. + (\delta + t_0^{-1/4}) \int_0^t \left\| \left( \frac{\phi_x}{v^{3/2}}, \frac{\xi_x}{v^{1/2}} \right) \right\|^2 d\tau + \right. \\ \left. + \int_0^t (t_0^{-1/8} + \delta^2)(1+t)^{-9/8} \left\| \left( \sqrt{\Phi \left( \frac{v}{V} \right)}, \psi, \frac{1}{\sqrt{\delta}} \xi \right) \right\|^2 dt \right\}.$$

Following the same procedure as in lemma 2.3, we have

$$(3.26) \quad \left\| \left( \sqrt{\Phi \left( \frac{v}{V} \right)}, \psi, \frac{1}{\sqrt{\delta}} \xi \right) (t) \right\|^2 + \left\| \frac{\tilde{v}_x}{\tilde{v}} \right\|^2 + \\ + \int_0^t \left\{ \left\| \left( \frac{\phi_x}{v^{3/2}}, \frac{\psi_x}{v^{1/2}}, \frac{\xi_x}{v^{1/2}} \right) \right\|^2 \right\} dt \leq C(M_0) \left\{ \|(\phi_0, \psi_0, \sqrt{\delta}\varphi_0)\|_1^2 + \right. \\ \left. + C(M_1) \left[ \delta + t_0^{-1/16} + (\delta + t_0^{-1/4}) \int_0^t \left\| \left( \frac{\phi_x}{v^{3/2}}, \frac{\xi_x}{v^{1/2}} \right) \right\|^2 d\tau + \right. \right. \\ \left. \left. + \int_0^t (t_0^{-1/8} + \delta^2)(1+t)^{-9/8} \left\| \left( \sqrt{\Phi \left( \frac{v}{V} \right)}, \psi, \frac{1}{\sqrt{\delta}} \xi \right) \right\|^2 dt \right] \right\}.$$

We now choose some constants  $\delta_0 < 1$  and  $t_0 > 1$  such that

$$(3.27) \quad C(M_0) C(M_1) \times \max \{ \delta_0, t_0^{-1/16} \} < \frac{1}{2}.$$

Then for any  $\delta \leq \delta_0$ , the Gronwall's inequality yields

$$(3.28) \quad \left\| \left( \sqrt{\Phi\left(\frac{v}{V}\right)}, \psi, \frac{1}{\sqrt{\delta}} \xi \right) (t) \right\|^2 + \left\| \frac{\tilde{v}_x}{\tilde{v}} \right\|^2 + \int_0^t \left\| \left( \frac{\phi_x}{v^{3/2}}, \frac{\psi_x}{v^{1/2}}, \frac{\xi_x}{v^{1/2}} \right) (\tau) \right\|^2 d\tau \leq C(M_0) \{1 + \|(\phi_0, \psi_0, \sqrt{\delta} \varphi_0)\|_1^2\}.$$

Due to (2.31)-(2.33), there exists a positive constant  $M_2$  which only depends on  $M, M_0$  and the initial data such that

$$(3.29) \quad M_2^{-1} \leq v(x, t) \leq M_2.$$

Let  $M_1 = M_2$ , then we obtain (2.8)<sub>1</sub> and

$$(3.30) \quad \left\| \left( \phi, \psi, \frac{1}{\sqrt{\delta}} \xi \right) (t) \right\|^2 + \|\phi_x\|^2 + \int_0^t \{ \|(\phi_x, \psi_x, \xi_x)(\tau)\|^2 \} d\tau \leq C(M_0) \{1 + \|(\phi_0, \psi_0, \varphi_0)\|_1^2\}.$$

The estimates on  $\xi_x$  are omitted here because it is similar to lemma 2.4. Thus we obtain the same a priori estimate as proposition 2.1. Theorem 1.4 is easy from the local existence and the a priori estimate.

*Acknowledgements.* The authors would like to thank Profs. A. Matsumura and K. Nishihara for their constant encouragements and helpful discussions. The authors are supported in part by the JSPS Research Fellowship for foreign researchers. The second author is also supported by the National Natural Science Foundation of China under contracts 10001036 and 10041003 respectively, and the grant from the Chinese Academy of Sciences entitled by «Yin Jin Guo Wai Jie Chu Ren Cai Ji Jin».

#### REFERENCES

- [1] F. V. ATKINSON - L. A. PELETIER, *Similarity solutions of the nonlinear diffusion equation*, Arch. rational Mech. Anal., **54** (1974), pp. 373-392.
- [2] C. T. DUYN - L. A. PELETIER, *A class of similarity solution of the nonlinear diffusion equation*, Nonlinear Analysis, T.M.A., **1** (1977), pp. 223-233.
- [3] F. HUANG - A. MATSUMURA - X. SHI, *On the stability of contact discontinuity for compressible Navier-Stokes equations with free boundary*, to appear in Osaka J. Math.

- [4] F. HUANG - A. MATSUMURA - X. SHI, *A gas-solid free boundary problem for a compressible viscous gas*, to appear in SIAM J. Math. Anal., 2003.
- [5] L. HSIAO - T. P. LIU, *Nonlinear diffusive phenomena of nonlinear hyperbolic systems*, Chin. Ann. Math. B, **14:4** (1993), pp. 465-480.
- [6] S. KAWASHIMA - A. MATSUMURA, *Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion*, Commun. Math. Phys., **101** (1985), pp. 97-127.
- [7] S. KAWASHIMA - A. MATSUMURA - K. NISHIHARA, *Asymptotic behavior of solutions for the equations of a viscous heat-conductive gas*, Proc. Japan Acad., Ser. A, **62** (1986), pp. 249-252.
- [8] T. LIU, *Shock waves for compressible Navier-Stokes equations are stable*, Comm. Pure Appl. Math., **XXXIX** (1986), pp. 565-594.
- [9] T. LIU - Z. XIN, *Nonlinear stability of rarefaction waves for compressible Navier-Stokes equations*, Comm. Math. Phys., **118** (1988), pp. 451-465.
- [10] T. P. LIU - Z. P. XIN, *Pointwise decay to contact discontinuities for systems of viscous conservation laws*, Asian J. Math., **1**, no. 1 (1997), pp. 34-84.
- [11] A. MATSUMURA - M. MEI, *Convergence to travelling fronts of solutions of the p-system with viscosity in the presence of a boundary*, Arch. Rat. Mech. Anal., **146** (1999), pp. 1-22.
- [12] A. MATSUMURA - K. NISHIHARA, *On the stability of traveling wave solutions of a one-dimensional model system for compressible viscous gas*, Japan J. Appl. Math., **2** (1985), pp. 17-25.
- [13] A. MATSUMURA - K. NISHIHARA, *Asymptotics toward the rarefaction wave of the solutions of a one-dimensional model system for compressible viscous gas*, Japan J. Appl. Math., **3** (1986), pp. 1-13.
- [14] A. MATSUMURA - K. NISHIHARA, *Global stability of the rarefaction wave of a one-dimensional model system for compressible viscous gas*, Commun. Math. Phys., **144** (1992), pp. 325-335.
- [15] A. MATSUMURA - K. NISHIHARA, *Global asymptotics toward the rarefaction wave for solutions of viscous p-system with boundary effect*, Q. Appl. Math., Vol. **LVIII** (2000), pp. 69-83.
- [16] Y. NIKKUNI - S. KAWASHIMA, *Stability of stationary solutions to the half-space problem for the discrete Boltzmann equation with multiple collisions*, Kyushu J. Math., **54**, no. 2 (2000), pp. 233-255.
- [17] K. NISHIHARA - T. YANG - H. ZHAO, *Nonlinear stability of strong rarefaction waves for compressible Navier-Stokes equations*, Preprint 2002.
- [18] K. NISHIHARA - H. ZHAO, *Global stability of strong rarefaction waves for compressible Navier-Stokes equations with boundary effects*, in preparation.
- [19] T. PAN - H. X. LIU - K. NISHIHARA, *Asymptotic behavior of a one-dimensional compressible viscous gas with free boundary*, to appear in SIAM J. Math. Anal.
- [20] Z. P. XIN, *On nonlinear stability of contact discontinuities*, *Proceeding of 5th international conferences on hyperbolic problems: theory, numerics and applications*, edited by Glimm, etc., World Scientific, 1996, pp. 249-257.