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Endomorphisms of Logarithmic Schemes.

MARK KISIN (*)

ABSTRACT - Let (X, M) be a log. smooth log. scheme over a discrete valuation ring V, and (X_s, M_s) its special fibre. Generalising a result of Illusie, we show that (X_s, M_s) depends only on a certain explicit nilpotent neighbourhood of $X_s \subset X$. As a consequence, the sheaf of vanishing cycles $R\Psi\mathbb{Q}_l$ depends only on this nilpotent neighbourhood, thereby refining a result of Berkovich in a special case. We prove a Lefschetz Trace Formula for endomorphisms of (X_s, M_s) in the case that (X, M) is a proper log. curve. We prove two versions of this formula using log. étale and log. crystalline cohomology. This suggests that there should exist an intersection theory of log. schemes.

Introduction.

Introduction Let k be a separably closed field of characteristic p, equipped with the log. structure $\mathbb{N} \to k$, $1 \mapsto 0$. The aim of this paper is to study endomorphisms of log. smooth log. schemes over $(k, \mathbb{N})$. Examples of such log. schemes include the special fibres of semi-stable schemes, or more generally log. smooth log. schemes over a discrete valuation ring V with residue field k.

This work grew out of the following rather naive question: Suppose that X is a semi-stable scheme over V, and denote by X_s its special fibre. If one is given an automorphism σ of X_s, when does σ act on the complex of nearby cycles $R\Psi\mathbb{Q}_l$? That such a question might be reasonable is suggested by the fact that σ evidently acts on the cohomology sheaves $R^i\Psi\mathbb{Q}_l$ of $R\Psi\mathbb{Q}_l$ [Il, 3.2]. One motivation for this question is that even when the cohomology of the generic fibre of X is very complicated, the

(*) Indirizzo dell'A.: SFB 478, Westfälische Wilhelms-Universität, Münster.
E-mail: kisin@math.uni-muenster.de

special fibre may nevertheless be highly degenerate, making it possible to construct many automorphisms of the special fibre «by hand». For example, when X is a relative curve, the reader should think of the case where the components of the special fibre all have genus zero. This raises the possibility of constructing interesting representations in the cohomology of the generic fibre. Given this motivation, what we would like from the theory is a simple criterion to recognise when an automorphism of the special fibre acts on cohomology, and a Lefschetz type trace formula to be able to compute the representation on cohomology. These are the two questions we try to address in this paper.

It turns out that our problem can be most naturally analysed in the setting of log. schemes. Namely, X_s is equipped with a log. structure M_s, and σ acts on $R\Psi\mathbb{Q}_l$ provided it lifts to (X_s, M_s). We show that the existence of a lifting of σ to (X_s, M_s) is in fact a *purely local* question. This is somewhat surprising as $R\Psi\mathbb{Q}_l$ is a rather global object.

In fact we work in somewhat more generality. We consider log. smooth V-schemes (X, M), and their special fibres (X_s, M_s). A result of C. Nakayama shows that the complex of nearby cycles $R\Psi\mathbb{Q}_l$ is determined by (X_s, M_s). If the smooth locus of X_s is dense then we show, in particular, that an automorphism σ of X_s lifts to (X_s, M_s) if and only if locally on X it lifts to X modulo π^{m+1}, where $\pi \in V$ is a uniformiser, and m is an integer which may in practice be easily computed using the log. structure M_s. The key point is that although the local liftings of σ to X modulo π^{m+1} are not unique they all give rise to the same endomorphism on the log. scheme (X_s, M_s).

As another application of our theory we show that (still assuming the smooth locus is dense in X_s) the log. scheme (X_s, M_s) depends only on the reduction of X modulo π^{m+1} (Corollary (2.5)). This generalises a result of Illusie for semi-stable schemes, in which case $m = 1$. The condition on the smooth locus of X_s is not a strong one, since it is always satisfied after a finite base extension of V.

We also obtain, in a special case, a refined version of a conjecture of Deligne, which was proved by Berkovich in [Ber]. Namely that for any flat V-scheme X, the nearby cycle sheaves $R\Psi\Lambda$ depend only on the completion of X along its special fibre. When X is log. smooth over V, our results show that these sheaves in fact depend only on the reduction of X modulo an explicitly computable power of π (see (2.6)).

The final, and perhaps most interesting, application of the ideas we develop is a Lefschetz type trace formula for log. curves. We explain a special case, and refer the reader to Theorem (3.3) for the precise statement. Suppose that (X, M) is a log. smooth proper curve over V, and let (X_s, M_s) denote its special fibre. Consider an endomorphism σ of (X_s, M_s) such that σ acting on the underlying scheme X_s has only isolated fixed points. As we explained above, σ acts on the nearby cycles $R\Psi\mathbb{Q}_l$ of X, and we may compute the alternating sum of traces of σ on $\mathbb{H}^*(X_s, R\Psi\mathbb{Q}_l)$ (which is of course the cohomology of the geometric generic fibre of X). On the other hand if $x \in X_s$ is a fixed point of σ, then we show that σ lifts to an endomorphism $\widehat{\sigma}$ of the formal neighbourhood of x in X. The «generic fibre» of this formal neighbourhood is a rigid analytic space, and we may count the number of fixed points with multiplicities of $\widehat{\sigma}$ acting on this generic fibre. Adding up over all such $x \in X_s$, we show that this sum is equal to the alternating sum of traces on cohomology computed earlier. In fact, we can only prove this result when X is semi-stable or the endomorphism σ is finite on the underlying scheme X_s. This finiteness assumption is needed to reduce to the semi-stable case, but the result should be true without it. We also prove an analogous result using crystalline cohomology.

We conjecture that this formula should be true in higher dimensions also. One of the striking consequences of the trace formula is that if $x \in X_s$ is a fixed point of σ, then the number of rigid analytic fixed points of $\widehat{\sigma}$ specialising to x is independent of the choice of $\widehat{\sigma}$, because $\widehat{\sigma}$ can be chosen independently in the formal neighbourhood of each such fixed point x. This suggests that there should be an intersection theory for log. schemes giving, in particular, an *a priori* definition of this number. Moreover, such a theory should yield the trace formula in higher dimensions, since Poincaré Duality, and the Künneth Formula are already available in log. étale cohomology by work of Nakayama [Na].

The paper is organised as follows. In § 1, we review the results of Nakayama [Na], which give a description of nearby cycles in terms of log. structures, and derive some simple consequences.

In § 2 we study the question of when a map between schemes over k, extends to a map of log. schemes. Using our results we obtain the generalisation of Illusie's result mentioned above, and also the refinement, in this special case, of the result of Berkovich.

In § 3, we prove our trace formula using log. étale cohomology. The proof is quite long and technical. The general strategy is as follows. First we prove the result in the case of a semi-stable curve. This is done by, more or less explicitly, computing both sides of the formula. For general log. smooth curves, we show that after making a base change one can reduce to the semi-stable case. One of the key ingredients here is a result of de Jong [de J2, 4.18] which says after a base change a normal flat curve over V can be modified into a semi-stable one (T. Saito has informed us that he also obtained this result some time ago). We were unable to prove what we needed using the semi-stable reduction theorem, since there one in general has also to blow down (not just blow up) in order to reach a semi-stable situation. Using this result, and some extra arguments, we show that if (X, M) is any log. smooth curve over V, and σ an endomorphism of (X_s, M_s) then after a finite base change, we can modify X into a semi-stable curve by a sequence of blow ups which are, in a suitable sense, equivariant with respect to σ. By this method we are able to reduce the trace formula for a proper log. smooth curve to the semi-stable case.

Finally, in § 4, we briefly explain how our results can be extended to log. crystalline cohomology.

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1. Nearby Cycles and the Logarithmic étale Site.

(1.1) Throughout the paper F will denote a discretely valued field, with uniformiser π, valuation ring $\mathcal{O}_F$, and residue field k. We assume that k is separably closed. (This will not restrict the generality of our results because one can always make a base change to this situation). Spec $(\mathcal{O}_F)$ carries a canonical log. structure given by

$$\mathbb{N} \to \mathcal{O}_F;\ 1 \mapsto \pi.$$

We are going to consider log. schemes (X, M) over $\mathcal{O}_F$, which are log. smooth with *vertical* log. structure. This means that étale locally on X there is a finitely generated, integral, saturated monoid P, and an $x \in P$, such that the torsion part of $P^{\mathrm{gp}}/\langle x \rangle$ has prime to p order, and an étale map $X \to \mathrm{Spec}\,(\mathcal{O}_F[P]/(x-\pi))$, such that the log structure on X is induced by P. Moreover we require that for any $a \in P$ there exists $b \in P$ with $ab = x^m$ for some integer m, depending on a and b. This condition guarantees that the log. structure is vertical, which means that it becomes trivial away from the special fibre.

Unless otherwise stated, we assume throughout the paper that the log. structures on log. schemes over $\mathcal{O}_F$ are vertical.

We say that a log. scheme (X, M) is fine saturated, or fs if étale locally it admits a chart given by a finitely generated, integral saturated monoid. We also say that the log. structure M is fs.

DEFINITION (1.2). *A map of fs log. schemes $(X, M) \to (Y, N)$ is called Kummer if it is log. étale, and if the underlying map of schemes is quasi-finite.*

Define the Kummer site $(X, M)_{\mathrm{Kum}}$ of (X, M) by declaring the neighbourhoods to be log. schemes, which are Kummer over (X, M), and by declaring coverings to be collections of neighbourhoods, which together surject onto the underlying scheme X.

There is another equivalent definition of Kummer maps of log. schemes. Namely a map of log. schemes $(X, M) \to (Y, N)$ is Kummer if it is log. étale, and (étale locally) has a chart $\mathbb{Z}[Q] \to \mathbb{Z}[P]$ with P, Q finitely generated, integral, saturated monoids, and $Q^n \subset P \subset Q$, for some prime to p integer n. Here $Q^n = \{q^n : q \in Q\}$.

The two definitions are equivalent by [Ki, 1.2]. We prefer to use the one here, since in practice is it a little easier to check. For example suppose that (Y_0, N_0) is the reduction of (Y, N) modulo a nilpotent ideal. Then any log. étale covering of (Y_0, N_0) lifts to a log. étale covering of (Y, N). Using our definition it is clear that Kummer coverings lift to Kummer coverings, while this is not quite obvious using the alternative definition.

(1.3) We are going to review the results of [Na] describing the nearby cycles functor $R\Psi$ for a scheme X equipped with a log smooth log. structure over $\mathcal{O}_F$.

Denote by $j : X_\eta \hookrightarrow X$, and $i : X_s \hookrightarrow X$, the inclusion of the generic fibre and closed fibre of X, respectively, and set $R\Theta_F = i^* Rj_*$ where Rj_* is the total derived functor of j_*. For a finite extension F' of F, we denote by $R\Theta_{F'}$ the composite of the nearby cycles functor for the normalisation of $X \otimes_{\mathcal{O}_F} \mathcal{O}_{F'}$, followed by the direct image of the morphism of sites

$$(X \otimes_{\mathcal{O}_F} \mathcal{O}_{F'})_{s,\,\text{ét}} \to X_{s,\,\text{ét}}.$$

Thus $R\Theta_{F'}\Lambda$ is a complex of sheaves on $X_{s,\,\text{ét}}$. The complex of nearby cycles $R\Psi\Lambda$ is defined by

$$R\Psi\Lambda = \varinjlim R\Theta_{F'}\Lambda,$$

where F' runs over all finite extensions of F.

We equip X_s with the log. structure $M_s = i^*(M)$, and we suppose that the log. structure M is trivial over the generic point of Spec $(\mathcal{O}_F)$.

Consider the log. structure on Spec k given by $\mathbb{N} \to k$, $1 \mapsto 0$. For each positive integer n, denote by (Spec k, $\mathbb{N})_n$ the log. scheme (Spec k, $\mathbb{N}$) viewed as a log. scheme over (Spec k, $\mathbb{N}$) via the multiplication by n map $\mathbb{N} \xrightarrow{n} \mathbb{N}$. Write

$$(X_s^n, M_s^n) = (X_s, M_s) \times_{(\text{Spec}\, k,\, \mathbb{N})} (\text{Spec}\, k, \mathbb{N})_n,$$

where the fibre product is taken in the category of fs log. schemes. Write

$$p_n : (X_s^n, M_s^n)_{\text{Kum}} \to X_{s,\,\text{ét}}$$

for the canonical projection. Using [Na, 3.2(v)] we have

PROPOSITION (1.4). *For Λ a finite abelian group of prime to p order, there is a canonical isomorphism*

$$\varinjlim Rp_{n*}\Lambda \xrightarrow{\sim} R\Psi\Lambda, \tag{1.6.1}$$

where n runs through integers which are prime to p.

(1.5) Consider a log. smooth scheme (X_s, M_s) over $(k, \mathbb{N})$. Although the notation X_s is suggestive, we do not assume that X_s is the special fibre of a log. smooth $\mathcal{O}_F$-scheme. Recall [Ka, 3.14] that locally on X_s there is an exact, closed immersion $X_s \to X$ into an X which is log. smooth over $\mathcal{O}_F$, with special fibre X_s. Locally, X is unique up to isomorphism, since the set of embeddings is a torsor under H^1 of a quasi-coherent sheaf.

The obstruction to the existence of *global* embedding of the above type lies in H^2 of a certain quasi-coherent sheaf on X_s, hence it vanishes if X_s is a curve.

Even if there is no such global embedding, we can make the definitions of (1.3), and define $R\Psi\Lambda$ for each finite group Λ with order invertible in k, by the formula

$$R\Psi\Lambda = \varinjlim Rp_{n*}\Lambda.$$

We then define $R\Psi\mathbb{Z}_l$, $R\Psi\mathbb{Q}_l$ in the obvious way. By (1.4) this is consistent with the usual definition in the case that a global embedding $X_s \to X$ does exist.

We say that the log. structure M_s is *vertical* if each $x \in X$ has a neighbourhood U_s, which admits an exact closed immersion $U_s \to U$ into a log. smooth $\mathcal{O}_F$-scheme U, with vertical log. structure, and special fibre U_s. Unless explicitly stated otherwise smooth log. schemes over k will be assumed to have vertical log. structure.

PROPOSITION (1.6). *Let (X_s, M_s) and (Y_s, N_s) be log. smooth and proper over k. Assume we are given a map of log. schemes*

$$f^{\log}: (X_s, M_s) \to (Y_s, N_s).$$

Then $f^{\log}$ induces a canonical map

$$\mathbb{H}^*(f^{\log}): \mathbb{H}^*(Y_s, R\Psi\Lambda) \to \mathbb{H}^*(X_s, R\Psi\Lambda).$$

In particular, if $X_s \to X$ and $Y_s \to Y$ are exact closed immersions into log. smooth $\mathcal{O}_F$-schemes, then f induces a map

$$H^*_{\bar\eta}(f^{\log}): H^*(Y_{\bar\eta}, \Lambda) \to H^*(X_{\bar\eta}, \Lambda),$$

where $\bar\eta$ denotes the geometric generic fibre.

These constructions are compatible with composition of morphisms.

PROOF. By (1.4) the cohomology $\mathbb{H}^*(X_s, R\Psi\Lambda)$ is the cohomology of the sheaf Λ on the topos which is the inverse limit of Kummer sites $\varprojlim (X_s^n, M_s^n)_{\mathrm{Kum}}$, where n runs over prime to p integers (cf. [Na § 3]). Thus, the result follows from the functoriality, with respect to X_s, of the cohomology of such a topos. ■

LEMMA (1.7). *Let (X_{1s}, M_{1s}) and (X_{2s}, M_{2s}) be proper, and log. smooth over k, and $f: (X_{1s}, M_{1s}) \to (X_{2s}, M_{2s})$ a map of log schemes over k. Suppose we are given endomorphisms σ_1, σ_2 of (X_{1s}, M_{1s}) and (X_{2s}, M_{2s}) respectively, such that $\sigma_2 \circ f = f \circ \sigma_1$. Then we have a commu-*

tative diagram

$$\begin{array}{ccc} \mathbb{H}^*(X_{1s}, R\Psi\Lambda) & \xrightarrow{\mathbb{H}^*(f)} & \mathbb{H}^*(X_{1s}, R\Psi\Lambda) \\ {\scriptstyle \mathbb{H}^*(\sigma_1)}\downarrow & & {\scriptstyle \mathbb{H}^*(\sigma_2)}\downarrow \\ \mathbb{H}^*(X_{1s}, R\Psi\Lambda) & \xrightarrow{\mathbb{H}^*(f)} & \mathbb{H}^*(X_{1s}, R\Psi\Lambda) \end{array}$$

PROOF. This is immediate by applying the compatibility of the construction of (1.6) with compositions to the map $\sigma_2 \circ f = f \circ \sigma_1$. ■

LEMMA (1.8). *Let* (X_s, N_s) *be log. smooth over* k. *Let* F' *be a finite extension of* F, *and write* $(X_s', N_s') = (X_s, N_s) \times_{\mathcal{O}_F} \mathcal{O}_{F'}$, *where the product is taken in the category of fs log. schemes. Denote by* q *the natural morphism*

$$q : X'_{s,\, \text{ét}} \to X_{s,\, \text{ét}}.$$

If Λ *is a finite abelian group of prime to* p *order, then there is a canonical isomorphism*

$$R\Psi_{X_s}\Lambda \xrightarrow{\sim} q_* R\Psi_{X_s'}\Lambda.$$

PROOF. From the definitions, one sees easily that there is a natural map $R\Psi_{X_s}\Lambda \to q_* R\Psi_{X_s'}\Lambda$. To show that it is an isomorphism is a local problem, so we may assume that (X_s, N_s) is the special fibres of smooth $\mathcal{O}_F$-log. scheme (X, N). Set $(X', N') = (X, N) \times_{\mathcal{O}_F} \mathcal{O}_{F'}$. In this case, (1.4) shows that $R\Psi_{X_s}\Lambda$ and $R\Psi_{X_s'}\Lambda$ are isomorphic to the nearby cycle sheaves of X and X' respectively. That our morphism is an isomorphism now follows from the fact that X and X' have isomorphic geometric generic fibres: $X'_{\bar{\eta}} \xrightarrow{\sim} X_{\bar{\eta}}$. ■

COROLLARY (1.9). *With the notation of* (1.8), *suppose that* σ *is an endomorphism of* (X_s, N_s). *Denote by* σ' *the endomorphism of* X_s' *induced by* σ. *Then we have*

$$\mathbb{H}^*(X_s', R\Psi\Lambda) \xrightarrow{\sim} \mathbb{H}^*(X_s, R\Psi\Lambda)$$

and the endomorphisms $\mathbb{H}^*(\sigma)$ *and* $\mathbb{H}^*(\sigma')$ *are compatible with this isomorphism.*

PROOF. The first claim follows from (1.8), and the second from the construction in (1.6) and (1.7). ■

2. Maps between schemes and log. schemes.

The purpose of this section is to study morphisms between log. schemes which are log. smooth over $(k, \mathbb{N})$ and the maps they induce on log. étale cohomology. In particular, we show that under certain circumstances, a map between log. smooth log. schemes is completely determined by the map on the underlying schemes, and that an endomorphism of schemes induces an endomorphism of log. schemes provided a local condition is satisfied. In particular this means that «reasonable» maps between the special fibres of semi-stable schemes, induce maps on the cohomology of the *generic fibres*. The criterion makes it quite feasible to construct maps of log. schemes «by hand,» since one has only to give a map between the underlying schemes, and then check a local condition - see example (2.7).

The following lemma will be used to show that under quite general conditions a map of schemes $X_s \to Y_s$ extends to map of log. schemes in at most one way.

LEMMA (2.1). *Let* (X, M) *and* (Y, N) *be fs log. schemes over* $(\mathrm{Spec}\, k, \mathbb{N})$, *with* (X, M) *log. smooth over* $(\mathrm{Spec}\, k, \mathbb{N})$. *Let* $f : X \to Y$ *a map of k-schemes and* $U \subset Y$ *an open subset such that* $f^{-1}(U)$ *is dense in* X.

Suppose that $f|_U$ *extends to a map of log. schemes*

$$g : (f^{-1}(U), M|_{f^{-1}(U)}) \to (U, N|_U).$$

Then there is at most one map of log. schemes $(X, M) \to (Y, N)$ *over* $(\mathrm{Spec}\, k, \mathbb{N})$, *compatible with* f *and* g.

In particular, if $(U, N|_U) \to (\mathrm{Spec}\, k, \mathbb{N})$ *is strict, then there is at most one map* $f^{\log} : (X, M) \to (Y, N)$ *of log. schemes over* $(\mathrm{Spec}\, k, \mathbb{N})$ *inducing the map* f *on the underlying schemes.*

PROOF OF (2.1). The first claim implies the second, since if $(U, N|_U)$ is strict over $(\mathrm{Spec}\, k, \mathbb{N})$, then there is at most one possibility for g.

We prove the first claim. If $\bar{x}$ is a point of $X_{\acute{e}t}$, with image $\bar{y}$ in $Y_{\acute{e}t}$, we have to show there is at most one map $f^*(N_{\bar{y}}) \to M_{\bar{x}}$, which makes the following diagram commute

$$\begin{array}{ccc} f^*(N_{\bar{y}}) & \longrightarrow & M_{\bar{x}} \\ \downarrow & & \downarrow \\ f^*(\mathcal{O}_{Y,\bar{y}}) & \longrightarrow & \mathcal{O}_{X,\bar{x}} \end{array}$$

Let $\tilde{x}$ be a point of $X_{\text{ét}}$ whose image in Spec $\mathcal{O}_{X,\bar{x}}$ lies in $f^{-1}(U)$, and denote by $\tilde{y}$ the image of $\tilde{x}$ in $Y_{\text{ét}}$. The conditions guarantee that $\tilde{y}$ lies over a point of U, and g determines a map $f^*(N_{\tilde{y}}) \to M_{\tilde{x}}$, making the diagram

$$\begin{array}{ccc} f^*(N_{\tilde{y}}) & \longrightarrow & M_{\tilde{x}} \\ \downarrow & & \downarrow \\ f^*(\mathcal{O}_{Y,\tilde{y}}) & \longrightarrow & \mathcal{O}_{X,\tilde{x}} \end{array}$$

commute.

Now for each $\tilde{x}$ as above, we have the co-specialisation map $M_{\bar{x}} \to M_{\tilde{x}}$, and I claim that the direct sum of these maps $M_{\bar{x}} \to \oplus_{\tilde{x}} M_{\tilde{x}}$ taken over all choices of $\tilde{x}$ is injective. Accepting this claim for a moment, we see that a map making the first diagram commute exists if and only if the composite

$$f^*(N_{\bar{y}}) \to f^*(N_{\tilde{y}}) \to \oplus_{\tilde{x}} M_{\tilde{x}}$$

has image contained in $M_{\bar{x}} \subset \oplus_{\tilde{x}} M_{\tilde{x}}$, and in this case it is uniquely determined.

To see the claim, suppose that $a_1, a_2 \in M_{\bar{x}}$ have the same image in $\oplus_{\tilde{x}} M_{\tilde{x}}$. Observe that X is Cohen-Macaulay [Ka 2, 4.1], so any dense subscheme is scheme-theoretically dense [Mat, 17.6]. In particular, this holds for $f^{-1}(U)$, so the natural map $\mathcal{O}_{X,\bar{x}} \to \oplus_{\tilde{x}} \mathcal{O}_{X,\tilde{x}}$ is injective. Thus, if $\alpha : M \to \mathcal{O}_X$ denote the map defining the log. structure, then we must have $\alpha(a_1) = \alpha(a_2)$. By (2.2) below, this implies that either both a_1 and a_2 are divisible by x (the image of 1 in the monoid $\mathbb{N}$ defining the log. structure on Spec k), or $a_1 = a_2 + w$ for some $w \in \mathcal{O}_{X,\bar{x}}^{\times}$. In the second case, since all our monoids are integral, we have that the image of w in $\mathcal{O}_{X,\tilde{x}}$ is 1 for all $\tilde{x}$, so that $w = 1$, and $a_1 = a_2$. In the first case, we may replace a_1 and a_2 by $a_1 - x$ and $a_2 - x$, and repeat the argument. Since our log. structures are coherent, a_1 and a_2 cannot be infinitely divisible by x, and this procedure must terminate after finitely many steps. ∎

LEMMA (2.2). *Let* (Spec A, M) *be a log. smooth log. scheme over* (Spec k, $\mathbb{N}$), *and denote by* $\alpha: M \to A$ *the map defining the log. structure. If* $a_1, a_2 \in M$ *and* $\alpha(a_1) = \alpha(a_2)$, *then either* a_1 *and* a_2 *are divisible by* x, *so that* $\alpha(a_1) = \alpha(a_2) = 0$, *or* $a_1 = a_2 + w$ *for some* $w \in M^\times \xrightarrow{\sim} A^\times$.

PROOF. The problem is étale local, so, using [Ka, 3.14], we may assume that A is a local ring, and that $h: (\mathrm{Spec}\, A, M) \to (\mathrm{Spec}\, k, \mathbb{N})$ admits a chart $\mathbb{N} \to P$ such that the map $\mathrm{Spec}\, A \to \mathrm{Spec}\, k[P]/(x)$ is strict and smooth.

We may invert any elements of P that become invertible in A, so we assume that any element of P which is not a unit maps to the maximal ideal of A. Let $Q = P/P^\times$. Since Q^{gp} is a free abelian group, we may choose a section to the projection $P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$. Such a section induces a decomposition $P \xrightarrow{\sim} P^\times \oplus Q$. Then $k[Q] \to A$ is also a chart for (Spec A, M).

We have $Q^\times = \{1\}$, and the image of Q in A consists of non-units. This implies that $M \xrightarrow{\sim} Q \oplus A^\times$, and that $k[Q]_{(Q)} \to A$ is a local map, where $k[Q]_{(Q)}$ is the localisation at the maximal ideal generated by Q. Moreover, if $x' \in Q$ denotes an element which differs from (the image of) $x \in P$ by a unit of P then, since $k[Q] \to k[Q \oplus P^\times]$ is smooth, A is smooth over $k[Q]/(x')$.

For $i = 1, 2$ write $a_i = q_i u_i$ where $q_i \in Q$ and $u_i \in A^\times$. Since $\alpha(a_1) = \alpha(a_2)$, the image of q_2 in $A/(q_1)$ is 0. Since the local, smooth map $k[Q]_{(Q)}/(x', q_1) \to A/(q_1)$ is faithfully flat, and hence injective, the image of q_2 in $k[Q]_{(Q)}/(x', q_1)$ is trivial. However, it is easy to see that if $q \in Q \backslash (x' + Q \cup q_1 + Q)$ then q has non-zero image in $k[Q]_{(Q)}/(x', q_1)$. Thus either $q_2 \in x' + Q$ or $q_2 \in q_1 + Q$. In the first case we are done. In the second, note that, repeating the above argument with q_1 and q_2 interchanged we may assume that $q_1 \in q_2 + Q$, so that $q_2 - q_1 \in Q^\times = \{1\}$. Thus, $q_1 = q_2$, which implies that a_1 and a_2 differ by a unit. ■

(2.3) Let (Y, N) be a log. smooth, log. scheme over $\mathcal{O}_F$, with vertical log. structure. We assume that the underlying scheme Y is Noetherian, and define an integer $m = m(Y)$ as follows: étale locally $(Y, N) \to (\mathrm{Spec}\,(\mathcal{O}_F), \mathbb{N})$ has a chart given by a map of finitely generated, integral, saturated monoids $\mathbb{N} \to P$. Denote by x the image in P of $1 \in \mathbb{N}$. Consider a presentation for P

$$\mathcal{P}: \mathbb{N}^s \overset{\phi, \psi}{\rightrightarrows} \mathbb{N}^r \xrightarrow{\gamma} P\,,$$

such that one of the canonical generators $\tilde{x}$ of $\mathbb{N}^r$ maps to $x \in P$.

Such a presentation always exists. Indeed, as P is finitely generated, we may choose a surjection $\mathbb{N}^r \to P$ mapping one of the canonical generators of $\mathbb{N}^r$ to $x \in P$. Denote by K the kernel of the induced map $\mathbb{Z}^r \to P^{\mathrm{gp}}$. Define

$$A = \{(\kappa, z) \in K \oplus \mathbb{Z}^r \colon \kappa + z,\ z \in \mathbb{N}^r\}.$$

Now we have

$$K \oplus \mathbb{Z}^r \overset{(\kappa, z) \mapsto (\kappa + z, z)}{\hookrightarrow} \mathbb{Z}^r \oplus \mathbb{Z}^r,$$

and under this embedding A is simply the intersection of $K \oplus \mathbb{Z}^r$ with the rational cone $(\mathbb{N}^r \oplus \mathbb{N}^r) \otimes_{\mathbb{N}} \mathbb{Q}^+ \cup \{0\} \subset (\mathbb{Z}^r \oplus \mathbb{Z}^r) \otimes_{\mathbb{Z}} \mathbb{Q}$. Thus by Gordon's Lemma [Ful, p12] A is a finitely generated monoid. Hence there is a surjection $\chi : \mathbb{N}^s \to A$ for some s. If $\phi', \psi' : A \to \mathbb{N}^r$, denote the maps given respectively by projection onto the second factor, and addition in $A \subset K \oplus \mathbb{Z}^r$, then one checks easily, using the fact that P is integral, that we may take $\phi = \phi' \circ \chi$, $\psi = \psi' \circ \chi$.

Denote by $e_1, \ldots, e_s$ the canonical generators of $\mathbb{N}^s$. We set $m(\mathcal{P})$ equal to the smallest positive integer m such that $x^m \in P$ is divisible by $\gamma \circ \phi(e_i)$ and $\gamma \circ \psi(e_i)$ for $i = 1, 2 \ldots s$. Such an m exists, as the log. structure on Y is vertical. We set $m(P) = \min_{\mathcal{P}} m(\mathcal{P})$, where $\mathcal{P}$ runs over all presentations satisfying the conditions above. Finally we set $m(Y)$ equal to the smallest positive integer m such that each point of Y has an étale neighbourhood which admits a chart given by a monoid P, as above, with $m(P) \leqslant m$.

For concrete Y the integer $m(Y)$ is easily computed. For example if Y is semi-stable, or more generally regular (in the classical sense), then $m = 1$.

We remark that if u is a canonical generator of $\mathbb{N}^r$, then there exists $v \in P$ with $\gamma(u)\, v = x^{m(P)}$. If $u = \tilde{x}$ this is obvious. If $u \neq \tilde{x}$ it suffices to show that there exists i such that either $\phi(e_i)$ or $\psi(e_i)$ is divisible by u. However, as P induces a vertical log. structure, there exist $u' \in \mathbb{N}^r$, and such that $\gamma(uu') = \gamma(\tilde{x}^t)$ for some positive integer t. Thus there exists $w \in \mathbb{N}^s$ such that $\psi(w) uu' = \phi(w) \tilde{x}^t$ (or with ϕ and ψ interchanged). Since $u \neq \tilde{x}$, u divides $\phi(w)$, whence it divides $\phi(e_i)$ for some i.

If (Y_s, N_s) is any log. smooth scheme over k, we define $m(Y_s)$ as the smallest integer m_0 such that for each $y \in Y_s$ there exists a neighbourhood U_s, and an exact closed immersion into a log. smooth $\mathcal{O}_F$-scheme with special fibre U_s, and $m(U) \leqslant m_0$.

PROPOSITION (2.4). *Let (X_s, M_s) and (Y_s, N_s) be log. smooth log. schemes over k. Set $m = m(Y_s)$. Let $f\colon X_s \to Y_s$ be a map of the underlying* *schemes. We consider exact closed immersions* $X_s \to X$ *and* $Y_s \to Y$ *into log. smooth* $\mathcal{O}_F$*-schemes, which are defined locally on* X_s *and* Y_s *respectively. Consider the following conditions*

(1) *Étale locally on* X_s *and* Y_s f *lifts to a map* $X^{(m+1)} \to Y^{(m+1)}$, *where for* $n \in \mathbb{N}$, $X^{(n)}$ *and* $Y^{(n)}$ *denote the reduction modulo* π^n *of* X *and* Y *respectively.*

(2) *For each* $n \in \mathbb{N}$, *étale locally on* X_s *and* Y_s, f *lifts to a map* $X^{(n)} \to Y^{(n)}$.

(3) *Étale locally on* X *and* Y, f *is induced by a map* $(X_s, M_s) \to \to (Y_s, N_s)$ *of log schemes over* (Spec $(\mathcal{O}_F)$, $\mathbb{N}$).

(4) f *is induced by a unique map* $f^{\log} : (X_s, M_s) \to (Y_s, N_s)$ *of log. schemes over* (Spec $(\mathcal{O}_F)$, $\mathbb{N}$).

We have (4) $\Rightarrow$ (3) $\Leftrightarrow$ (2) $\Leftrightarrow$(1). *If the condition*

(*) *For* $U \subset Y_s$ *the open subset where* Y_s *is smooth,* $f^{-1}(U)$ *is dense in* X_s

holds, then (1)-(4) *are equivalent, and these are all equivalent to* (1)-(3) *with «étale locally» replaced by «locally» in* (1)-(3).

PROOF. (4) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1) are obvious. If the condition (*) is satisfied, then we have (3) $\Rightarrow$ (4) by Lemma (2.1). Moreover if (4) is true, then of course (3) is true with «locally» in place of «étale locally,» and the arguments below show that this implies (2) with «locally» in place of «étale locally».

(3) $\Rightarrow$ (2). Denote by $M^{(n)}$ and $N^{(n)}$ the induced log. structures on $X^{(n)}$ and $Y^{(n)}$ respectively. Since $(X_s, N_s) \hookrightarrow (X^{(n)}, M^{(n)})$ is an exact closed immersion of fs log. schemes, and $(Y^{(n)}, M^{(n)})$ is log. smooth over (Spec $(\mathcal{O}_F/(\pi^{m+1}))$, $\mathbb{N}$), the composite

$$(X_s, M_s) \to (Y_s, N_s) \hookrightarrow (Y^{(n)}, M^{(n)})$$

lifts to map of log. schemes $(X^{(n)}, M^{(n)}) \to (Y^{(n)}, M^{(n)})$, and we may take the the underlying map of schemes for the map of (2).

(1) $\Rightarrow$ (3). We may assume that X and Y are the spectra of strictly henselian local rings, A and B respectively. Suppose that $\beta : P \to B$ is a chart for Y, and for each positive integer n denote by $\beta_n : P \to B/\pi^n$ the induced map. We abuse notation slightly, and write $\alpha : M \to A$ for the log.

structure on X, and $\alpha_n \colon M \to A/\pi^n$ for the induced map. Denote by h the map $B/\pi^{m+1} \to A/\pi^{m+1}$ induced by the given lift $X^{(m+1)} \to Y^{(m+1)}$.

Consider a presentation

$$\mathcal{P} \colon \mathbb{N}^s \overset{\phi,\,\psi}{\rightrightarrows} \mathbb{N}^r \xrightarrow{\gamma} P,$$

with $m(\mathcal{P}) \leqslant m$, and such that one of the generators $\tilde{x}$ of $\mathbb{N}^r$ maps to $x \in P$.

Since $M \xrightarrow{\sim} (A[1/\pi])^\times \cap A$ ([Ka 2, 11.6]), and the log. structure on Y is vertical, we have that $h(\beta_{m+1}(P)) \subset \alpha_{m+1}(M)$. Indeed, it is enough to show that $h(\beta_{m+1}(\gamma(u))) \in \alpha_{m+1}(M)$ for any generator u of $\mathbb{N}^r$. We saw in (2.3) that there exists $v \in P$ such that $\gamma(u)\, v = x^m$. Let $\tilde{u}, \tilde{v}$ be lifts to A of $h(\beta_{m+1}(\gamma(u)))$ and $h(\beta_{m+1}(v))$ respectively. Then we must have $\tilde{u}\tilde{v} = \pi^m + \pi^{m+1} w = \pi^m(1 + \pi w)$ for some $w \in A$, and in particular $\tilde{u} \in M$, as required.

It follows that there exists a map of monoids $\tilde{h}^{\log} \colon \mathbb{N}^r \to M$ which sends $\tilde{x}$ to $\pi \in M \xrightarrow{\sim} (A[1/\pi])^\times \cap A$, and which makes the following diagram commute.

$$\begin{array}{ccccc}
\mathbb{N}^r & \xrightarrow{\gamma} & P & \longrightarrow & B/\pi^{m+1} \\
{\scriptstyle \tilde{h}^{\log}} \big\downarrow & & & & \big\downarrow {\scriptstyle h} \\
M & \xrightarrow{\alpha} & A & \longrightarrow & A/\pi^{m+1}
\end{array}$$

Denote, by M_s the reduction of M modulo π. I claim that the composite

$$\mathbb{N}^r \xrightarrow{\tilde{h}^{\log}} M \longrightarrow M_s$$

factors through P, and hence induces a map of log. schemes $f_{\log.} \colon (X_s, M_s) \to (Y_s, N_s)$, as required.

To see the claim let e be one of the canonical generators of $\mathbb{N}^s$. We have to show that $y = \tilde{h}^{\log}(\phi(e))$ and $z = \tilde{h}^{\log}(\psi(e))$ differ by a unit of A congruent to 1 modulo π. Since $\alpha_{m+1}(y) = \alpha_{m+1}(z)$, we have $\alpha(y) = \alpha(z) + \pi^{m+1} w$ for some $w \in A$. By the choice of m and $\mathcal{P}$, $\pi^m \in A/\pi^{m+1}$ is a multiple of $\alpha_{m+1}(z)$, whence we obtain easily that there exists $u \in A$ with $\alpha(z) u = \pi^m$. This gives

$$\alpha(y) = \alpha(z) + \alpha(z)\, u\pi w = \alpha(z)(1 + u\pi w),$$

which proves the result, as α is an injection. ■

The above result implies a generalisation of a result of Illusie:

COROLLARY (2.5). *Let* (Y_1, N_1) *and* (Y_2, N_2) *be log. smooth over* $\mathcal{O}_F/\pi^{m+1}$, *where* $m = \max(m(Y_1), m(Y_2))$. *Suppose that the underlying schemes* Y_1 *and* Y_2 *are Noetherian, and smooth over* $\mathcal{O}_F/\pi^{m+1}$ *outside a nowhere dense, Zariski closed subset.*

If there is an isomorphism

$$Y_1 \otimes_{\mathcal{O}_F} \mathcal{O}_F/\pi^{m+1} \xrightarrow{\sim} Y_2 \otimes_{\mathcal{O}_F} \mathcal{O}_F/\pi^{m+1},$$

then the reductions of (Y_1, N_1) *and* (Y_2, N_2) *modulo* π *are canonically isomorphic.*

In particular, if Y_1 *and* Y_2 *are semi-stable, and the reductions of* Y_1 *and* Y_2 *modulo* π^2 *are isomorphic then the reductions of the log. schemes* (Y_1, N_1) *and* (Y_2, N_2) *modulo* π *are isomorphic.*

PROOF. The case where Y_1 and Y_2 are semi-stable is due to L. Illusie [Na, A.4]. It follows from the more general claim, since in this case $m(Y_1) = m(Y_2) = 1$.

In general, the given isomorphism and its inverse induce maps

$$(Y_1, N_1) \otimes_{\mathcal{O}_F} \mathcal{O}_F/\pi \to (Y_2, N_2) \otimes_{\mathcal{O}_F} \mathcal{O}_F/\pi$$

and

$$(Y_2, N_2) \otimes_{\mathcal{O}_F} \mathcal{O}_F/\pi \to (Y_1, N_1) \otimes_{\mathcal{O}_F} \mathcal{O}_F/\pi$$

respectively, by (2.4). Their composites must be the identity by the uniqueness in (2.4)(4). ■

REMARKS (2.6). (1) Let X, Y be log. smooth and proper over $\mathcal{O}_F$, and consider a map $f : X_s \to Y_s$ between the special fibres of the underlying schemes. Suppose that f satisfies 2.4(4).

Then, by (1.6) f induces a natural map on cohomology.

$$H_{\bar{\eta}}(f) : H^*(Y_{\bar{\eta}}, \Lambda) \to H^*(X_{\bar{\eta}}, \Lambda).$$

These maps on cohomology are compatible with composition. Indeed, by the uniqueness in (2.4)(4), associating the map of log. schemes $f^{\log}$ to f respects composition. However, the association to $f^{\log}$ of the map $H_{\bar{\eta}}(f^{\log})$ constructed in (1.6) respects composition of maps of log. schemes, so our construction respects compositions.

(2) Suppose that X_s, Y_s are proper, log. smooth $(k, \mathbb{N})$-schemes, and

that $f: X_s \to Y_s$ is a map of the underlying k-schemes. (We do not assume that X_s, Y_s are special fibres of log. smooth $\mathcal{O}_F$-schemes.) Assume that f satisfies 2.4(1), a purely local condition. Assume that each irreducible component of X_s dominates an irreducible component of Y_s. For example, if $X_s = Y_s$ and f is an automorphism, then this is automatic. Then there exists a positive, integer n, such that if we pull back the situation in (2.6) by the base change $(k, \mathbb{N}) \to (k, \mathbb{N})$ given by $\mathbb{N} \xrightarrow{n} \mathbb{N}$, then f satisfies 2.4(4).

Indeed, by [Ka 2, 2.1(ii)], there is a dense open subset $U \subset Y$, such that $N|_U$ is étale locally generated by a single element t, with $t^n = ux$, for some $u \in N^\times$, the generator $x \in \mathbb{N}$, and n a positive integer. Set $(U', N') = (U, N|_U) \times_{(k, \mathbb{N})} (k, \mathbb{N})$ the base change mentioned above. After this base change the above equation becomes $t^n = uy^n$, with $y \in \mathbb{N}$ the generator; $y^n = x$. Now u is a section of N' and N' is saturated, so we also have $t' = ty^{-1}$ is a section of N', and in fact of $N'^\times$ since u is a unit. On the other hand, one checks easily that N' is locally generated by t', y and $\mathcal{O}_{U'}^*$. As t' is a unit this means that $(U', N') \to (k, \mathbb{N})$ is strict.

Our assumptions now insure that 2.4(*) is satisfied, whence we also have 2.4(4). Now (1.6) shows that we get a map

$$\mathbb{H}(f^{\log}): \mathbb{H}^*(Y_{\bar{s}}, R\Psi\Lambda) \to \mathbb{H}^*(X_{\bar{s}}, R\Psi\Lambda).$$

By (1.8) this map is independent of the choice of n.

In particular, if X_s, Y_s are the special fibres of log. smooth $\mathcal{O}_F$-schemes X, Y respectively, then *the purely local condition* (2.4(1)), guarantees that f induces a map between the cohomology of $X_{\bar{\eta}}$ and $Y_{\bar{\eta}}$.

(3) Suppose that (X, M) is log. smooth over $\mathcal{O}_F$. By [Ka 2, 11.6] the log. structure M is determined by the underlying $\mathcal{O}_F$-scheme X. If the special fibre X_s of X is smooth outside a nowhere dense Zariski closed subset, then by (2.5) the special fibre (X_s, M_s) of (X, M) is determined by the reduction $X^{(m+1)}$ of X modulo π^{m+1} (notation as in (2.4)). By (1.4) the sheaf of nearby cycles $R\Psi\Lambda$ on X_s depends only on (X_s, M_s), so that $R\Psi\Lambda$ depends only on $X^{(m+1)}$.

In general, after a finite base change, the smooth locus of X_s becomes dense, as in (2.6)(2), above, so we may apply the arguments of the previous paragraph to this base change. On the other hand, note that such a base change will in general *not* leave the integer m unchanged.

The above is a refinement for log. smooth schemes of a conjecture of Deligne, proved by Berkovich [Ber], which says that for any flat $\mathcal{O}_F$-scheme X, the vanishing cycles depend only on the completion of X along its special fibre.

EXAMPLE (2.7). The condition (2.4)(1) is not hard to check in concrete situations, and can sometimes be made more explicit. Suppose for example that $X = Y$ is a semi-stable curve, and that f is an automorphism, so that the condition (*) of (2.4) is satisfied.

Over smooth points of X_s the condition (2.4)(1) is automatic. If $x \in X_{\text{ét}}$ is an étale point over a double point of X, then $\mathcal{O}_{X_{\text{ét}}, x}$ is the strict henselisation of $A = W[X, Y]/(XY - \pi)$ in the ideal (X, Y). If $f(y) = x$, choose Z, W in $B = \mathcal{O}_{X_{\text{ét}}, y}$ such that Z and W span the tangent space of B/π, and $ZW = \pi$. Then after interchanging Z and W if necessary, f induces $A/\pi \to B/\pi$, $X \mapsto aW$, $Y \mapsto bZ$, with $a, b \in (B/\pi)^\times \times f$ lifts (étale locally) to the first nilpotent neighbourhood of X_s in X if and only if a, b may be lifted to $\tilde{a}, \tilde{b} \in \in (B/\pi^2)^\times$ respectively, with $\tilde{a}\tilde{b}WZ = \pi$. That is if and only if $ab = 1$.

3. Lefschetz trace formula.

Let X be a proper log. smooth $\mathcal{O}_F$-curve. In the previous section we studied endomorphisms of the closed fibre X_s of X, equipped with its log. structure. The main result of this section is to show that one sometimes has a Lefschetz type trace formula for such endomorphisms.

(3.1) Before stating the main theorem, we recall the construction of analytic spaces associated to certain types of formal schemes [deJ, § 7]. Write A_n for the π-adic completion of a polynomial ring in n variables over $\mathcal{O}_F$. Write $B_{m,n}$ for a power series ring in m variables over A_n. Then one attaches to Spf $(B_{m,n})$ a π-adic analytic unit ball of dimension $m + n$, which is the product of an m-dimensional open unit ball, and an n dimensional closed unit ball. Similarly, if R is a quotient of $B_{m,n}$ then one can attach to Spf (R) a suitable subspace of the unit ball above. This construction is functorial. If a formal scheme has a finite covering by such Spf (R)'s then one can glue the associated analytic spaces. We refer the reader to [deJ] for further details.

(3.2) Let $\mathfrak{Y}$ be a formal scheme which has a finite covering by formal schemes of the form Spf (R), with R as above. Denote by $\mathfrak{Y}_\eta$ the associated rigid analytic space. If σ_η is an endomorphism of $\mathfrak{Y}_\eta$ consider the subspace $\mathfrak{Y}_\eta^{\sigma_\eta}$ of $\mathfrak{Y}_\eta$ which is fixed by σ_η. This is obtained, in the usual way, by intersecting the graph of σ_η with the diagonal in $\mathfrak{Y}_\eta \times \mathfrak{Y}_\eta$. Suppose that $\mathfrak{Y}_\eta^{\sigma_\eta}$ is zero dimensional. Since $\mathfrak{Y}$ is Noetherian, this implies that $\mathfrak{Y}_\eta^{\sigma_\eta}$ is attached to a finite dimensional F-algebra $\mathcal{R}$. We define *the number of fixed points of*

σ_η *counted with multiplicity* to be $\dim_F \mathcal{R}$, and denote this number by $\#\mathrm{Fix}(\sigma_\eta)$.

THEOREM (3.3). *Let (X_s, M_s) be a proper log. smooth curve over $(k, \mathbb{N})$. We denote by $\sigma:(X_s, M_s)\to(X_s, M_s)$ an endomorphism such that*

(1) *σ is not the identity on any connected component of X_s.*

(2) *Let $X_s^\sigma \subset X_s$ denote the fixed subscheme of X_s. We assume that there is a flat, log. smooth $\mathcal{O}_F$-scheme Y, whose special fibre Y_s is an open neighbourhood of X_s, which contains X_s^σ. Write $\mathfrak{Y}$ for the completion of Y along X_s^σ. We assume that σ lifts to an endomorphisms $\widehat{\sigma}: \mathfrak{Y}\to\mathfrak{Y}$.*

(3) *Either (X_s, M_s) is the special fibre of a semi-stable $\mathcal{O}_F$-curve or σ is finite on the underlying scheme X_s.*

Then $\mathfrak{Y}_\eta^{\widehat{\sigma}_\eta}$ is zero dimensional, and for any prime $l\neq p$ we have

$$\#\mathrm{Fix}(\widehat{\sigma}_\eta) = \sum_{i=0}^{2}(-1)^i \operatorname{tr}(\mathbb{H}^i(\sigma)\,|\,\mathbb{H}^i(X_s, R\Psi\mathbb{Q}_l)). \tag{3.3.1}$$

Here the left hand side is given by the preceding discussion, and the right hand side by (1.6), *$\mathbb{H}^i(\sigma)$ denoting the map on cohomology constructed in* (1.6) *from σ.*

In particular, if X is a log. smooth curve over $\mathcal{O}_F$, with special fibre X_s, then σ induces a map $H^*(\sigma_{\overline{\eta}})$ on the cohomology of the geometric generic fibre $X_{\overline{\eta}}$, and

$$\#\mathrm{Fix}(\widehat{\sigma}_\eta) = \sum_{i=0}^{2}(-1)^i \operatorname{tr}(H^i(\sigma_{\overline{\eta}})\,|H^i(X_{\overline{\eta}}, \mathbb{Q}_l)).$$

If σ has only isolated (but not necessary simple) fixed points then the condition (2) *above is automatic.*

PROOF. *Step* 1: If σ has isolated fixed points the condition (2) is automatic. Indeed, choose any log. smooth $\mathcal{O}_F$-scheme Y such that Y_s is isomorphic to an open subset of X_s containing the fixed points of σ. If x is an étale point of Y_s, then (2.4)(2) implies that σ lifts to an endomorphism $\mathcal{O}_{Y_{\text{ét}}, x}\to \to \mathcal{O}_{Y_{\text{ét}}, x}$, whence to an endomorphism on the completions $\widehat{\mathcal{O}}_{Y_{\text{ét}}, x}\to\widehat{\mathcal{O}}_{Y_{\text{ét}}, x}$. If x lies over a a fixed point of σ, then the completion of Y at this fixed point is equal to $\mathrm{Spf}\,(\widehat{\mathcal{O}}_{Y_{\text{ét}}, x})$, as k is separably closed, so the result follows.

Now assume that X_s is the special fibre of a semi-stable curve. We first show the theorem in this special case. Note that by (1.5) this is a purely local condition, and by [Ka, 3.14] it implies that Y is semi-stable.

Step 2: $\mathfrak{Y}_\eta^{\hat{\sigma}_\eta}$ is zero dimensional. To see this consider $T \subset (X_s)^\sigma$ a connected component of $(X_s)^\sigma$, and denote by $\mathfrak{Y}_T$ the corresponding connected component of $\mathfrak{Y}$.

I claim that $(\mathfrak{Y}_T)_\eta$ is connected. If T is an isolated fixed point then $(\mathfrak{Y}_T)_\eta$ is either a disc or an annulus depending on whether T is a smooth or a double point. Otherwise T is a union of irreducible components of X_s. In this case, suppose we knew that for every point $y \in \mathfrak{Y}_T$ the quotient $\mathcal{O}_{\mathfrak{Y}_T, y}/\pi$ were reduced. Then it follows easily that any idempotent on $(\mathfrak{Y}_T)_\eta$ extends to $\mathfrak{Y}_T$, and hence is equal to 1, so that $(\mathfrak{Y}_T)_\eta$ is connected as required. To see that the $\mathcal{O}_{\mathfrak{Y}_T, y}/\pi$ are all reduced, note that if T contains all the components of X_s passing through y, then $\mathcal{O}_{\mathfrak{Y}_T, y}/\pi \xrightarrow{\sim} \mathcal{O}_{T, y}$, and the result is clear. Otherwise the completion $\widehat{\mathcal{O}}_{\mathfrak{Y}_T, y}$ of $\mathcal{O}_{\mathfrak{Y}_T, y}$ is isomorphic to $\mathcal{O}_F[\![U, V]\!]/(UV - \pi)$, so that

$$\mathcal{O}_{\mathfrak{Y}_T, y}/\pi \hookrightarrow \widehat{\mathcal{O}}_{\mathfrak{Y}_T, y}/\pi \xrightarrow{\sim} \mathcal{O}_F[\![U, V]\!]/(UV),$$

whence the result.

Since $(\mathfrak{Y}_T)_\eta$ is connected (and smooth), if $\mathfrak{Y}_\eta^{\hat{\sigma}_\eta} \cap (\mathfrak{Y}_T)_\eta$ is not zero dimensional, we must have $(\mathfrak{Y}_T)_\eta \subset \mathfrak{Y}_\eta^{\hat{\sigma}_\eta}$. This implies that $\mathfrak{Y}_T$ is fixed by $\hat{\sigma}$, which implies that if $y \in T$, then σ fixes $\mathcal{O}_{X_s, y}$, whence σ fixes an open set $H \subset X_s$ containing T. Replacing H by a connected component, we may assume that $H = T$, whence T is both open and closed in X_s, hence equal to a connected component of X_s, a contradiction, as $\sigma \neq 1$ on such components. So $\mathfrak{Y}_\eta^{\hat{\sigma}_\eta}$ is zero dimensional, as required.

Step 3: Let $y \in X_s$ be a double point (i.e a point where $\mathrm{rk}_{\mathbb{Z}}(M_s/\mathcal{O}^*_{X_s, y})^{\mathrm{gp}} = 2$) which is fixed by σ. The hypotheses insure that σ lifts to an endomorphism of $\widehat{\mathcal{O}}_{Y, y}$, the completion of $\mathcal{O}_{Y, y}$. Abusing notation, we will denote this endomorphism by $\hat{\sigma}$. As Y is semi-stable, $\widehat{\mathcal{O}}_{Y, y} \xrightarrow{\sim} \mathcal{O}_F[\![U, V]\!]/(UV - \pi)$ is a regular ring, hence a unique factorisation domain. Thus we see that either $\hat{\sigma}(U) = uU$ and $\hat{\sigma}(V) = u^{-1}V$ or $\hat{\sigma}(U) = uV$ and $\hat{\sigma}(V) = u^{-1}U$, for a suitable unit $u \in \widehat{\mathcal{O}}_{Y, y}$. In particular any lift of σ necessarily induces an automorphism of $\widehat{\mathcal{O}}_{Y, y}$.

Step 4: We compute the right hand side of the formula (3.3.1), in the theorem. This, and the calculation to be done in Step 5 is inspired by an argument of Faltings [Fa, § 4], where it is carried out in a special case (see also [Sa]).

Let $\widetilde{X}_s$ be the normalisation of X_s. The action of σ on X_s lifts to $\widetilde{X}_s$. For each point $y \in X_s$ which is a double point and is fixed by σ, set $\varepsilon_y = 1$ if σ stabilises the two components of X_s (or more precisely the two components of the completion of X_s at y) passing through y, and $\varepsilon_y = -1$ if it interchanges them. This corresponds to the two cases considered in Step 3. I claim that the right hand side of (3.3.1) is given by

$$\sum_T \sum_{i=0}^{2} (-1)^i \operatorname{tr}(\sigma | H^i(T, \mathbb{Q}_l)) - 2 \sum_y \varepsilon_y, \tag{3.3.2}$$

where T runs over the components of $\widetilde{X}_s$ stabilised by σ, and y runs over the double points of X_s fixed by σ.

Our formula differs from the one given in [Fa, p. 475]. There it is claimed that the contribution from a double point y with $\varepsilon_y = -1$ is 0 rather than -1. However Faltings assumes that none of the fixed points of $\widehat{\sigma}_\eta$ specialise to a double point of X. On the other hand, we will see below that if $\varepsilon_y = -1$, then there are precisely 2 fixed points of $\widehat{\sigma}_\eta$ specialising to y, so Faltings' assumption implies that there are no points with $\varepsilon_y = -1$, and the discrepancy does not show up!

Let us prove the formula (3.3.2). If $K^\bullet$ is a complex of finite dimensional $\mathbb{Q}_l$-vector spaces, and ϕ is an endomorphism of $K^\bullet$ then the alternating sum of traces of ϕ on the terms of $K^\bullet$ is equal to the alternating sum of traces of ϕ on the cohomology of $K^\bullet$. Thus, using the spectral sequence

$$H^i(X_s, R^j \Psi\mathbb{Q}_l) \Rightarrow H^{i+j}(X_s, R\Psi\mathbb{Q}_l),$$

we see that the cohomological sum we need to compute is equal to

$$\sum_{i=0}^{2} (-1)^i \operatorname{tr}(\sigma | H^i(X_s, \mathbb{Q}_l)) - \operatorname{tr}(\sigma | H^0(X_s, R^1 \Psi\mathbb{Q}_l)). \tag{3.3.3}$$

Now $R^1 \Psi\mathbb{Q}_l$ is supported at the double points of X_s. If y is such a double point, then $(R^1 \Psi\mathbb{Q}_l)_y$ is a one dimensional $\mathbb{Q}_l$-vector space, and if σ fixes y, then σ acts on $(R^1 \Psi\mathbb{Q}_l)_y$ by multiplication by ε_y (see [Il, 2.1.5]). Applying a standard excision argument, we see that

$$\sum_{i=0}^{2} (-1)^i \operatorname{tr}(\sigma | H^i(X_s, \mathbb{Q}_l)) = \sum_T \sum_{i=0}^{2} (-1)^i \operatorname{tr}(\sigma | H^i(T, \mathbb{Q}_l)) - \sum_y \varepsilon_y, \tag{3.3.4}$$

where T and y run over the same range as in (3.3.2). Combining this with the calculation of $R^1 \Psi\mathbb{Q}_l$ above gives (3.3.2).

Step 5: We want to compute the left hand side of (3.3.1). This will involve considering fixed points of $\widehat{\sigma}_\eta$ which specialise to the various types of fixed points of σ. For a point $y \in X_s$ which is fixed by σ consider the part of the analytic space $\mathfrak{Y}_\eta^{\widehat{\sigma}_\eta}$ which specialises to y. This analytic space is associated to a finite dimensional F-algebra R. We set $a_y = \dim_F R$. We begin by computing a_y for y a double point of X_s which is fixed by σ.

First of all we remark that if T is an irreducible component of X_s, then we can consider the multiplicity n_T with which T appears in $\mathfrak{Y}^{\widehat{\sigma}}$. Namely, $\mathfrak{Y}^{\widehat{\sigma}}$ is a union of (possibly non-reduced) points, and of a divisor on $\mathfrak{Y}$, and the multiplicity of T in $\mathfrak{Y}^{\widehat{\sigma}}$ is simply the multiplicity of T in this divisor.

Now consider a double point y of X_s, which is fixed by σ. As in Step 3, the completion $\widehat{\mathcal{O}}_{\mathfrak{Y}, y}$ of $\mathcal{O}_{\mathfrak{Y}}$ at y is isomorphic to $A = \mathcal{O}_F[\![U, V]\!]/(UV - \pi)$. Suppose that $\widehat{\sigma}(U) = uV$ and $\widehat{\sigma}(V) = u^{-1}U$, with $u \in A^{\times}$. Then the part of $\mathfrak{Y}_{\eta}^{\widehat{\sigma}_{\eta}}$ specialising to y, is given by the analytic space associated to Spf $(A/(U - uV))$. Since π does not divide $U - uV$ in A, $A/(U - uV)$ is flat over $\mathcal{O}_F$. Thus

$$a_y = \dim_F A/(U - uV)[1/\pi] = \dim_k A/(U - uV, \pi) =$$

$$= \dim_k k[\![U, V]\!]/(U^2, V^2, U - uV, UV) = 2\,,$$

where the third equality uses, $U^2 = U(U - uV) + u\pi \in A$. Thus there two fixed points of $\widehat{\sigma}_{\eta}$ specialising to y.

If y is not of the type above, then by Step 3, we must have $\widehat{\sigma}(U) = uU$ and $\widehat{\sigma}(V) = u^{-1}V$, with $u \in A^{\times}$. Denote by T, S the two (possibly equal) components of X_s passing through y. Suppose that U and V are local parameters at y for T and S respectively. Then we must have $u - 1 = U^{n_S} V^{n_T} W$ with W not divisible by U or V. Write $W = u_0 + U^{b_T} u_1 + V^{b_S} u_2 + \pi W'$, with $u_0, u_1, u_2 \in A^{\times} \cup \{0\}$, $W' \in A$, and $b_T = b(y, T)$ and $b_S = b(y, S)$ non-negative integers. We may assume that $u_1 = 0$ (resp. $u_2 = 0$) if and only if $b_T = 0$ (resp. $b_S = 0$). If $u_0 = 0$ we must have $u_1, u_2 \in A^{\times}$. Since W is coprime to π, A/W is π-torsion free. Thus, if $u_0 = 0$, we have

$$a_y = \dim_F A/W[1/\pi] = \dim_k A/(W, \pi) = \tag{3.3.5}$$

$$= \dim_k k[\![U, V]\!]/(U^{b_T+1}, V^{b_S+1}, U^{b_T}u_1 + V^{b_S}u_2) = b_T + b_S,$$

where the second equality uses $U^{b_T+1}u_1 = WU \in A/\pi$. If $u_0 \neq 0$, then $W \in A$ is a unit, and the first equality above shows $a_y = 0$.

Step 6: Let T be a component of X_s which is stabilised by σ. Denote by T^0 the open part of T which consists of points at which X_s is smooth. For

$y \in T - T^0$, we adopt the notation of Step 5, and write

$$h(y, T) = \begin{cases} b(y, T) & u_0 = 0, \\ 0 & u_0 \neq 0. \end{cases}$$

Then I claim we have

$$\sum_{i=0}^{2} (-1)^i \operatorname{tr}(\sigma | H^i(T, \mathbb{Q}_l)) = \sum_{y \in T^0} a_y + \sum_y (n_S - n_T + h(y, T) + 1), \tag{3.3.6}$$

where in the second sum y runs over the points of $T - T^0$ which are fixed by σ, and S denotes the other component of X_s passing through the point y. Of course the first sum on the right has only finitely many non-zero summands, by Step 2, hence is finite.

To prove the formula, first suppose that T is not fixed by σ. Then $n_T = 0$, and the left hand side of (3.3.6) is just the number of fixed points, counted with multiplicity, of σ acting on T. If $y \in T^0$ is a fixed point of σ, then the completion of Y at y is isomorphic to Spf $(\mathcal{O}_F[\![U]\!])$, and $\widehat{\sigma}$ induces a map $\mathcal{O}_F[\![U]\!] \to \mathcal{O}_F[\![U]\!]$. As σ does not fix T, y is an isolated fixed point on T, so that $U - \widehat{\sigma}(U) \neq 0$ modulo π. By Weierstrass preparation, we may write $\widehat{\sigma}(U) - U = fw$, where f is a monic polynomial in U, and w is a unit. Therefore

$$a_y = \dim_F \mathcal{O}_F[\![U]\!]/(U - \widehat{\sigma}(U))[1/\pi] = \dim_k k[\![U]\!]/(U - \sigma(U)) = \deg f,$$

and $\deg f$ is equal to the multiplicity of y as a fixed point of $\sigma|_T$. So the first sum in the right hand side of (3.3.6) accounts for the fixed points of $\sigma|_T$ which lie in T^0.

Now suppose that $y \in T$ is a point which corresponds to a double point of X_s. Using the notation of Step 5, since the parameter V vanishes on T, the multiplicity of $y \in T$ is equal to

$$\dim_k k[\![U]\!]/(uU - U) =$$
$$= \dim_k k[\![U]\!]/(u_0 U^{n_S+1} + u_1 U^{n_S + b(y, T) + 1}) = n_S + h(y, T) + 1.$$

This gives the second term in the right hand side of (3.3.6).

Next suppose that T is fixed by σ. Then the left hand side of (3.3.6) is simply $2 - 2g$, where $g = g(T)$ is the genus of T. We construct a meromorphic derivation δ on $\mathcal{O}_T$ as follows. If s is a local section of $\mathcal{O}_T$, define

$$\delta(s) = \pi^{-n_T}(\widehat{\sigma}(\tilde{s}) - \tilde{s}) \quad \text{modulo } \pi,$$

where $\tilde{s}$ is a local section of $\mathcal{O}_{\mathcal{Y}}$ which lifts s. One checks easily that $\delta(s)$ is independent of the choice of lift, and that δ is indeed a derivation. We may regard δ as a meromorphic section of the tangent bundle of T.

At smooth points of $\mathcal{Y}$, which correspond to points of T, π is a generator for the ideal which cuts out T, so δ is holomorphic on T^0. In general it will have poles at points of $T - T^0$.

Suppose that $y \in T^0$. The completion of Y at y is isomorphic to Spf (A), with $A \xrightarrow{\sim} \mathcal{O}_F[\![U]\!]$, and $\widehat{\sigma}$ induces an endomorphism of A, given by $\widehat{\sigma}(U) = U + \pi^{n_T} f$ with $f \in A$, not divisible by π. Now δ vanishes at y with order $v_y(\bar{f})$, where $\bar{f}$ denotes the reduction of f modulo π, and v_y denotes the valuation on the completion of $\mathcal{O}_{T,y}$. On the other hand, A/f is π-torsion free, so that

$$a_y = \dim_F A/f[1/\pi] = \dim_k k[\![U]\!]/\bar{f} = v_y(\bar{f}).$$

So δ vanishes with order a_y at y.

Now suppose that $y \in T - T^0$. Using the notation of Step 5, we see that

$$\pi^{-n_T}(\widehat{\sigma}(U) - U) = \pi^{-n_T}(u-1)\, U = \pi^{-n_T} U^{n_S+1} V^{n_T} W = U^{n_S - n_T + 1} W.$$

As $W = u\,0 + u_1 U^{b(y,\,T)}$ modulo V we obtain

$$\delta(U) = u_0 U^{n_S - n_T + 1} + u_1 U^{n_S - n_T + 1 + b(y,\,T)}.$$

So δ has a «zero» of order $n_S - n_T + h(y, T) + 1$ at y (i.e a pole if this number is negative), since $b(y, T) = 0$ implies $u_1 = 0$, $u_0 \neq 0$. As the tangent bundle on T has degree $2 - 2g$, summing up the order of the zeroes of δ for all points of T gives the result.

Step 7: Denote by X°_s the smooth part of X_s, and by Q the set of points $y \in X_s$ which are double points of X_s which are fixed by σ, and such that $\varepsilon_y = 1$.

To finish the calculation we sum (3.3.6) over all the components T of X_s which are stable under σ. We obtain

(3.3.7) $$\sum_T \sum_{i=0}^{2} (-1)^i \operatorname{tr}(\sigma | H^i(T, \mathbb{Q}_l)) =$$

$$= \sum_{y \in X^\circ_s} a_y + \sum_{y \in Q} (h(y, T) + h(y, S) + 2) = \sum_{y \in X^\circ_s \cup Q} a_y + 2 \sum_{y \in Q} \varepsilon_y,$$

where the second equality follows from (3.3.5), and as usual the two components of X_s through a point $y \in Q$ are denoted by S and T. The formula

(3.3.1) follows immediately from (3.3.7) and Step 4, keeping in mind that if $y \notin Q$ is a double point of X_s, which is fixed by σ, then $a_y = 2 = -2\varepsilon_y$ by Step 5.

This completes the proof when X_s is the special fibre of a semi-stable curve.

(3.4) To finish the proof of (3.3), we proceed by reducing to the semi-stable case. The idea is as follows: If (X_s, M_s) is the special fibre of a log. smooth $\mathcal{O}_F$-curve X, then after a suitable base change we show that, by repeatedly blowing up the singular points of X, we obtain a semi-stable curve, and we manage to reduce (3.3) for the original curve to this semi-stable one. It is in order to be able to perform this reduction that we need to assume that σ is finite on the underlying scheme X_s. Although this is a natural condition, the result should be true without it.

For the argument we need some preparation.

LEMMA (3.5). *With the notation of* (3.3) (*assuming* σ *is finite*) *we have*

(1) *If* $x, y \in X_s$ *and* $\bar{x}, \bar{y}$ *are étale points over* x *and* y *respectively, with* $\bar{x} = \sigma(\bar{y})$, *then the map* $(M_s/\mathcal{O}^*_{X_s})^{\mathrm{gp}}_{\bar{x}} \to (M_s/\mathcal{O}^*_{X_s})^{\mathrm{gp}}_{\bar{y}}$ *is injective.*

(2) $\mathrm{rk}_{\mathbb{Z}}(M_s/\mathcal{O}^*_{X_s})^{\mathrm{gp}}_{\bar{y}} \geq \mathrm{rk}_{\mathbb{Z}}(M_s/\mathcal{O}^*_{X_s})^{\mathrm{gp}}_{\bar{x}}$.

PROOF. First note that (2) follows immediately from (1).

We prove (1). As was remarked in (1.5), we may assume that X_s is the special fibre of a log. smooth $\mathcal{O}_F$-curve X. By (2.1) σ lifts to a map $\hat{\sigma}$ of formal neighbourhoods of x and y in X. Such a lift induces a map $\hat{\sigma}^{\#}: \widehat{\mathcal{O}}_{X,x} \to \widehat{\mathcal{O}}_{X,y}$. Now $\widehat{\mathcal{O}}_{X,y}$ and $\widehat{\mathcal{O}}_{X,x}$ are normal local rings [Ka 2, 4.1], hence domains, and $\widehat{\mathcal{O}}_{X,x}[1/\pi]$ is a one dimensional domain whose maximal primes correspond to the points of the generic fibre of X specialising into x. Thus, if $\hat{\sigma}^{\#}$ is not injective, then the induced map $\widehat{\mathcal{O}}_{X,x}[1/\pi] \to \widehat{\mathcal{O}}_{X,y}[1/\pi]$ factors through the residue field at one of these points. However one easily sees that this means σ is constant at y, whence it cannot be finite on the underlying scheme X_s, a contradiction. It follows that $\hat{\sigma}^{\#}$ is injective.

To show (1) it suffices to show that $(M_s/\mathcal{O}^*_{X_s})_{\bar{x}} \to (M_s/\mathcal{O}^*_{X_s})_{\bar{y}}$ is injective. If $\widehat{M}_{\bar{x}}$ and $\widehat{M}_{\bar{y}}$ denote the log. structures on $\widehat{\mathcal{O}}_{X,x}$ and $\widehat{\mathcal{O}}_{X,y}$ then we have $(M_s/\mathcal{O}^*_{X_s})_{\bar{x}} = \widehat{M}_{\bar{x}}/\widehat{\mathcal{O}}^*_{X,x}$, and similarly with y in place of x. Thus, we have to show that

$$\widehat{M}_{\bar{x}}/\widehat{\mathcal{O}}^*_{X,x} \to \widehat{M}_{\bar{y}}/\widehat{\mathcal{O}}^*_{X,y}$$

is injective. By [Ka 2, 11.6] we have $\widehat{M_{\bar{x}}} = \widehat{\mathcal{O}}_{X,x}[1/\pi]^* \cap \widehat{\mathcal{O}}_{X,x}$, and similarly for y. If $t, s \in \widehat{M_{\bar{x}}}$ and $\alpha \in \widehat{\mathcal{O}}^*_{X,y}$ with $t = \alpha s$ then we have $\alpha = ts^{-1} \in \widehat{\mathcal{O}}_{X,x}[1/\pi]^* \cap \mathcal{O}^*_{X,y}$, so that $\alpha \in \widehat{\mathcal{O}}_{X,x}$ as $\widehat{\mathcal{O}}_{X,x}$ is normal, and the map $\widehat{\sigma}^{\#}$ is finite. Since the same argument applies to α^{-1}, we see that $\alpha \in \widehat{\mathcal{O}}^*_{X,x}$. This shows the required injectivity. ■

LEMMA (3.6). *Keeping the notation of the previous lemma, suppose that (X_s, M_s) is the special fibre of a log. smooth $\mathcal{O}_F$ curve (X, M). Let $S \subset X_s$ be a set of points stable by the action of σ, and such that $\mathrm{rk}_{\mathbb{Z}}(M^{\mathrm{gp}}_{\bar{x}}/\mathcal{O}^*_{X,\bar{x}}) \geq 2$, for $x \in S$, and $\bar{x}$ an étale point over x. Consider the scheme X' obtained from X by blowing up along S, and normalising.*

(1) *S is a finite set, and X' has a natural structure of a log. smooth $\mathcal{O}_F$-scheme (X', M').*

(2) *If σ is an endomorphism of the special fibre (X_s, M_s) of (X, M), then σ lifts uniquely to an endomorphism σ' of (X'_s, M'_s).*

PROOF. (1) That S is a finite set follows from that fact that X is log. regular [Ka 2, 2.1]. To see that X' has a natural log. structure M', we claim that X' is a log. blow up. In the terminology of [Ka 2, § 9] this means that for each point $x \in S$ and $\bar{x}$ an étale point over x, there is a chart $P_{\bar{x}} \to \mathcal{O}_{X,\bar{x}}$ for $M_{\bar{x}}$, and $X' \times_X \mathrm{Spec}\,(\mathcal{O}_{X,\bar{x}})$ is obtained by taking a subdivision of $\mathrm{Spec}\,(\mathbb{Z}[P_{\bar{x}}])$. To see this denote by $I_{\bar{x}}$ the ideal spanned by the image of $P_{\bar{x}} \backslash P^{\times}_{\bar{x}}$. Then $\mathcal{O}_{X,\bar{x}}/I_{\bar{x}}$ is a regular ring. If it has dimension 1, then $M^{\mathrm{gp}}_{\bar{x}}/\mathcal{O}^*_{X,\bar{x}}$ must have rank 1 [Ka 2, 2.1(ii)], and $\mathcal{O}_{X,\bar{x}}$ must be regular. Thus $\mathcal{O}_{X,\bar{x}}/I_{\bar{x}}$ is zero dimensional, and $I_{\bar{x}}$ is the maximal ideal of $\mathcal{O}_{X,\bar{x}}$. Since the maximal ideal of $\mathcal{O}_{X,\bar{x}}$, is generated by elements of $P_{\bar{x}}$, one sees from the definition of blowing up, that $X' \times_X \mathrm{Spec}\,(\mathcal{O}_{X,\bar{x}})$ is obtained by taking a subdivision of $P_{\bar{x}}$ (See (2) below for a more detailed description).

Finally the morphism $X' \to X$ is log. smooth by [Ka 2, 9.5(ii)], so X' is log. smooth over $\mathcal{O}_F$.

(2) Let $x \in S$ and $y \in X_s$ with $x = \sigma(y)$. Choose étale points $\bar{x}, \bar{y}$ over x and y with $\bar{x} = \sigma(\bar{y})$. We have to show that there is a unique way to choose a map of log. schemes in the top row of the diagram

$$\begin{array}{ccc} X'_s \times_{X_s} \mathrm{Spec}\,(\mathcal{O}_{X_s,y}) & \longrightarrow & X'_s \times_{X_s} \mathrm{Spec}\,(\mathcal{O}_{X_s,x}) \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,(\mathcal{O}_{X_s,y}) & \longrightarrow & \mathrm{Spec}\,(\mathcal{O}_{X_s,x}) \end{array}$$

making it commute. By uniqueness, it is enough to prove this after the base change $\mathrm{Spec}\,(\mathcal{O}_{X_s, \bar{x}}) \to \mathrm{Spec}\,(\mathcal{O}_{X_s, x})$. Since σ is finite on underlying schemes we have, by definition of $\bar{x}$ and $\bar{y}$ a canonical isomorphism

$$\mathcal{O}_{X_s, y} \otimes_{\mathcal{O}_{X_s, x}} \mathcal{O}_{X_s, \bar{x}} \xrightarrow{\sim} \mathcal{O}_{X_s, \bar{y}}\,.$$

Thus, it is enough to construct a map as in the diagram above, but with $\bar{x}, \bar{y}$ in place of x, y.

With the notation of the proof of (1) we may choose a chart $P_{\bar{x}} \to P_{\bar{y}}$ for the map of k-log. schemes $\mathrm{Spec}\,(\mathcal{O}_{X_s, \bar{y}}) \to \mathrm{Spec}\,(\mathcal{O}_{X_s, \bar{x}})$.

Now if $e_1, \ldots, e_k \in P_{\bar{x}}$ are a set of generators and $1 \leqslant j \leqslant k$, denote by $Q_j \subset P_{\bar{x}}^{\mathrm{gp}}$ the monoid generated by $e_i e_j^{-1}$ for $i = 1, \ldots, k$. Let X'' denote the blow up of S on X. Then $X_s'' \times_{X_s} \mathrm{Spec}\,(\mathcal{O}_{X_s, \bar{x}})$ is covered by the union of the $\mathrm{Spec}\,(\mathcal{O}_{X_s, \bar{x}}) \times_{\mathrm{Spec}\,(\mathbb{Z}[P_{\bar{x}}])} \mathrm{Spec}\,(\mathbb{Z}[Q_j])$ for $j = 1, \ldots, k$. It follows that $X_s' \times \times_{X_s} \mathrm{Spec}\,(\mathcal{O}_{X_s, \bar{x}})$ is covered by the union of the $\mathrm{Spec}\,(\mathcal{O}_{X_s, \bar{x}}) \times \times_{\mathrm{Spec}\,(\mathbb{Z}[P_{\bar{x}}])} \mathrm{Spec}\,(\mathbb{Z}[Q_j^{\mathrm{sat}}])\, j = 1, \ldots, k$, where Q_j^{sat} denotes the saturation of Q_j. Now if we extend $e_1 \ldots e_k$ to a set of generators for $P_{\bar{y}}$, (which contains $P_{\bar{x}}$ by (3.5)), we get an analogous covering of $X_s' \times_{X_s} \mathrm{Spec}\,(\mathcal{O}_{X_s, \bar{y}})$, and one sees immediately from these two coverings (and the fact that the chosen set of generators for $P_{\bar{y}}$ extends the one for $P_{\bar{x}}$) that σ lifts to a unique map of log. schemes $X_s' \times_{X_s} \mathrm{Spec}\,(\mathcal{O}_{X_s, \bar{y}}) \to X_s' \times_{X_s} \mathrm{Spec}\,(\mathcal{O}_{X_s, \bar{x}})$. Repeating this argument for each $x \in S$ shows that σ lifts to a map of log. schemes $X_s' \to X_s'$. ■

LEMMA (3.7). *Keep the notation of the previous lemma, but suppose that the underlying scheme X is regular and that X_s is connected. Let $x, y \in X_s$, and $\bar{x}, \bar{y}$ étale points over x, y with $\bar{x} = \sigma(\bar{y})$. Then σ is surjective on the underlying scheme X_s, and if $\mathrm{rk}_{\mathbb{Z}}(M^{\mathrm{gp}}/\mathcal{O}_X^*)_{\bar{x}} = 2$, there exists a lift of σ to the formal neighbourhoods of $x, y \in X$. Such a lift induces an isomorphism of complete local $\mathcal{O}_F$-algebras $\widehat{\mathcal{O}}_{X,x} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,y}$.*

PROOF. Since σ is not constant, its image contains at least one component $C \subset X_s$. Choose $x \in C$, and $y \in X_s$ with $x = \sigma(y)$. First we claim that the image of σ contains all the other components passing through x. Note that this claim implies the first claim of the lemma, as then repeating the argument with the other components, and using the fact that X_s is connected shows that σ is surjective.

Denote by $\hat{\sigma}$ a lift of σ to the formal neighbourhoods of $x, y \in X$. This exists by (2.1). Write $\hat{\sigma}^{\#}: \widehat{\mathcal{O}}_{X,x} \to \widehat{\mathcal{O}}_{X,y}$ for the induced map of local, complete $\mathcal{O}_F$-algebras. It is injective by (3.5).

If C is the only component passing through x, there is nothing to prove. Otherwise we must have $\mathrm{rk}_{\mathbb{Z}}(M^{\mathrm{gp}}/\mathcal{O}_X^*)_{\bar{x}} = 2$, whence also $\mathrm{rk}_{\mathbb{Z}}(M^{\mathrm{gp}}/\mathcal{O}_X^*)_{\bar{y}} = 2$ by (3.5). As σ is finite, the map $\widehat{\sigma}^{\#}: \widehat{\mathcal{O}}_{X,x} \to \widehat{\mathcal{O}}_{X,y}$ is finite also. Such a map between local regular rings is flat [Mat, 23.1], whence so is its reduction modulo π. In particular the reduction of $\widehat{\sigma}^{\#}$ modulo π is injective, which shows that $\sigma(X_s)$ contains the other components passing through x.

Note that as σ is surjective, it permutes the finite set S of points $z \in X_s$ such that $\mathrm{rk}_{\mathbb{Z}}(M^{\mathrm{gp}}/\mathcal{O}_X^*)_{\bar{z}} = 2$, by (3.5). As usual, $\bar{z}$ is an étale point over z. Choose a lift $\widehat{\sigma}^{\#}$, as above, for each pair of points $x, y \in S$ with $x = \sigma(y)$. We saw in (3.5) that these lifts are all injective, and we have to show that they are surjective. If $n \in \mathbb{N}^+$, it is enough to prove this with σ^n in place of σ. Thus we may assume that σ fixes S pointwise, and that $x = y$. Now there is an isomorphism $\widehat{\mathcal{O}}_{X,x} \xrightarrow{\sim} \mathcal{O}_F[\![u, v]\!]/(u^r v^s - \pi)$, and it is easy to see that any endomorphism of this $\mathcal{O}_F$-algebra is an automorphism, the key point being that this ring is a unique factorisation domain. ■

(3.8) *End of proof of* (3.3). First note that, by the remarks of (1.5), we may assume that X_s is the special fibre of a proper log. smooth $\mathcal{O}_F$-curve X.

Step 8: For any finite extension F'/F, we may replace X by $X' = X \times_{\mathcal{O}_F} \mathcal{O}_{F'}$, the product in the category of fs log. schemes. Indeed the special fibre X_s' of X' is equal to $X_s \times_{\mathcal{O}_F} \mathcal{O}_{F'}$ (product as fs log. schemes), so σ lifts to an endomorphism σ' of X_s' by functoriality of fibre products. On the other hand $Y' = Y \times_{\mathcal{O}_F} \mathcal{O}_{F'}$ contains $(X_s')^{\sigma'}$, and its underlying scheme is equal to the normalisation of the scheme theoretic product $Y \times_{\mathcal{O}_F} \mathcal{O}_{F'}$. As all our rings are excellent, the normalisation $\mathfrak{Y}''$ of $\mathfrak{Y} \times_{\mathcal{O}_F} \mathcal{O}_{F'}$ (product as formal schemes) is equal to the completion of Y' along the preimage of X_s^{σ} in X_s', so σ' lifts to $\mathfrak{Y}''$. It follows that σ' also lifts to the completion $\mathfrak{Y}'$ of Y' along $(X_s')^{\sigma'}$, as this is simply the completion of $\mathfrak{Y}''$ along $(X_s')^{\sigma'}$, and our hypotheses are preserved. Moreover by (1.9), and since on rigid analytic fibres $\mathfrak{Y}_\eta' \hookrightarrow \mathfrak{Y}_\eta''$ is an open immersion, neither side of the formula (3.3.1) changes (compare Step 9 below).

In particular after a making a base extension, as above, we may assume that the generic fibre of X has geometrically connected connected components. This implies that if T is a connected component of X, then $T \times_{\mathcal{O}_F} \mathcal{O}_{F'}$ remains connected (product as fs log. schemes). If $T_1, \ldots T_k$ are the connected components of X_s, then only components which are mapped to themselves by σ contribute to either side of (3.3.1). Thus we may assume that X_s is connected, and that this remains true after replacing X by X' as in the previous paragraph.

Step 9: Denote by $S \subset X_s$ a set of points satisfying the conditions of (3.6), and consider the scheme X' obtained from X by blowing up S and normalising. We want to replace X by X'. By (3.6) X' is equipped with the structure of a smooth log. scheme over $\mathcal{O}_F$, and σ lifts to an endomorphism σ' of its special fibre X_s'. Denote by Y' the normalisation of the blow up of Y in $Y \cap S$, and by $\mathfrak{Y}''$ the completion of Y' along $Y' \cap \cap p^{-1}(X_s^{\sigma})$. Then $\mathfrak{Y}''$ is the normalisation of the formal blow up of $\mathfrak{Y}$ along $S \cap X_s^{\sigma}$, so that the $\widehat{\sigma}$ lifts to an endomorphism $\widehat{\sigma}''$ of $\mathfrak{Y}''$. One checks easily that $\widehat{\sigma}''$ lifts σ''. (Note that we have used here the fact that all our rings are excellent, so that normalisation commutes with completion).

Hence $\widehat{\sigma}$ lifts to the completion $\mathfrak{Y}'$ of Y' along $Y' \cap (X_s')^{\sigma'}$ as this is the completion of $\mathfrak{Y}''$ along $(X_s')^{\sigma'}$. Finally note that on rigid analytic fibres we have an open immersion $\mathfrak{Y}_\eta' \hookrightarrow \mathfrak{Y}_\eta''$, an isomorphism $\mathfrak{Y}_\eta'' \xrightarrow{\sim} \mathfrak{Y}_\eta$, and that the fixed locus of $\widehat{\sigma}_\eta$ is equal to that of $\widehat{\sigma}_\eta''$, and is therefore contained in $\mathfrak{Y}_\eta'$.

Thus, using (1.7), we may replace X by X' and $\mathfrak{Y}$ by $\mathfrak{Y}'$. Note that in particular, since σ is finite, (3.5)(2) implies that σ permutes the set of points $x \in X_s$ with $\mathrm{rk}_{\mathbb{Z}}(M^{\mathrm{gp}}/\mathcal{O}_X^*)_{\bar{x}} \geqslant 2$, so we may take S equal to this set.

Now by [Ka 2, 10.4], if we repeatedly blow up X in the set of points x where $\mathrm{rk}_{\mathbb{Z}}(M^{\mathrm{gp}}/\mathcal{O}_X^*)_{\bar{x}} \geqslant 2$, and then normalise, we eventually get X regular (note that blowing up points which are already regular does not hurt regularity). Hence using the previous paragraph, we may assume that X is regular. (The procedure of blowing up and normalising is also Lipman's algorithm for resolving singularities on an excellent surface, but the proof becomes simpler in the log. smooth situation). Thus by (3.7) we have

(3.8.1) σ is surjective. If $y \in X_s$ $x = \sigma(y)$, and $\mathrm{rk}_{\mathbb{Z}}(M_x^{\mathrm{gp}}/\mathcal{O}_{X,\bar{x}}^*) > 1$

then there exists an isomorphism of $\mathcal{O}_F$-algebras $\widehat{\mathcal{O}}_{X,x} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,y}$.

Step 10: If $x \in X_s$ is a point where $\mathrm{rk}_{\mathbb{Z}} M_{X_s,\bar{x}}^{\mathrm{gp}}/\mathcal{O}_{X_s,\bar{x}}^* = 1$, then we have $\widehat{\mathcal{O}}_{X,x} \xrightarrow{\sim} \mathcal{O}_F[[u, v]]/(u^n - \pi)$, for some integer n, and the (unique) component of X_s passing through x appears with multiplicity n in $X_s \subset X$. Now the discussion in (2.6)(2) shows that there is a finite extension F'/F such that the components of $X_s \times_{\mathcal{O}_F} \mathcal{O}_{F'} \subset X \times_{\mathcal{O}_F} \mathcal{O}_{F'}$ passing through the points which lie over x have multiplicity 1. Hence, we may assume that the components of X_s appear with multiplicity 1 in $X_s \subset X$. In particular, this implies that the smooth points of X_s are dense. Note that although this base change destroys the regularity of X the condition (3.8.1) is still satisfied, as one sees easily using the arguments with formal neighbourhoods which appeared in Step 8.

Step 11: By [deJ2, 4.18], after making a further extension of scalars, we may assume that there is a semi-stable curve X' and a dominant map $X' \to X$, which is an isomorphism on generic fibres. First we claim that X' can be obtained from X by repeatedly blowing up points, and normalising. (Such an operation is called a «log. blow up of a point» or a «formal log. blow up of a point» for the corresponding operation on a formal scheme). To see this, note that if $C \subset X'$ is a component of the special fibre which maps to a point $x \in X_s$, then $X' \to X$ factors through the log. blow up of X in x, using the universal property of blow ups, that $C \subset X'$ is a Cartier divisor, and the fact that X' is a normal scheme. Repeating the argument, we see that there is a scheme X'' obtained from X by repeatedly log. blowing up points and a finite dominant map $X' \to X''$ which is an isomorphism on generic fibres. However, as both sides are normal, and the map is generically an isomorphism, it must be an isomorphism.

Now suppose that $X' \to X'' \to X$, with X'' obtained from X by log. blowing up points. X'' is log. smooth by (3.6). If $z \in X''$ is a point of the special fibre X''_s with $\mathrm{rk}_{\mathbb{Z}}(M_X^{\mathrm{gp}}/\mathcal{O}_X^*)_{\bar{z}} = 1$, then the calculation in Step 10, and the fact that X' is semi-stable implies that X'' is already smooth at z. Thus we may assume X' is obtained from X by only log. blowing up points z where $\mathrm{rk}_{\mathbb{Z}}(M_X^{\mathrm{gp}}/\mathcal{O}_X^*)_{\bar{z}} \geq 2$.

Finally we explain how to modify the algorithm which produces X' from X in such a way that it produces a semi-stable scheme $\widetilde{X}'$ such that σ lifts to $\widetilde{X}'_s$ equipped with its canonical log. structure. For this suppose that we have $X' \to X'' \to X$ as above. I claim that there exists a scheme $\widetilde{X}'' \to X$, obtained from X by repeatedly blowing up points and such that

(1) If M'' denotes the log. structure on $\widetilde{X}''$, then σ lifts to the an endomorphism σ'' of the special fibre $(\widetilde{X}''_s, M''_s)$ of $(\widetilde{X}'', M'')$.

(2) For each $z \in \widetilde{X}''_s$ there exists a point $z^+ \in X''$ and an isomorphism of $\mathcal{O}_F$-algebras $\widehat{\mathcal{O}}_{X'', z^+} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\widetilde{X}'', z}$.

(3) σ'' satisfies the condition (3.8.1).

To see this note that if $X'' = X$, there is nothing to prove. We proceed by induction on the number of log. blow ups necessary to obtain X'' from X. Suppose that the result is known for a given X'' as above. For each $z \in$

$\in \widetilde{X}''$ fix a $z^+ \in X''$ satisfying (2) above. Let $y \in X''$, and suppose that in the algorithm for obtaining X' from X by log. blow ups we are required to log. blow up y. If there is no $z \in \widetilde{X}''$ with $z^+ = y$, then we may replace X'' by its log. blow up in y, and leave $\widetilde{X}''$ unchanged. Suppose that we have a $z \in \widetilde{X}''$ with $z^+ = y$. Let $S \subset \widetilde{X}''_s$ be the finite set of points w with $\mathrm{rk}_{\mathbb{Z}}(M''^{\mathrm{gp}}/\mathcal{O}^*_{\widetilde{X}''})_{\overline{w}} > 1$. As σ'' satisfies (3.8.1) it permutes the points of S by (3.5). Let $T \subset S$ be the set of points $w \in S$ such that the orbit of w under σ'' contains a point z with $z^+ = y$. We claim that if we replace X'' by its log. blow up W in y, and $\widetilde{X}''$ by its log. blow up $\widetilde{W}$ in T, then the conditions (1), (2), (3) above are still satisfied.

Indeed, (1) follows from (3.6) and the fact that T is stable under σ''. To see (2) note that condition (3.8.1), and the fact that each σ'' orbit of T contains a point z with $z^+ = y$ implies that if $x \in T$ there is an isomorphism $\widehat{\mathcal{O}}_{\widetilde{X}'', x} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X'', y}$, so that the formal blow ups of $\mathrm{Spf}\,(\widehat{\mathcal{O}}_{\widetilde{X}'', x})$ and $\mathrm{Spf}\,(\widehat{\mathcal{O}}_{X'', y})$, in x and y respectively are isomorphic. Thus if $\widetilde{w} \in \widetilde{W}$ lies over x, then there exists a point $w \in W$ lying over y with $\widehat{\mathcal{O}}_{\widetilde{W}, \widetilde{w}} \xrightarrow{\sim} \widehat{\mathcal{O}}_{W, w}$, as required.

Finally to check (3) let $\sigma^{\widetilde{W}}$ be the lift of σ'' to the special fibre $\widetilde{W}_s$ of $\widetilde{W}$ equipped with its log. structure, and denote by $r : \widetilde{W} \to \widetilde{X}''$ the canonical projection. Take $u, v \in \widetilde{W}_s$ with $u = \sigma^{\widetilde{W}}(v)$. If $r(u), r(v) \notin T \subset \widetilde{X}''$, then (3.8.1) follows from the corresponding property for $\widetilde{X}''$. If $r(u), r(v) \in T$, then (3.8.1) for $\widetilde{X}''$ implies that there is an isomorphism between the formal neighbourhoods of $r^{-1}(r(u)) \subset \widetilde{W}$ and $r^{-1}(r(v)) \subset \widetilde{W}$, which lifts $\sigma^{\widetilde{W}}$ (cf. Step 9). In particular such an isomorphism induces an isomorphism $\widehat{\mathcal{O}}_{\widetilde{W}, u} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\widetilde{W}, v}$, whence (3) also holds.

If we apply the above result with $X'' = X'$, then we see we may assume that $\widetilde{X}''$ is semi-stable. Replacing X' with such a $\widetilde{X}''$ we see that we may assume that σ lifts to the special fibre X'_s of X', equipped with its canonical log. structure. Now the theorem follows from Step 9, and the semi-stable case, which has already been proved. ■

(3.9) We will see in the next section that Theorem 3.3 has an analogue using log. crystalline cohomology. For higher dimensions, the following conjecture seems reasonable.

CONJECTURE (3.10). *Let (X_s, M_s) be proper and log smooth over k, of dimension d. Suppose we are given an endomorphism σ of (X_s, M_s). Let $X_s^\sigma \subset X_s$ denote the fixed subscheme of σ. Let Y be a log. smooth $\mathcal{O}_F$-scheme, such that Y_s is an open neighbourhood of X_s containing X_s^σ. De-*

note by $\mathfrak{Y}$ the completion of Y along X_s^{σ}. Suppose that σ lifts to an endomorphism $\widehat{\sigma}$ of $\mathfrak{Y}$, and that the induced endomorphism $\widehat{\sigma}_\eta$ of the rigid fibre $\mathfrak{Y}_\eta$ of $\mathfrak{Y}$ has a fixed point subspace which consists of isolated points.

Then

$$\#\mathrm{Fix}(\widehat{\sigma}_\eta) = \sum_{i=0}^{2d} (-1)^i \operatorname{tr}(\mathbb{H}^i(\sigma) \mid \mathbb{H}^i(X_s, R\Psi\mathbb{Q}_l)),$$

where l is any prime different from p, and $H^i(\sigma)$ denotes the map on cohomology constructed in (1.6) *from σ.*

(3.11) As noted in the introduction, Theorem 3.3 implies, in particular that the number of rigid fixed points of $\widehat{\sigma}_\eta$ specialising to a given isolated fixed point of σ is independent of the choice of $\mathfrak{Y}$, and $\widehat{\sigma}$. Thus, this number should have an *a priori* definition, which suggests the existence of an intersection theory for log. schemes. Such a theory would quite likely lead to a proof of Conjecture 3.10.

4. The crystalline realization of the theory.

(4.1) The theory of § 1-3, uses étale cohomology. However, the constructions, and results have other «realizations.» In this section we want to sketch how to obtain similar results using log. crystalline cohomology.

We work first of all with the usual situation of a log. smooth (of Cartier type) scheme over $\mathcal{O}_F$, where $\mathcal{O}_F$ is a *mixed characteristic* valuation ring. We compare log. crystalline cohomology of the special fibre, and de Rham cohomology of the general fibre, and prove a Lefschetz Trace Formula in the style of § 3.

(4.2) Suppose we are given a log. scheme (S, N), with $\mathcal{O}_S$ killed by some positive integer, and S equipped with a coherent ideal $\mathfrak{I}$ which is equipped with a divided power structure γ. We consider a log. scheme (X, M) over (S, N), such that γ extends to X. We refer to [HK, 2.15] for the definition of the logarithmic crystalline site $((X, M)/(S, L, \gamma))_{\mathrm{crys}}$. We often abbreviate this to $(X/S)_{\mathrm{crys}}$.

Since we do not use the definition of $(X/S)_{\mathrm{crys}}$ explicitly in the following, we do not explain it here, but instead list some of the properties of this site, which will be used in the following.

(1) $(X/S)_{\mathrm{crys}}$ is equipped with a sheaf of rings $\mathcal{O}_{X/S}$.

(2) There is a canonical morphism of topoi (but not of ringed topoi!) [HK, 2.20].

$$u^{\log}_{X/S}:(X/S)_{\mathrm{crys}}\to X_{\text{ét}}.$$

(3) Consider a discrete valuation ring $\mathcal{O}_F$, as above, with perfect residue field k of characteristic p, and quotient field F. We set $S_n = \mathrm{Spec}\,(W(k)/p^n)$, equipped with the log. structure induced from $W(k)$. Let (X_s, M_s) be a proper smooth log. scheme over $(k, \mathbb{N}^+)$ which is of Cartier type [Ka, 4.8]. The *Hyodo-Kato* cohomology $H^m_{HK}(X_s/W(k))$ of (X_s, M_s) is defined by

$$H^m_{HK}(X_s/W(k)) = \varprojlim H^m((X_s/S_n)_{\mathrm{crys}}, \mathcal{O}_{X_s/S_n}).$$

If (X_s, M_s) is the special fibre of a proper log. smooth $\mathcal{O}_F$-scheme (X, M), then there is a canonical isomorphism

$$F\otimes_{W(k)} H^m(X_s, W(k)) \xrightarrow{\sim} H^m_{dR}(X_\eta/F),$$

the latter being the de Rham cohomology of the generic fibre X_η of X [HK, 5.1].

(4.3) Now suppose that we have the situation of Proposition 1.9.

Arguments analogous to those of §1 show that f induces a map on cohomology

$$H^i_{HK}(f): H^i_{HK}(Y_s/W(k)) \to H^i_{HK}(X_s/W(k)).$$

In particular, any map $f: X_s\to Y_s$ satisfying the condition (1) and (*) of (2.4) induces a map on Hyodo-Kato cohomology. If X_s and Y_s are the special fibres of log. smooth, proper $\mathcal{O}_F$-schemes X and Y respectively, then (4.2)(3) shows that f induces a map between the de Rham cohomology of X and Y.

If $f: X_s\to Y_s$ is as above, but only satisfies the condition (2.4)(1), then as in the remark (2.6)(2), if each irreducible component of X_s dominates an irreducible component of Y_s, then after a finite extension F'/F, we may assume that f satisfies the condition (*) of (2.4). Thus, in any case, if $\overline{F}$ denotes the algebraic closure of F, then f induces a map

$$H^i_{HK}(Y_s/W(k))\otimes_{W(k)}\overline{F} \to H^i_{HK}(X_s/W(k))\otimes_{W(k)}\overline{F}.$$

We also have an analogue of (3.3):

THEOREM 4.4. *Suppose that we are in the situation of theorem* 3.3. *Then*

$$\#\mathrm{Fix}\,(\widehat{\sigma}_\eta) = \sum_{i=0}^{2} (-1)^i \operatorname{tr}(H^i_{HK}(\sigma)\,|\,H^i_{HK}(X_s/W(k)). \tag{4.4.1}$$

PROOF. Formally the proof is the same as that of (3.3), so we just sketch it, with emphasis on the points which are slightly different. As in the proof (3.3), we can reduce to the case where X_s is the special fibre of a semi-stable curve. The key point is that both sides of (4.4.1) are invariant under log. blowing up: for the left hand side this is a purely geometric fact, and was already discussed in the proof of (3.3), and for the right hand side this follows for example from the comparison with de Rham cohomology in (4.2)(3). In the semi-stable case, the calculation of the left hand side of (4.4.1) is the same as in (3.3), being purely geometric. To calculate the right hand side we need an analogue of (3.3.2).

We claim that the right hand side of (4.4.1) is equal to the following expression:

$$\sum_T \sum_{i=0}^{2} (-1)^i \operatorname{tr}(\sigma|H^i_{\mathrm{cris}}(T/W(k))) - 2\sum_y \varepsilon_y, \tag{4.4.2}$$

Here H^i_{cris} denotes usual crystalline cohomology, and the rest of the notation is as for (3.3.2). Assuming the claim, we may finish the proof as in (3.3), by applying the Lefschetz Trace Formula for crystalline cohomology to each smooth component T above (see [GM] - which covers the case of intersections with multiplicity).

It remains to show the claim. Denote by $\widetilde{X}_s$ the normalisation of X_s, and by $X^\circ_s \subset X_s$ the smooth part of X_s. We may also view X°_s as an open subset of $\widetilde{X}_s$, and we denote by $j : X^\circ_s \to X_s$, and $\tilde{j} : X^\circ_s \to \widetilde{X}_s$ the two open immersions. We need the fact that there is a complex of sheaves $W^\bullet_{X_s/W}$ on $X_{s,\,\text{ét}}$ (which we may take to be the so called logarithmic de Rham-Witt complex [HK]), such that there is a canonical isomorphism

$$H^m_{HK}(X_s/W(k)) \xrightarrow{\sim} H^m(X_{s,\,\text{ét}}, W^\bullet_{X_s/W}). \tag{4.4.3}$$

We have a similar statement with $\widetilde{X}_s$ in place of X_s, and then the left hand side is simply the usual crystalline cohomology of $\widetilde{X}_s$. Set

$$K_1 = \mathrm{Cone}\,(j_! j^* W^\bullet_{X_s/W} \to W^\bullet_{X_s/W}),$$

$$K_2 = \mathrm{Cone}\,(\tilde{j}_! \tilde{j}^* W^\bullet_{\widetilde{X}_s/W} \to W^\bullet_{\widetilde{X}_s/W}).$$

Using $j^* W^\bullet_{X_s/W} \tilde{\to} \tilde{j}^* W^\bullet_{\tilde{X}_s/W}$, we obtain exact triangles

$$R\Gamma_c(X^\circ{}_{s,\text{ét}}, j^* W^\bullet_{X_s/W}) \to R\Gamma(X_{s,\text{ét}}, W^\bullet_{X_s/W}) \to R\Gamma(X_{s,\text{ét}}, K_1),$$

$$R\Gamma_c(X^\circ{}_{s,\text{ét}}, j^* W^\bullet_{X_s/W}) \to R\Gamma(\tilde{X}_{s,\text{ét}}, W^\bullet_{\tilde{X}_s/W}) \to R\Gamma(\tilde{X}_{s,\text{ét}}, K_2).$$

These exact triangles will allow us to compare the right hand side of (4.4.1) with (4.4.2), using (4.4.3) once we have computed the trace of σ on $H^m(X_{s,\text{ét}}, K_1)$ and $H^m(\tilde{X}_{s,\text{ét}}, K_2)$. As the K_i are supported on isolated points, these cohomology groups are non-zero if and only if $m = 0$. In this case consider a double point $y \in X_s - X^\circ{}_s$, and denote by $\tilde{y}_1, \tilde{y}_2 \in \tilde{X}_s$ the two points over x. Using the formulas of [HK, 4.6] one computes that the cohomology of K_2 is concentrated in degree 0 and that of K_1 in degrees 0 and 1. One also sees that if σ fixes y then we have

$$\operatorname{tr}(\sigma | H^0(K_{2,\tilde{y}_1}) \oplus H^0(K_{2,\tilde{y}_2})) = 1 + \varepsilon_y,$$

$$\operatorname{tr}(\sigma | H^i(K_{1,y})) = \begin{cases} 1 & i = 0, \\ \varepsilon_y & i = 1. \end{cases}$$

Taking an alternating sum of traces on suitable cohomology groups gives (4.4.2) ∎

Of course we expect that Conjecture 3.10 also holds with crystalline cohomology.

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