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Abstract - We show the existence of fundamental solutions for $p$-adic pseudo-differential operators with polynomial symbols.

1. Introduction.

Let $K$ be a $p$-adic field, i.e. a finite extension of $\mathbb{Q}_p$, the field of $p$-adic numbers. Let $R_K$ be the valuation ring of $K$, $P_K$ the maximal ideal of $R_K$, and $\mathcal{K} = R_K / P_K$ the residue field of $K$. The cardinality of $\mathcal{K}$ is denoted by $q$. For $z \in K$, $\nu(z) \in \mathbb{Z} \cup \{ + \infty \}$ denotes the valuation of $z$, $|z|_K = q^{-\nu(z)}$ and $ac(z) = z\pi^{-\nu(z)}$ where $\pi$ is a fixed uniformizing parameter for $R_K$. Let $\Psi'$ denote an additive character of $K$ trivial on $R_K$ but not on $P_K^{-1}$.

A function $\Phi : K^n \to \mathbb{C}$ is called a Schwartz-Bruhat function if it is locally constant with compact support. We denote by $\mathcal{S}(K^n)$ the $C$-vector space of Schwartz-Bruhat functions over $K^n$. The dual space $\mathcal{S}'(K^n)$ is the space of distributions over $K^n$. Let $f = f(x) \in K[x]$, $x = (x_1, \ldots, x_n)$, be a non-zero polynomial, and $\beta$ a complex number satisfying $\text{Re}(\beta) > 0$. If $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in K^n$, we set $[x, y] := \sum_{i=1}^{n} x_i y_i$.

A $p$-adic pseudo-differential operator $f(\partial, \beta)$, with symbol $|f|_{\mathcal{K}}$, is an

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operator of the form
\[ f(\tilde{\partial}, \beta): \mathcal{S}(K^n) \rightarrow \mathcal{S}(K^n) \]
\[ \Phi \rightarrow \mathcal{F}^{-1}(|f|_K \mathcal{F}(\Phi)) \]
where
\[ \mathcal{F}: \mathcal{S}(K^n) \rightarrow \mathcal{S}(K^n) \]
\[ \Phi \rightarrow \int_{K^n} \Psi([-x, y]) \Phi(x) \, dx \]
is the Fourier transform. The operator \( f(\tilde{\partial}, \beta) \) has self-adjoint extension with dense domain in \( L^2(K^n) \). We associate to \( f(\tilde{\partial}, \beta) \) the following \( p \)-adic pseudo-differential equation:
\[ f(\tilde{\partial}, \beta) u = g, \quad g \in \mathcal{S}(K^n). \] (1.3)

A fundamental solution for (1.3) is a distribution \( E \) such that \( u = E * g \) is a solution.

The main result of this paper is the following.

**Theorem 1.1.** Every \( p \)-adic pseudo-differential equation \( f(\tilde{\partial}, \beta) u = g \), with \( f(x) \in K[x_1, \ldots, x_n] \backslash K \), \( g \in \mathcal{S}(K^n) \), and \( \beta \in \mathbb{C} \), \( \text{Re}(\beta) > 0 \), has a fundamental solution \( E \in \mathcal{S}'(K^n) \).

The \( p \)-adic pseudo-differential operators occur naturally in \( p \)-adic quantum field theory [11], [6]. Vladimirov showed the existence of a fundamental solutions for symbols of the form \( |\xi|^\alpha \), \( \alpha > 0 \) [10], [11]. In [7], [6] Kochubei showed explicitly the existence of fundamental solutions for operators with symbols of the form \( |f(\xi_1, \ldots, \xi_n)|_K^\alpha \), \( \alpha > 0 \), where \( f(\xi_1, \ldots, \xi_n) \) is a quadratic form satisfying \( f(\xi_1, \ldots, \xi_n) \neq 0 \) if \( |\xi_1|_K + \ldots + |\xi_n|_K \neq 0 \). In [8] Khrennikov considered spaces of functions and distributions defined outside the singularities of a symbol, in this situation he showed the existence of a fundamental solution for a \( p \)-adic pseudo-differential equation with symbol \( |f|_K \neq 0 \). The main result of this paper shows the existence of fundamental solutions for operators with polynomial symbols. Our proof is based on a solution of the division problem for \( p \)-adic distributions. This problem is solved by adapting the ideas developed by Atiyah for the archimedean case [1], and Igusa’s theorem on the meromorphic continuation of local zeta functions [3], [4]. The connection between local zeta functions (also called Igusa’s local ze-
ta functions) and fundamental solutions of $p$-adic pseudo-differential operators has been explicitly showed in particular cases by Jang and Sato [5], [9]. In [9] Sato studies the asymptotics of the Green function $G$ of the following pseudo-differential equation

$$(f(\partial, 1) + m^2) u = g, \quad m > 0.$$ (1.4)

The main result in [9, theorem 2.3] describes the asymptotics of $G(x)$ when the polynomial $f$ is a relative invariant of some prehomogeneous vector spaces (see e.g. [3, Chapter 6]). The key step is to establish a connection between the Green function $G(x)$ and the local zeta function attached to $f$.

All the above mentioned results suggest a deep connection between Igusa’s work on local zeta functions (see e.g. [3]) and $p$-adic pseudo-differential equations.

2. Local zeta functions and division of distributions.

The local zeta function associated to $f$ is the distribution

$$(|f|^s_K, \Phi) = \int_{K^n} \Phi(x) |f(x)|_K^s dx,$$ (2.1)

where $\Phi \in S(K^n)$, $s \in \mathbb{C}$, $\text{Re}(s) > 0$, and $dz$ is the Haar of $K^n$ normalized so that $\text{vol}(R^n_K) = 1$. The local zeta functions were introduced by Weil [12] and their basic properties for general $f$ were first studied by Igusa [3], [4]. A central result in the theory of local zeta functions is the following.

\textbf{Theorem 2.1 (Igusa, [3, Theorem 8.2.1]).} The distribution $|f|^s_K$ admits a meromorphic continuation to the complex plane such that $(|f|^s_K, \Phi)$ is a rational function of $q^{-s}$ for each $\Phi \in S(K^n)$. In addition the real parts of the poles of $|f|^s_K$ are negative rational numbers.

The archimedean counterpart of the previous theorem was obtained jointly by Bernstein and Gelfand [2], independently by Atiyah [1]. The following lemma is a consequence of the previous theorem.

\textbf{Lemma 2.1.} Let $f(x) \in K[x_1, \ldots, x_n]$ be a non-constant polynomial, and $\beta$ a complex number satisfying $\text{Re}(\beta) > 0$. Then there exists a distribution $T \in S'(K^n)$ satisfying $|f|^\beta_K T = 1$. 

PROOF. By theorem 2.1 $|f|_k$ has a meromorphic continuation to $C$ such that $\langle |f|_k, \Phi \rangle$ is a rational function of $q^{-s}$ for each $\Phi \in \mathcal{S}(K^n)$. Let

$$|f|_k = \sum_{m \leq \zeta} c_m (s + \beta)^m$$

be the Laurent expansion at $-\beta$ with $c_m \in \mathcal{S}(K^n)$ for all $m$. Since the real parts of the poles of $|f|_k$ are negative rational numbers by theorem 2.1, it holds that $|f|_k^\beta = c_0 |f|_k + \sum_{m=1}^\infty c_m |f|_k^\beta (s + \beta)^m$.

By using the Lebesgue lemma and (2.3) it holds that

$$\lim_{s \to -\beta} \langle |f|_k^\beta, \Phi \rangle = \int_{K^*} \Phi(x) \, dx = \langle 1, \Phi \rangle$$

Therefore we can take $T = c_0$. $\blacksquare$

If $T \in \mathcal{S}'(K^n)$ we denote by $\mathcal{F}T \in \mathcal{S}'(K^n)$ the Fourier transform of the distribution $T$, i.e. $\langle \mathcal{F}T, \Phi \rangle = \langle S, \mathcal{F}(\Phi) \rangle$, $\Phi \in \mathcal{S}(K^n)$.

3. Proof of the main result.

By lemma 2.1 there exists a $T \in \mathcal{S}'(K^n)$ such that $|f|_k^\beta T = 1$. We set $E = \mathcal{F}^{-1} T \in \mathcal{S}'(K^n)$ and assert that $E$ is a fundamental solution for (1.3). This last statement is equivalent to assert that $\mathcal{F}(\Phi) = \langle \mathcal{F}E, \mathcal{F}(g) \rangle$ satisfies $|f|_k^\beta \mathcal{F}(\Phi) = \mathcal{F}(g)$. Since $|f|_k^\beta \mathcal{F}(\Phi) = |f|_k^\beta (\mathcal{F}E) \mathcal{F}(g) = |f|_k^\beta T \mathcal{F}(g) = = \mathcal{F}(g)$, we have that $E$ is a fundamental solution for (1.3).

4. Operators with twisted symbols.

Let $\chi : R_k^x \to \mathbb{C}$ be a non-trivial multiplicative character, i.e. a homomorphism with finite image, where $R_k^x$ is the group of units of $K$. We put formally $\chi(0) = 0$. If $f(x) \in K[x_1, \ldots, x_n] \setminus K$, we say that $\chi(\text{act}(f)) |f|_k^\beta$, with $\beta \in \mathbb{C}$, $\Re(\beta) > 0$, is a twisted symbol, and call the
pseudo-differential operator

\[(4.1) \ \Phi \to f(\tilde{\sigma}, \beta, \chi) \Phi = \mathcal{F}^{-1} (\chi(ac(f)) |f|^b_k \mathcal{F}(\Phi)), \quad \Phi \in \mathcal{S}(K^n), \]

a twisted operator. Since the distribution \(\chi(ac(f)) |f|^b_k\) satisfies all the properties stated in theorem 2.1 (cf. [3, Theorem 8.2.1]), theorem 1.1 generalizes literally to the case of twisted operators. In [6, chapter 2] Kochubei showed explicitly the existence of fundamental solutions for twisted operators in some particular cases.

REFERENCES


Manoscritto pervenuto in redazione il 2 dicembre 2002.