

Painlevé's Theorem Extended.

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ABSTRACT - We extend Painlevé's determinateness theorem from the theory of ordinary differential equations in the complex domain allowing more general «multiple-valued» Cauchy's problems. We study C^0 -continuability (near singularities) of solutions.

Foreword and preliminaries.

In this paper we slightly improve Painlevé's determinateness theorem (see [HIL], th. 3.3.1), investigating the C^0 -continuability of the solutions of finitely «multiple-valued», meromorphic Cauchy's problems. In particular, we shall be interested in phenomena taking place when the attempt of continuing a solution along an arc leads to singularities of the known terms: we shall see that, under not too restricting hypotheses, this process will converge to a limit. Of course we shall formalize «multiple valuedness» by means of Riemann domains over regions in \mathbb{C}^2 (see also [GRO], p. 43 ff). Branch points will be supposed to lie on algebraic curves. We recall that in the classical statement of the theorem «multiple valuedness» in the known term is allowed with respect to the independent variable only.

The following theorem extends to the complex domain the so called «single-sequence criterion» from the theory of real o.d.e.'s (see e.g. [GIU], th. 3.2); a technical lemma ends the section; the local existence-and-uniqueness theorem is reported in the appendix.

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THEOREM 1. *Let W be a \mathbb{C}^N -valued holomorphic mapping, solution of the equation $W'(z) = F(W(z), z)$ in $\mathcal{V} \subset \mathbb{C}$, where F is a \mathbb{C}^N -valued holomorphic mapping in a neighbourhood of $\text{graph}(W)$. Let $z_\infty \in \partial \mathcal{V}$, suppose that there exists a sequence $\{z_n\} \rightarrow z_\infty$, such that, set $W(z_n) := W_n$, $\lim_{n \rightarrow \infty} W_n = W_\infty \in \mathbb{C}^N$ and that F is holomorphic at (W_∞, z_∞) : then W admits analytical continuation up to z_∞ .*

PROOF. We deal only with the case $N = 1$: we can find $a > 0$ and $b > 0$ such that the Taylor's developments $\sum_{k, l=0}^{\infty} c_{kln} (W - W_n)^k (z - z_n)^l$ at (W_n, z_n) of F are absolutely and uniformly convergent in all closed bidiscs $\overline{\mathbb{D}((W_n, z_n), a, b)}$. By means of Cauchy's estimates we can find an upper bound T for $\sum_{k, l=0}^{\infty} |c_{kln}| a^k b^l$; by theorem 2.5.1 of [HIL] the solutions S_n of $W' = F(W, z)$, $W(z_n) = W_n$ have radius of convergence at least $a(1 - e^{-b/2aT}) := \sigma$. Thus there exists M such that $z_\infty \in \mathbb{D}(z_M, \sigma)$; by continuity, $S_M(z_\infty) = W_0$, and, by uniqueness, $S_M = S_\infty$ in $\mathbb{D}(z_M, \sigma) \cap \mathbb{D}(z_\infty, \sigma)$, i.e. W admits analytical continuation up to z_∞ . ■

LEMMA 2. *Let X be a metric space, $\gamma : [a, b) \rightarrow X$ a continuous arc and suppose that there does not exist $\lim_{t \rightarrow b} \gamma(t)$: then, for every N -tuple $\{x_1 \dots x_N\} \subset X$ there exists a sequence $\{t_i\} \rightarrow b$ and neighbourhoods U_k of x_k such that $\{\gamma(t_i)\} \subset X \setminus \bigcup_{k=1}^N U_k$. ■*

The main theorem.

We recall that a *Riemann domain* over a region $\mathcal{U} \subset \mathbb{C}^N$ is a complex manifold Δ with an everywhere maximum-rank holomorphic surjective mapping $p : \Delta \rightarrow \mathcal{U}$; Δ is *proper* provided that so is p (see [GRO] p. 43); we also recall that $\mathfrak{C} \subset \mathbb{C}^N$ is *algebraic* if it is the common zero set of K polynomial functions on \mathbb{C}^N , with $0 < K < N$.

Let now \mathcal{N} be a curve in $\mathbb{C}_{(u, v)}^2$, (Δ, p) a proper Riemann domain over $\mathbb{C}^2 \setminus \mathcal{N}$, F a meromorphic function on Δ , holomorphic outside a curve \mathcal{N} , $X_0 \in \Delta \setminus \mathcal{N}$, $(u_0, v_0) = p(X_0)$ and η a local inverse of p , defined in a bidisc $\mathbb{D}_1 \times \mathbb{D}_2$ around (u_0, v_0) . Let $u : \mathbb{D}(v_0, r) \rightarrow \mathbb{D}_1$ (with r like in the existence-and-uniqueness theorem in the appendix) be the solution of the Cauchy's problem: $u'(v) = F \circ \eta(u(v), v)$, $u(v_0) = u_0$ and $\mathfrak{C} = \mathcal{N} \cup \cup p(\mathcal{N})$.

THEOREM 3 (Painlevé's determinateness theorem). *Suppose that \mathcal{A} is algebraic; let $\gamma : [0, 1] \rightarrow \mathbb{C}$ be an embedded C^1 arc starting at v_0 such that, for each $t \in [0, 1]$, the complex line $v = \gamma(t)$ is not contained in \mathcal{A} ; suppose that an analytical continuation ω of u may be got along $\gamma|_{[0, 1]}$: then there exists $\lim_{t \rightarrow 1} \omega \circ \gamma(t)$, within \mathbb{P}^1 .*

PROOF. Suppose, on the contrary, that such limit does not exist: for every $v \in C_v$, set $W_v = pr_1(\mathcal{A} \cap (C \times \{v\}))$: by hypothesis W_{v_1} is finite or empty. The former case is trivial; as to the latter, say, $W_{v_1} = \{\lambda_k\}_{k=1 \dots q}$. For each $k = 1 \dots q$ and every $\varepsilon > 0$, set $D_{k\varepsilon} = \mathbb{D}(\lambda_k, \varepsilon)$, $T_{k\varepsilon} = \partial(\mathbb{D}_{k\varepsilon})$; then there exists $\varrho_\varepsilon > 0$ such that $v \in \overline{\mathbb{D}(v_1, \varrho_\varepsilon)} \Rightarrow W_v \subset \bigcup_{k=1}^q D_{k\varepsilon}$: set now

$$\left\{ \begin{array}{l} M_\varepsilon = \max_{X \in p^{-1}(\bigcup_{k=1}^q T_k \times \overline{\mathbb{D}(v_1, \varrho_\varepsilon)})} |F(X)| \\ M_{R\varepsilon} = \max_{X \in p^{-1}(\partial(\mathbb{D}(0, R)) \times \overline{\mathbb{D}(v_1, \varrho_\varepsilon)})} |F(X)| \quad \text{for each } R > 0; \end{array} \right.$$

Introduce the compact set $\Theta_{R\varepsilon} = \overline{\mathbb{D}(O, R)} \setminus \bigcup_{k=1}^q D_{k\varepsilon}$: for every $v \in \overline{\mathbb{D}(v_1, \varrho_\varepsilon)}$, $p^{-1}(C_u \times \{v\})$ is a Riemann surface, hence, by maximum principle, and by the arbitrariness of v in $\overline{\mathbb{D}(v_1, \varrho_\varepsilon)}$,

$$X \in p^{-1}(\Theta_{R\varepsilon} \times \overline{\mathbb{D}(v_1, \varrho_\varepsilon)}) \Rightarrow |F(X)| \leq \max(M_\varepsilon, M_{R\varepsilon}).$$

Now we have assumed that $\omega \circ \gamma(t)$ does not admit limit as $t \rightarrow 1$, hence, by lemma 2 (with $X = \mathbb{P}^1$, $\{x_k\} = \{\lambda_k\} \cup \{\infty\}$), there exist: a sequence $\{t_i\} \rightarrow 1$, ε small enough and R large enough such that $\{\omega(\{\gamma(t_i)\})\} \subset \Theta_{R\varepsilon}$. Without loss of generality, we may suppose that $\{\gamma(t_i)\} \subset \mathbb{D}(v_1, \varrho_\varepsilon)$. Since p is proper, $p^{-1}(\Theta_u \times \overline{\mathbb{D}(v_1, \varrho)})$ is compact, hence we can extract a convergent subsequence Ω_k from $p^{-1}\{\omega(\gamma(t_i)), \gamma(t_i)\}$, whose limit we shall call Ω . By hypothesis there exists a holomorphic function element $(\mathcal{V}, \tilde{\omega})$ such that $\mathcal{V} \supset \gamma([0, 1])$ and $\tilde{\omega}(v_0) = u(v_0)$; moreover, $F \circ \eta$ could be analytically continued across $\omega \circ \gamma \times \gamma|_{[0, 1]}$, since $\tilde{\omega}'$ is finite at each point of $\gamma([0, 1])$; by construction, F is holomorphic at Ω and $rk(p_*(\Omega)) = 2$, hence $F \circ \eta$ admits analytical continuation up to Ω . Therefore, by theorem 1, $\tilde{\omega}$ admits analytical continuation up to v_1 , hence there exists $\lim_{t \rightarrow 1} \omega \circ \gamma(t) = \lim_{t \rightarrow 1} \tilde{\omega} \circ \gamma(t)$, which is a contradiction. ■

A simple example.

Consider $w' = -\sqrt[4]{8} \sqrt{3z + w^2} / 4 \sqrt[4]{(z + w^2)^3}$, $w(1) = 1$, where the branches of the roots are those ones which take positive values on the

positive real axis (this is in fact the choice of η); the problem is solved by $w(z) = \sqrt{z}$, which admits analytical continuation, for example, on the backwards oriented semi-closed interval $[1, 0)$: note that nor $3z + w^2 = 0$ nor $z + w^2 = 0$ contains any complex line $z = \text{const}$; hence, as expected, $\lim_{t \rightarrow 0^+, t \in \mathbb{R}} \sqrt{t}$ exists, being in fact 0.

REMARK. The algebraicity assumption about \mathcal{C} in the main theorem could be weakened (we thank the referee for pointing out this argument): indeed it would be enough to suppose that, for every $\nu \in \mathbb{C}_\nu$, $pr_1(\mathcal{C} \cap (\mathbb{C} \times \{\nu\}))$ is finite.

Appendix: the existence-and-uniqueness theorem.

Let W_0 be a complex N -tuple, $z_0 \in \mathbb{C}$; let F be a \mathbb{C}^N -valued holomorphic mapping in $\prod_{j=1}^N D(W_0^j, b) \times D(z_0, a)$, ($a, b \in \mathbb{R}$) with C^0 -norm M and C^0 -norm of each $\partial F / \partial w^j$ ($j = 1 \dots N$) not exceeding $K \in \mathbb{R}$.

THEOREM. If $r < \min(a, b/M, 1/K)$, there exists a unique holomorphic mapping $W : D(z_0, r) \rightarrow \prod_{j=1}^N D(W_0^j, b)$ such that $W' = F(W(z), z)$ and $W(z_0) = W_0$. (See e.g. [HIL], th 2.2.2, [INC] p. 281-284).

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Manoscritto pervenuto in redazione il 16 maggio 2002.