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A Global Existence Result in Sobolev Spaces for MHD System in the Half-Plane.

EMANUELA CASELLA (*) - PAOLA TREBESCHI (*)

ABSTRACT - The main result of this paper is a global existence theorem in suitable Sobolev spaces for 2D incompressible MHD system in the half-plane. The existence result derives by the existence of a global classical solution in Hölder spaces, by proving some a-priori estimates in Sobolev spaces and, finally, by applying the Banach-Caccioppoli fixed point theorem. Hence, the uniqueness of the solution follows.

1. Introduction.

Let Ω be the half-plane $\mathbb{R}^2_+ := \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 > 0\}$, and let Γ be the boundary of Ω. In $Q_T := \Omega \times (0, T)$, with $T > 0$, we consider the equations of magneto-hydrodynamics for 2D incompressible ideal fluid

$$u_t + (u \cdot \nabla) u + \nabla \pi + \frac{1}{2} \nabla |B|^2 - (B \cdot \nabla) B = 0 \quad \text{in} \quad Q_T, \tag{1}$$

$$B_t + (u \cdot \nabla) B - (B \cdot \nabla) u - \mu \Delta B = 0 \quad \text{in} \quad Q_T, \tag{2}$$

$$\operatorname{div} u = 0 \quad \text{in} \quad Q_T, \tag{3}$$

$$\operatorname{div} B = 0 \quad \text{in} \quad Q_T, \tag{4}$$

(*) Indirizzo degli AA.: Dip. di Matematica, Università di Brescia, Facoltà di Ingegneria, Via Valotti 9, 25133 Brescia.

(5) $$u\cdot\nu = 0 \text{ on } \Gamma\times(0, T),$$

(6) $$B\cdot\nu = 0 \text{ on } \Gamma\times(0, T),$$

(7) $$\operatorname{rot} B = 0 \text{ on } \Gamma\times(0, T),$$

(8) $$u(x, 0) = u_0(x) \text{ in } \Omega,$$

(9) $$B(x, 0) = B_0(x) \text{ in } \Omega.$$

Here $u = u(x, t) = (u^1(x, t), u^2(x, t))$, $B = B(x, t) = (B^1(x, t), B^2(x, t))$ and $\pi = \pi(x, t)$ denote the unknown velocity field, the magnetic field and the pressure of the fluid respectively. The functions $u_0 = (u_0^1(x), u_0^2(x))$ and $B_0 = (B_0^1(x), B_0^2(x))$ denote the given initial data, ν the unit outward normal on Γ and μ a real positive constant. Moreover, we use the notation

$$f_t = \frac{\partial f}{\partial t}, \quad \partial_i = \frac{\partial}{\partial x_i}, \quad \nabla = (\partial_1, \partial_2), \quad u\cdot\nabla = u^1\partial_1 + u^2\partial_2,$$

$$\partial_{ij}^2 = \frac{\partial^2}{\partial_i\,\partial_j}, \quad \Delta = \partial_{11}^2 + \partial_{22}^2.$$

In case the magnetic field B is identically equal to zero, i.e. in the case of Euler equations, such a problem for global *classical* solutions was studied by many authors, starting from Lichtenstein [10] and Wolibner [15]. The existence of global solutions in Hölder spaces in bounded domains has been proven by Kato [6]. This result was extended to the exterior domain case by Kikuchi [8]. On the other hand, the existence of a classical solution for MHD system was shown by Kozono [9] and by Casella, Secchi and Trebeschi [5] in the bounded and unbounded case, respectively.

Existence results in Sobolev spaces were proved by several authors. For the Euler equation we refer to Temam [14], Kato and Lai [7] and Beirão Da Veiga [3], [4]. Existence and uniqueness results in W^k-spaces for the equations of magneto-hydrodynamics, when $\mu = 0$, have been proved by Alexseev [1]. Moreover, in this case, Secchi [12] and Schmidt [11] proved not only existence and uniqueness results, but also the continuous dependence on the data. In this paper we prove a global existence result in suitable Sobolev spaces for MHD system in the half-plane case. To prove this result, firstly, we show a local existence theorem in Sobolev spaces. Then we derive some a-priori estimates, global in time, which come from the all-time existence of classical solution of system (1)-

(9) in Hölder-spaces. We underline that energy-method works well, since the classical solution (u, B) is such that $\|u(t)\|_{L^\infty(\Omega)}$, $\|B(t)\|_{L^\infty(\Omega)}$, $\|\nabla u(t)\|_{L^\infty(\Omega)}$, $\|\nabla B(t)\|_{L^\infty(\Omega)}$ and $\|B_t(t)\|_{L^\infty(\Omega)}$ are uniformly bounded in time on the whole interval $[0, T]$.

We observe that the main result obtained in the present paper is a necessary first step in the analysis of slightly compressible MHD fluids, which will be the object of a forecoming work.

The plan of the paper is the following. In next section we fix some notations and we introduce some preliminary results and the main theorem. In Section 3 we show some a-priori estimates, and finally in Section 4 we prove the main result.

2. – Notations and results.

For a scalar-valued function ϕ, we set

$$\mathrm{Rot}\,\phi = (\partial_2 \phi,\ -\partial_1 \phi),$$

for a vector-valued function $u = (u^1, u^2)$, we use the notation

$$\mathrm{rot}\,u = \partial_1 u^2 - \partial_2 u^1 \quad \text{and} \quad \mathrm{div}\,u = \nabla \cdot u = \partial_1 u^1 + \partial_2 u^2.$$

We denote the norm of $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, by $\|\cdot\|_{L^p}$. $H^m(\Omega)$ denotes the usual Sobolev space of order $m \geqslant 1$, and $\|\cdot\|_{H^m}$ denotes its norm. For simplicity we use the abbreviated notation L^p, H^m. We also use the same symbol for spaces of scalar and vector-valued functions.

Moreover, if X is a normed space, then $L^p(0, T; X)$, with $1 \leqslant p < +\infty$, denotes the set of all measurable functions $u(t)$ with values in X such that:

$$\|u\|_{L^p(X)} := \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{1/p} < +\infty,$$

where $\|\cdot\|_X$ is the norm in X.

Given $T > 0$ arbitrary, the set of all essentially bounded (with respect to the norm of X) measurable functions of t with values in X is denoted by $L^\infty(0, T; X)$. We equip this space with the usual norm

$$\|f\|_{L^\infty(X)} = \sup_{t \in [0, T]} \|f(t)\|_X.$$

In particular, the norm of $L^\infty(0, T; L^p)$, $1 \leqslant p < +\infty$, is denoted by $\|\cdot\|_{L^\infty(L^p)}$.

Let $\mathcal{C}^m([0, T]; X)$ denote the set of all X-valued m-times continuously differentiable functions of t, for $0 \leqslant t \leqslant T$.

We define $X^m(T) := \bigcap_{k=0}^{m-1} \mathcal{C}^k([0, T]; H^{m-k})$ equipped with the usual norm

$$\|u\|_{X^m}^2 := \sup_{[0, T]} \sum_{k=0}^{m-1} \|\partial_t^k u(t)\|_{H^{m-k}}^2.$$

We denote by $\mathcal{B}(\overline{\Omega})$ (resp. $\mathcal{B}(\overline{Q}_T)$) the Banach space of all real valued continuous and bounded functions on $\overline{\Omega}$ (resp. $\overline{Q}_T$), with the usual norm.

For $0 < \alpha < 1$, $C^\alpha(\overline{\Omega})$ denotes the usual space of functions in $\mathcal{B}(\overline{\Omega})$, uniformly Hölder continuous on $\overline{\Omega}$ with exponent α; the norm of $C^\alpha(\overline{\Omega})$ is $\|\cdot\|_{L^\infty} + [\cdot]_\alpha$, where

$$[\phi]_\alpha := \sup_{x \neq y,\, x,\, y \in \overline{\Omega}} \frac{|\phi(x) - \phi(y)|}{|x - y|^\alpha}.$$

For $0 < \alpha < 1$ and integer k, $C^{k+\alpha}(\overline{\Omega})$ denotes the space of functions ϕ with $D^\beta \phi \in \mathcal{B}(\overline{\Omega})$ for $|\beta| \leqslant k$, and $D^\gamma \phi \in C^\alpha(\overline{\Omega})$ for $|\gamma| = k$. The norm is

$$|\phi|_{k+\alpha} = \max_{|\beta| \leqslant k} \|D^\beta \phi\|_{L^\infty} + \max_{|\gamma| = k} [D^\gamma \phi]_\alpha.$$

With $\mathcal{C}^{k,j}(\overline{Q}_T)$ for integers $k, j \geqslant 0$ we mean the set of all functions ϕ for which every $\partial_x^q \partial_t^r \phi$ exists and is continuous on $\overline{Q}_T$, for $0 \leqslant |q| \leqslant k$, $0 \leqslant r \leqslant j$. $\mathcal{C}^{k+\alpha, j+\beta}(\overline{Q}_T)$, for integers $k, j \geqslant 0$ and $0 \leqslant \alpha, \beta < 1$ is the subset of $\mathcal{C}^{k,j}(\overline{Q}_T)$, consisting of Hölder continuous functions with exponents α in x and β in t.

For every function $\phi \in \mathcal{C}^{k+\alpha, j+\beta}(\overline{Q}_T)$, we consider the following seminorm:

$$[\phi]_{\alpha, \beta} := \sup_{x \neq y,\, t \in [0, T]} \frac{|\phi(x, t) - \phi(y, t)|}{|x - y|^\alpha} + \sup_{t \neq s,\, x \in \overline{\Omega}} \frac{|\phi(x, t) - \phi(x, s)|}{|t - s|^\beta},$$

and the norm

$$|\phi|_{k+\alpha, j+\beta} := \max_{|q| \leqslant k,\, r \leqslant j} \sup_{(x, t) \in \overline{Q}_T} |\partial_x^q \partial_t^r \phi(x, t)| + \max_{|q| = k} [\partial_x^q \partial_t^j \phi]_{\alpha, \beta}.$$

We shall denote by C and by C_i, $i \in \mathbb{N}$, some real positive constants which may be different in each occurrence, and by $C_\infty(t)$ a real function in $L^\infty(0, T)$ depending on $\|u(t)\|_{L^\infty}$, $\|B(t)\|_{L^\infty}$, $\|\nabla u(t)\|_{L^\infty}$, $\|\nabla B(t)\|_{L^\infty}$, $\|B_t(t)\|_{L^\infty}$ and some their suitable powers.

We now set $Z := \operatorname{rot} u$, $\xi := \operatorname{rot} B$, $Z_0 := \operatorname{rot} u_0$ and $\xi_0 := \operatorname{rot} B_0$. By applying rot to both sides of equations (1) and (2), we get

$$Z_t + u \cdot \nabla Z - B \cdot \nabla \xi = 0\,, \tag{10}$$

$$\xi_t + u \cdot \nabla \xi - B \cdot \nabla Z + 2\partial_1 u^1 \mathbb{D}B + 2\partial_2 B^2 \mathbb{D}u - \mu \Delta \xi = 0\,, \tag{11}$$

where $\mathbb{D}u = \partial_1 u^2 + \partial_2 u^1$, and $\mathbb{D}B = \partial_1 B^2 + \partial_2 B^1$. Finally, let F, ϕ, $\mathcal{F}$, ψ be defined as

$$F = -u \cdot \nabla \xi + B \cdot \nabla Z - 2\partial_1 u^1 \mathbb{D}B - 2\partial_2 B^2 \mathbb{D}u,$$

$$\phi(s) := \sum_{k=0}^{3} \|\partial_t^k \xi(s)\|_{H^1}^2,$$

$$\mathcal{F}(s) := \sum_{k=0}^{3} \|\partial_t^k F(s)\|_{H^{3-k}}^2,$$

$$\psi(s) := \sum_{k=0}^{3} \|\partial_t^k \xi(s)\|_{H^{5-k}}^2.$$

We now recall a result (see [5]) which will be fundamental to prove that, under suitable assumptions on initial data, the classical solution of problem (1)-(9) belongs to suitable Sobolev spaces.

THEOREM 2.1. *Let $T > 0$ be arbitrary. Let $u_0 \in \mathcal{C}^{1+\theta}(\overline{\Omega})$, $\operatorname{rot} u_0 \in L^1(\Omega)$, $B_0 \in \mathcal{C}^{2+\theta}(\overline{\Omega}) \cap H^1(\Omega)$ for some $0 < \theta < 1$, such that $\operatorname{div} u_0 = \operatorname{div} B_0 = 0$ in Ω and $u_0 \cdot \nu = B_0 \cdot \nu = 0$ on Γ. Then there exists a positive constant C_* such that, if $\|B_0\|_{H^1} + |B_0|_{2+\theta} \leq C_*$, then there exists a solution $\{u, B, \pi\} \in \mathcal{C}^{1,1}(\overline{Q}_T) \times \mathcal{C}^{2,1}(\overline{Q}_T) \times \mathcal{C}^{1,0}(\overline{Q}_T)$ of system (1)-(9). Such a solution is unique up to an arbitrary function of t which may be added to π.*

REMARK 2.2. In [5] Kozono's result obtained in [9] and Kikuchi's result, see [8], are extended to the exterior domain case and to the half-plane case, and to the MHD equations, respectively. In [5] the existence of the global classical solution for MHD system in Hölder spaces is proved by applying the Schauder fixed point theorem. The authors followed the idea of Kato [6], Kikuchi [8], Kozono [9]. The crucial step is the definition of a map, defined on a suitable class, the same already considered by Kozono in [9], which satisfies the conditions of the Schauder fixed point theorem. The uniqueness of the solutions of the studied problem is obtained by following standard tecniques, see Temam [13].

The main result, we are going to prove, is:

THEOREM 2.3. *Let $T > 0$ be arbitrary. Let the couple $(u_0, B_0) \in \in H^5 \cap L^1$, $\mathrm{rot}\, u_0 \in L^1$, $\mathrm{div}\, u_0 = \mathrm{div}\, B_0 = 0$ in Ω and $u_0 \cdot \nu = B_0 \cdot \nu = 0$ on Γ. Assume also that, for some $0 < \theta < 1$,*

$$\|B_0\|_{H^1} + |B_0|_{2+\theta} \leq C_*,$$

where C_* is the constant obtained in Theorem 2.1.

Then problem (1)-(9) has a unique solution (u, B, π) such that

$$u \in X^5(T), \quad B \in X^5(T) \cap L^2(0, T; H^6), \quad \nabla\pi \in X^4(T).$$

3. Some a-priori estimates.

We devote this section to prove

LEMMA 3.1. *The following energy-type estimate*

$$\frac{1}{2}\frac{d}{dt}(\phi(t) + \|Z(t)\|^2_{H^4}) + C_1\psi(t) \leq C_\infty(t)(\phi(t) + \|Z(t)\|^2_{H^4}) \tag{12}$$

holds in $[0, T]$.

Note that Theorem 2.1 ensures us that the time functions $\|u(t)\|_{L^\infty(\Omega)}$, $\|B(t)\|_{L^\infty(\Omega)}$, $\|\nabla u(t)\|_{L^\infty(\Omega)}$, $\|\nabla B(t)\|_{L^\infty(\Omega)}$, and $\|B_t(t)\|_{L^\infty(\Omega)}$ are uniformly bounded in time on the whole interval $[0, T]$. Consequently, the real function $C_\infty(t)$ (appearing in (12) and in some preliminary lemmata given below) belongs to $L^\infty(0, T)$. We shall prove (12) for regular solutions.

LEMMA 3.2. *The couple (u, B) satisfies the following energy-type estimate*

$$\|u\|^2_{L^\infty(L^2)} + \|B\|^2_{L^\infty(L^2)} + 2\mu\|\xi\|^2_{L^2(Q_T)} \leq \|u_0\|^2_{L^2} + \|B_0\|^2_{L^2}. \tag{13}$$

PROOF. We multiply equations (1) and (2) by u and B respectively. By standard calculations and by summing the resulting expressions, we get easily the thesis. ■

LEMMA 3.3. *The following inequality holds*

$$\|Z\|^2_{L^\infty(L^2)} + \|\xi\|^2_{L^\infty(L^2)} + \frac{\mu}{2}\|\nabla\xi\|^2_{L^2(Q_T)} \leq C(\|Z_0\|^2_{L^2} + \|\xi_0\|^2_{L^2}).$$

PROOF. We multiply equations (10) and (11) by Z and ξ respectively. Since

$$-\int_\Omega (B\cdot\nabla)\,\xi\cdot Z\,dx = \int_\Omega (B\cdot\nabla)\,Z\cdot\xi\,dx,$$

by summing the resulting expressions, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|Z(t)\|_{L^2}^2 + \|\xi(t)\|_{L^2}^2) + 2\int_\Omega (\partial_1 u_1\,\mathbb{D}B + \partial_2 B_2\,\mathbb{D}u)\,\xi\,dx + \mu\|\nabla\xi(t)\|_{L^2}^2 = 0. \tag{14}$$

Since $\|\nabla u(t)\|_{L^2} \leqslant C\|Z(t)\|_{L^2}$ and $\|\nabla B(t)\|_{L^4} \leqslant C\|\xi(t)\|_{L^2}^{1/2}\|\nabla\xi(t)\|_{L^2}^{1/2}$, we easily obtain that

$$2\int_\Omega |(\partial_1 u_1\,\mathbb{D}B + \partial_2 B_2\,\mathbb{D}u)\,\xi|\,dx \leqslant \frac{\mu}{2}\|\nabla\xi(t)\|_{L^2}^2 + C\|Z(t)\|_{L^2}^2\|\xi(t)\|_{L^2}^2. \tag{15}$$

By collecting (14) and (15), we get

$$\frac{1}{2}\frac{d}{dt}(\|Z(t)\|_{L^2}^2 + \|\xi(t)\|_{L^2}^2) + \frac{\mu}{2}\|\nabla\xi(t)\|_{L^2}^2 \leqslant C\|\xi(t)\|_{L^2}^2(\|Z(t)\|_{L^2}^2 + \|\xi(t)\|_{L^2}^2).$$

The thesis follows by using Lemma 3.2 and the Gronwall lemma. ■

The next step is to estimate the $L^\infty(0;T;L^2)$-norm of $\partial^\alpha Z$, where α is a multi-index such that $1 \leqslant |\alpha| \leqslant 4$. We get the following result.

LEMMA 3.4. *Let $\varepsilon > 0$. Then the following inequality*

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha Z(t)\|_{L^2}^2 \leqslant C_\infty(t)\|\partial^\alpha Z(t)\|_{L^2}^2 + \varepsilon\|\xi(t)\|_{H^5}^2,$$

holds in $[0, T]$, where C_∞ depends also on ε.

PROOF. By applying ∂^α to both sides of equation (10), we get

$$\partial^\alpha Z_t + (u\cdot\nabla)\,\partial^\alpha Z = -[\partial^\alpha, u\cdot\nabla]\,Z + \partial^\alpha((B\cdot\nabla)\,\xi), \tag{16}$$

where $[\cdot,\cdot]$ denotes the commutator operator. We now multiply (16) by $\partial^\alpha Z$ and we estimate term by term. We use the Hölder and Young inequalities and some suitable interpolation inequalities (obtained by the well-known Gagliardo-Nirenberg one). More precisely,

$$\|D^2 u\|_{L^4} \leqslant C \|\nabla u\|_{L^\infty}^{1/2} \|Z\|_{H^2}^{1/2}, \tag{17}$$

$$\|D^2 B\|_{L^4} \leqslant C \|\nabla B\|_{L^\infty}^{1/2} \|\xi\|_{H^2}^{1/2}, \tag{18}$$

$$\|D^3 u\|_{L^4} \leqslant C \|\nabla u\|_{L^\infty}^{1/2} \|Z\|_{H^4}^{1/2}, \tag{19}$$

$$\|D^3 B\|_{L^4} \leqslant C \|\nabla B\|_{L^\infty}^{1/2} \|\xi\|_{H^4}^{1/2}, \tag{20}$$

$$\|D^4 u\|_{L^4} \leqslant C \|\nabla u\|_{L^\infty}^{1/6} \|Z\|_{H^4}^{5/6}, \tag{21}$$

$$\|D^4 B\|_{L^4} \leqslant C \|\nabla B\|_{L^\infty}^{1/6} \|\xi\|_{H^4}^{5/6}, \tag{22}$$

$$\|D^2 B_t\|_{L^4} \leqslant C \|B_t\|_{L^\infty}^{1/2} \|\xi_t\|_{H^3}^{1/2}. \tag{23}$$

By using (17)-(23), we easily obtain the thesis. ■

LEMMA 3.5. *The following estimate*

$$\frac{1}{2}\frac{d}{dt}\phi(t) + C_1 \psi(t) \leqslant C_2(\mathcal{F}(t) + \phi(t))$$

holds in $[0, T]$.

PROOF. We write (11) in the form $\partial_t \xi - \mu \Delta \xi = F$. For each integer $k = 1, \ldots, 4$, we take $(k-1)$ time derivatives and we obtain the following problems

$$\begin{cases} \partial_t^k \xi - \mu \Delta \partial_t^{k-1} \xi = \partial_t^{k-1} F \text{ in } \Omega, \\ \partial_t^{k-1} \xi = 0 \text{ on } \partial\Omega. \end{cases} \tag{24}$$

For each fixed k, we multiply the first equation of (24) by $\partial_t^{k-1}\xi$ and by $-\Delta \partial_t^{k-1}\xi$. By using the Hölder inequality one has

$$\frac{1}{2}\frac{d}{dt}\|\partial_t^{k-1}\xi(t)\|_{H^1}^2 + \frac{\mu}{2}\left(\|\nabla \partial_t^{k-1}\xi(t)\|_{L^2}^2 + \|\Delta \partial_t^{k-1}\xi(t)\|_{L^2}^2\right) \leqslant$$
$$\leqslant C\left(\|\partial_t^{k-1} F(t)\|_{L^2}^2 + \|\partial_t^{k-1}\xi(t)\|_{L^2}^2\right). \tag{25}$$

We now write (24) in the form of the elliptic problem

$$\begin{cases} -\mu \Delta \partial_t^{k-1} \xi = -\partial_t^k \xi + \partial_t^{k-1} F \quad \text{in } \Omega, \\ \partial_t^{k-1} \xi = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{26}$$

By well-known results on the regularity of the solutions of problems

(26), we get

$$\|\partial_t^{k-1}\xi(t)\|_{H^{m+2}} \leqslant C(\|\partial_t^k \xi(t)\|_{H^m} + \|\partial_t^{k-1}\xi(t)\|_{H^m} + \|\partial_t^{k-1}F(t)\|_{H^m}). \tag{27}$$

We now sum (25) for $k = 1, \ldots, 4$, and we add to both sides of the resulting expression the term

$$\frac{\mu}{2}\left(\|\partial_t^3 \xi(t)\|_{L^2}^2 + \sum_{h=0}^{2} \|\partial_t^h \xi(t)\|_{H^{5-h}}^2\right).$$

By observing that $\|\xi_{ttt}(t)\|_{H^2}$ is equivalent to $\|\xi_{ttt}(t)\|_{H^1} + \|\Delta\xi_{ttt}\|_{L^2}$, we get

$$\frac{1}{2}\frac{d}{dt}\phi(t) + C_1\psi(t) \leqslant C_2\left(\mathcal{F}(t) + \|\partial_t^3\xi(t)\|_{L^2}^2 + \sum_{k=0}^{2}\|\partial_t^k \xi(t)\|_{H^{5-k}}^2\right).$$

We use inequality (27) firstly for $k = 1, 2, 3$ and $m = 4 - k$, and again in the cases $k = 1, 2$ and $m = 1$. By summing the resulting expressions we obtain the thesis. ■

We now estimate each term appearing in $\mathcal{F}(t)$. The result we are going to show is

LEMMA 3.6. *The following inequality*

$$\mathcal{F}(t) \leqslant C_\infty(t)(\phi(t) + \|Z(t)\|_{H^4}^2)$$

holds in $[0, T]$.

PROOF. We split the proof of the previous statement in several steps. As first step we write explicity $\|F(t)\|_{H^3}$. By using the Hölder and Gagliardo-Nirenberg inequalities, we easily obtain

$$\|F(t)\|_{H^3}^2 \leqslant C_\infty(t)(\|\xi(t)\|_{H^4}^2 + \|Z(t)\|_{H^4}^2).$$

By using (27) in the following cases $(k, m) = (1, 2), (k, m) = (2, 0)$, and finally $(k, m) = (1, 0)$, one has

$$\|\xi(t)\|_{H^4}^2 \leqslant C\left(\sum_{k=0}^{2}\|\partial_t^k\xi(t)\|_{L^2}^2 + \|F_t(t)\|_{L^2}^2 + \|F(t)\|_{H^2}^2\right).$$

By straightfull calculations, we get

$$\|F(t)\|_{H^2}^2 \leqslant C_\infty(t)(\|\xi(t)\|_{H^1}^2 + \|\xi_t(t)\|_{H^1}^2 + \|Z(t)\|_{H^4}^2),$$

$$\|F_t(t)\|_{L^2}^2 \leqslant C_\infty(t)(\|\xi(t)\|_{H^1}^2 + \|\xi_t(t)\|_{H^1}^2 + \|Z(t)\|_{H^4}^2).$$

Hence

$$\|F(t)\|_{H^3}^2 \leqslant C_\infty(t)(\phi(t) + \|Z(t)\|_{H^4}^2).$$

In order to estimate $\|F_t(t)\|_{H^2}$, $\|F_{tt}(t)\|_{H^1}$ and $\|F_{ttt}(t)\|_{L^2}$, we follow the same lines as in the previous step, and we consider the following interpolation inequalities

$$\|D^2 B\|_{L^8} \leqslant C\|\nabla B\|_{L^\infty}^{3/4} \|\xi\|_{H^4}^{1/4}, \tag{28}$$

$$\|D^2 u\|_{L^8} \leqslant C\|\nabla u\|_{L^\infty}^{3/4} \|Z\|_{H^4}^{1/4}, \tag{29}$$

$$\|D^3 B\|_{L^8} \leqslant C\|B\|_{L^\infty}^{5/16} \|\xi\|_{H^4}^{11/16}, \tag{30}$$

$$\|D^3 u\|_{L^8} \leqslant C\|u\|_{L^\infty}^{5/16} \|Z\|_{H^4}^{11/16}, \tag{31}$$

$$\|D^4 B\|_{L^4} \leqslant C\|\nabla B\|_{L^\infty}^{3/8} \|\xi\|_{H^5}^{5/8}, \tag{32}$$

$$\|D^4 B\|_{L^{8/3}} \leqslant C\|\nabla B\|_{L^\infty}^{1/4} \|\xi\|_{H^4}^{3/4}, \tag{33}$$

$$\|D^5 B\|_{L^{8/3}} \leqslant C\|\nabla B\|_{L^\infty}^{3/16} \|\xi\|_{H^5}^{13/16}, \tag{34}$$

$$\|D^3 B_t\|_{L^{8/3}} \leqslant C\|B_t\|_{L^\infty}^{1/4} \|\xi_t\|_{H^3}^{3/4}. \tag{35}$$

By virtue of the Hölder inequality, of (17)-(23) and of (28)-(35) we get

$$\|F_t(t)\|_{H^2}^2 + \|F_{ttt}(t)\|_{L^2}^2 \leqslant C_\infty(t)(\phi(t) + \|Z(t)\|_{H^4}^2),$$

$$\|F_{tt}(t)\|_{H^1}^2 \leqslant C_\infty(t)\big(\phi(t) + \|Z(t)\|_{H^4}^2\big) + \|D^3 B_t DZ\|_{L^2}^2.$$

By (35), by recalling that $B_t \in L^\infty(\Omega)$, and by using again (27), we get

$$\begin{aligned}\|F_{tt}(t)\|_{H^1}^2 &\leqslant C_\infty(t)\big(\phi(t) + \|Z(t)\|_{H^4}^2 + \|\xi_t(t)\|_{H^3}^2\big) \leqslant \\ &\leqslant C_\infty(t)(\phi(t) + \|Z(t)\|_{H^4}^2 + \|\xi_t(t)\|_{H^1}^2 + \\ &\quad + \|\xi_{tt}(t)\|_{H^1}^2 + \|F_t(t)\|_{H^1}^2) \leqslant \\ &\leqslant C_\infty(t)\big(\phi(t) + \|Z(t)\|_{H^4}^2\big).\end{aligned}$$

Hence, the claim follows. ■

By collecting Lemmata 3.3-3.6 we obtain inequality (12).

4. – Proof of Theorem 2.3.

The first topic which we treat is a local existence result in Sobolev spaces for system (1)-(9). Obtained that the classical solution of (1)-(9) belongs locally to $H^5(\Omega)$, from the a-priori estimates (12) and (13) we can extend such a solution on the whole time interval $[0, T]$.

PROOF. In order to show the local existence of a solution in Sobolev spaces, we apply the Banach-Caccioppoli theorem. In particular, let $0 < \tilde{t} \leq T$ be sufficiently small and let

$$S := \{(u, B) \in L^\infty(0, \tilde{t}; H^5(\Omega)) : \|(u, B)\|_{L^\infty(0, \tilde{t}; H^5)} \leq 2A\},$$

where A is a real positive constant such that $A > C_0(\|u_0\|^2_{H^5} + \|B_0\|^2_{H^5})$ for a suitable constant C_0, which will be fixed later.

Given the couple (u, B) in S and satisfying (3)-(9), let

$$\Lambda : S \to \Lambda(S)$$

be the map defined by

$$U := (u, B) \to \widetilde{U} := (\tilde{u}, \widetilde{B}),$$

where $\widetilde{U} := (\tilde{u}, \widetilde{B})$ is the solution of the following linear system

$$\tilde{u}_t + (u\cdot\nabla)\,\tilde{u} - (B\cdot\nabla)\,\widetilde{B} = \frac{1}{2}\nabla|B|^2 \text{ in } Q_T, \tag{36}$$

$$\widetilde{B}_t + (u\cdot\nabla)\,\widetilde{B} - (B\cdot\nabla)\,\tilde{u} - \mu\Delta\widetilde{B} = 0 \text{ in } Q_T, \tag{37}$$

$$\operatorname{div}\tilde{u} = 0 \text{ in } Q_T, \tag{38}$$

$$\operatorname{div}\widetilde{B} = 0 \text{ in } Q_T, \tag{39}$$

$$\tilde{u}\cdot\nu = 0 \text{ on } \Gamma\times(0, T), \tag{40}$$

$$\widetilde{B}\cdot\nu = 0 \text{ on } \Gamma\times(0, T), \tag{41}$$

$$\operatorname{rot}\widetilde{B} = 0 \text{ on } \Gamma\times(0, T), \tag{42}$$

$$\tilde{u}(x, 0) = u_0(x) \text{ in } \Omega, \tag{43}$$

$$\widetilde{B}(x, 0) = B_0(x) \text{ in } \Omega. \tag{44}$$

We now show that the map Λ satisfies all the assumptions of the Banach-Caccioppoli theorem. We now apply to both sides of equation (36)-(37) rot and ∂^α, where α is a multi-index with $|\alpha| \leq 4$. We multiply the resulting expressions by the test functions $\partial^\alpha \widetilde{Z}$ and $\partial^\alpha \tilde{\xi}$, where $\widetilde{Z} := \text{rot}\, \tilde{u}$ and $\tilde{\xi} := \text{rot}\, \widetilde{B}$. By suitable integrations by parts, and by using the Hölder and Young inequalities, an application of the Gronwall lemma yields that

$$\max_{t \in [0, \tilde{t}]} \| \widetilde{U}(t) \|_{H^5}^2 \leq \left\{ C_0 \| U(0) \|_{H^5}^2 + C \int_0^{\tilde{t}} \| U(t) \|_{H^5}^4 \, dt \right\} e^{C \int_0^{\tilde{t}} (1 + \| U(t) \|_{H^5})^2 \, dt},$$

where $\| U(0) \|_{H^5}^2 = \| u_0 \|_{H^5}^2 + \| B_0 \|_{H^5}^2$. Since U belongs to S, we get

$$\max_{t \in [0, \tilde{t}]} \| \widetilde{U}(t) \|_{H^5}^2 \leq (A + C\tilde{t}(2A)^4)\, e^{C\tilde{t}(1 + 2A)^2}.$$

Consequently, if $\tilde{t}$ is small enough, Λ maps the set S into itself. We now show that Λ is a contraction with respect to $L^\infty(0, \tilde{t}; L^2)$-norm. Let $\widetilde{U}_1 := (\tilde{u}_1, \widetilde{B}_1)$ and $\widetilde{U}_2 := (\tilde{u}_2, \widetilde{B}_2)$ be solutions of system (36)-(44). We now consider the difference between equations (36), written for $i = 1, 2$, and equations (37), again written for $i = 1, 2$. We use as test functions $\tilde{u}_1 - \tilde{u}_2$ and $\widetilde{B}_1 - \widetilde{B}_2$ respectively. By standard arguments, we get

$$\| \widetilde{U}_1 - \widetilde{U}_2 \|_{L^\infty(0, \tilde{t}; L^2)}^2 \leq C \tilde{t} e^{4A^2 \tilde{t}} \| U_1 - U_2 \|_{L^\infty(0, \tilde{t}; L^2)}^2 .$$

If $\tilde{t}$ is sufficiently small, Λ is a contraction and the unique fixed point of the map Λ is a solution of system (1)-(9). The thesis follows by the uniqueness of the classical solution and by using (12) and (13). ∎

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