Spherical Harmonics and Spherical Averages of Fourier Transforms.

PER SJÖLIN (*)

ABSTRACT - We give estimates for spherical averages of Fourier transforms of functions which are linear combinations of products of radial functions and spherical harmonics. This generalizes the case of radial functions.

1. Introduction.

We shall here study Fourier transforms in \( \mathbb{R}^n \) and we shall always assume \( n \geq 2 \). Let \( \theta \) denote the area measure on \( S^{n-1} \) and set

\[ \sigma(f)(R) = \int_{S^{n-1}} |\hat{f}(R\xi)|^2 d\theta(\xi), \quad R > 1, \]

where \( \hat{f} \) denotes the Fourier transform of a function \( f \in L^1(\mathbb{R}^n) \). We are interested in estimates of the type

\[ \sigma(f)(R) \leq CR^{-\beta} \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \frac{1}{|\xi|^{n-\alpha}} d\xi, \quad R > 1. \] (1)

For \( 0 < \alpha \leq n \) we shall consider the statement

\[ \text{there exists a constant } C = C_{\alpha, \beta} \text{ such that (1) holds for all } f \in C_0^\infty(\mathbb{R}^n) \text{ with supp } f \subset B_1 \text{ and } f \geq 0. \] (2)

Here \( B_1 \) denotes the unit ball in \( \mathbb{R}^n \).

(*) Indirizzo dell’A.: Department of Mathematics, Royal Institute of Technology, S-100 44 Stockholm, Sweden. E-mail: pers@math.kth.se
We set $\beta_+(\alpha) = \sup \{\beta; (2) \text{ holds}\}$. The number $\beta_+(\alpha)$ has been studied in Mattila [2], Sjölin [4], Bourgain [1], and Wolff [9]. In the case $n = 2$ it is known that

$$
\beta_+(\alpha) = \begin{cases} 
\alpha, & 0 < \alpha \leq 1/2 \\
1/2, & 1/2 < \alpha \leq 1 \\
\alpha/2, & 1 < \alpha \leq 2 
\end{cases}
$$

(see [2] and [9]). For $n \geq 3$ one knows that $\beta_+(\alpha) = \alpha$ for $0 < \alpha \leq (n - 1)/2$, max $((n - 1)/2, \alpha - 1) \leq \beta_+(\alpha) \leq \min (\alpha, \alpha/2 + n/2 - 1)$ for $(n - 1)/2 < \alpha < n$, and $\beta_+(n) = n - 1$ (see [2] and [4]).

Results of this type have applications in geometric measure theory in the study of distance sets.

In Sjölin and Soria [6], [7], $\theta$ is replaced by general measures and in these papers one also studies the case when the condition $f \geq 0$ in (2) is removed.

The case when $f$ is also assumed to be radial is studied in Sjölin [5]. We shall here generalize the case of radial functions. We recall that

$$
L^2(R^n) = \bigoplus_{k=0}^{\infty} H_k,
$$

where $H_k$ is the space of all linear combinations of functions of the form $fP$, where $f$ ranges over the radial functions and $P$ over the solid spherical harmonics of degree $k$, so that $fP$ belongs to $L^2(R^n)$ (see Stein and Weiss [8], p. 151).

Now fix $k \geq 0$ and let $P_1, P_2, \ldots, P_{a_k}$ be an orthonormal basis for the space of solid spherical harmonics of degree $k$ (where we use the inner product in $L^2(S^{k-1})$). The elements in $H_k$ can be written in the form

(3) \hspace{1cm} f(x) = \sum_{j=1}^{a_k} f_j(r) \ P_j(x) \hspace{1cm} (\text{here } r = |x|)

and

$$
\int_{R^n} |f(x)|^2 \, dx = \sum_{j=1}^{a_k} \int_0^\infty |f_j(r)|^2 r^{n+2k-1} \, dr.
$$

We let $\mathcal{R}$ denote the class of all functions $g$ on $[0, \infty)$, which satisfy the
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following conditions:

\[ g(r) \geq 0 \quad \text{for } r \geq 0 , \]
\[ g \text{ is } C^\infty \quad \text{on } (0, \infty) , \]
\[ g(r) = 0 \quad \text{for } r > 1 , \]

and

there exists \( \varepsilon > 0 \) such that \( g(r) = 0 \) for \( 0 \leq r \leq \varepsilon \).

We say that \( f \in \mathcal{S}_k \), \( k = 0, 1, 2, \ldots \), if \( f \) is given by (3) with all \( f_j \in \mathcal{R} \).

For \( 0 < \alpha \leq n \) we shall consider the statement:

\[ \text{(4) there exists } C = C_{\alpha, k} \text{ such that (1) holds for all } f \in \mathcal{S}_k. \]

We then set \( \beta(a) = \beta_{\alpha, k}(a) = \beta_{n,k}(a) = \sup \{ \beta; (4) \text{ holds} \} \).

We have the following result.

**THEOREM.** For \( k = 0, 1, 2, \ldots \), we have \( \beta_k(a) = a \) for \( 0 < a \leq n - 1 \), and \( \beta_k(a) = n - 1 \) for \( n - 1 < a \leq n \).

We shall first give a proof of the theorem which works directly for all \( k \geq 0 \). Another possibility is to first treat the case \( k = 0 \) (i.e. the case of radial functions), and then use the case \( k = 0 \) to study the case \( k \geq 1 \). We shall also say something about this second approach.

2. Proofs.

If \( f \) is a function on \( [0, \infty) \) we shall also use the notation \( f \) for the corresponding radial function in \( \mathbb{R}^n \). We also let \( \mathcal{F}_n \) denote the Fourier transformation in \( \mathbb{R}^n \).

**PROOF OF THE THEOREM.** Assume that \( f \) is given by (3) with all \( f_j \) belonging to \( \mathcal{R} \). It then follows from [8], p. 158, that

\[ \tilde{f}(x) = \sum_{j=1}^{n} F_j(r) P_j(x) \quad (\text{here } r = |x|), \]
where
\[
F_j(r) = c_k r^{1-n/2-k} \int_0^\infty f_j(s) J_{n/2+k-1}(rs) s^{n/2+k} ds, \quad r > 0,
\]
and \(J_m\) denotes Bessel functions.

For \(|\xi| = 1\) we obtain
\[
\tilde{f}(R\xi) = \sum_j F_j(R) P_j(R\xi) = R^k \sum_j F_j(R) P_j(\xi)
\]
and hence
\[
\sigma(f)(R) = R^{2k} \int_{S^{n-1}} \left| \sum_j F_j(R) P_j(\xi) \right|^2 d\theta(\xi) = R^{2k} \sum_j |F_j(R)|^2.
\]

We also have
\[
\int_{\mathbb{R}^n} |\tilde{f}(\xi)|^2 |\xi|^{-\nu} d\xi = \int_0^\infty \left( \int_{S^{n-1}} |\tilde{f}(r\xi')|^2 d\theta(\xi') \right) r^{\alpha-1} dr
\]
\[
= \int_0^\infty \left( \sum_j |F_j(r)|^2 \right) r^{\alpha-1} dr = \sum_j \int_{\mathbb{R}^n} |F_j(r)|^2 |\xi|^{2k+\nu} d\xi
\]
(where \(r = |\xi|\) in the last integral).

It follows that the statement (4) is equivalent to the statement:
if \(f \in \mathcal{R}\) and
\[
F(r) = c_k r^{1-n/2-k} \int_0^\infty f(s) J_{n/2+k-1}(rs) s^{n/2+k} ds, \quad r > 0,
\]
then
\[
R^{2k} |F(R)|^2 \leq c_k R^{-\beta} \int_{\mathbb{R}^n} |F(r)|^2 |\xi|^{2k+\nu} d\xi, \quad R > 1.
\]

Now assume that \(f \in \mathcal{R}\) and that \(F\) is given by (5). It is then clear that
\[
F(r) = c_k \mathcal{F}_{n+2k} f(r)
\]
and since \( f \in C_0^\infty(\mathbb{R}^{n+2k}) \) it follows that \( F \in \mathcal{S}(\mathbb{R}^{n+2k}) \), where \( \mathcal{S} \) denotes the Schwartz class. Assume that \( 2 - n - 2k \leq \beta \leq 1 \). Then \( n/2 + k - 1 \geq \beta/2 \) and it follows that

\[
|J_{n/2 + k - 1}(s)| \leq Cs^{-\beta/2}, \quad s > 0
\]

(cf. [8], p. 158).

Inserting this estimate in (5) we obtain

\[
|F(r)| \leq C r^{1 - n/2 - k} \int_0^1 f(s)(rs)^{-\beta/2}s^{n/2 + k} ds
\]

\[
= C r^{1 - n/2 - k} \int_0^\infty f(s) \varphi(s)(rs)^{-\beta/2}s^{n/2 + k} ds,
\]

where \( \varphi \in C_0^\infty(0, \infty) \), \( \varphi \geq 0 \) and

\[
\varphi(s) = \begin{cases}
1, & 0 < s \leq 1 \\
0, & s \geq 2
\end{cases}
\]

The Fourier inversion formula implies that

\[
f(s) = c_k s^{1 - n/2 - k} \int_0^\infty F(t) J_{n/2 + k - 1}(st) t^{n/2 + k} dt
\]

and hence

\[
|F(r)| \leq C r^{1 - n/2 - k} \left( \int_0^\infty s^{1 - n/2 - k} \int_0^\infty F(t) J_{n/2 + k - 1}(st) t^{n/2 + k} dt dt \right) \cdot r^{1 - n/2 - k} \int_0^\infty \varphi(s)(rs)^{-\beta/2}s^{n/2 + k} ds
\]

\[
= C r^{1 - n/2 - k - \beta/2} \int_0^\infty F(t) t^{n/2 + k} \left( \int_0^\infty \varphi(s)s^{1 - \beta/2} J_{n/2 + k - 1}(st) ds \right) dt.
\]

We conclude that

\[
r^k |F(r)| \leq C r^{1 - n/2 - \beta/2} \int_0^\infty F(t) t^{n/2 + k} I(t) dt,
\]

(8)
where
\[ I(t) = \int_0^\infty q(s) s^{1-\beta/2} J_{n/2+k-1}(st) \, ds, \quad t > 0. \]

We shall now estimate \( I(t) \). Setting \( \gamma = \beta/2 + n/2 + k - 1 \) we obtain
\[ I(t) = t^{n/2 + k - 1} t^{1-n/2-k} \int_0^\infty q(s) J_{n/2+k-1}(st) s^{n/2+k} s^{-n/2-k+1-\beta/2} \, ds \]
\[ = c_k t^{n/2+k-1} \mathcal{F}_{n+2k}(q(s)) s^{1-\beta/2-n/2-k}(t) \]
\[ = c_k t^{n/2+k-1} \mathcal{F}_{n+2k}(q(s)) s^{-\gamma}(t), \quad t > 0. \]

First assume \( 2 - n - 2k < \beta \leq 1 \). Then \( \gamma \leq 1/2 + n/2 + k - 1 < n + 2k \) and \( \gamma > 1 - n/2 - k + n/2 + k - 1 = 0 \). It follows that \( \mathcal{F}_{n+2k}(q(s)) s^{-\gamma} = c_k(\mathcal{F}_{n+2k}(q) \ast s^{-n-2k+\gamma}) \), where the convolution is taken in \( \mathbb{R}^{n+2k} \). Since \( \mathcal{F}_{n+2k} q \in \mathcal{S}(\mathbb{R}^{n+2k}) \) we obtain
\[ |\mathcal{F}_{n+2k}(q(s)) s^{-\gamma}(t)| \leq C(1+t)^{-n-2k+\gamma}, \quad t > 0. \]

In the remaining case \( \beta = 2 - n - 2k \) we have \( \gamma = 0 \) and it is clear that (9) holds also in this case.

We have \( n + 2k - \gamma = n + 2k - \beta/2 - n/2 - k + 1 = n/2 + k - \beta/2 + 1 \), and hence
\[ |J(t)| \leq C t^{n/2+k-1} (1+t)^{-n/2-k+\beta/2-1}, \quad t > 0. \]

Thus \( |I(t)| \leq C t^{n/2+k-1} \) for \( 0 < t \leq 1 \), and \( |I(t)| \leq Ct^{\beta/2-2} \) for \( t > 1 \).

Invoking (8) we then get
\[ r^t |F(r)| \leq Cr^{1-n/2-\beta/2} \int_0^\infty |F(t)| \psi(t) \, dt, \quad r > 0, \]

where \( \psi(t) = t^{n/2+2k-1} \) for \( 0 < t \leq 1 \), and \( \psi(t) = t^{n/2+k+\beta/2-2} \) for \( t > 1 \).

Using the Schwarz inequality we obtain
\[ \int_0^\infty |F| \psi \, dt \leq \left( \int_0^\infty |F(t)|^2 t^{2k+a-1} \, dt \right)^{1/2} \]
\[ \leq \left( \int_0^\infty \psi(t)^2 t^{-2k-a+1} \, dt \right)^{1/2}. \]
We have
\[
\int_0^1 \psi(t)^2 t^{-2k - \alpha + 1} dt = \int_0^1 t^{2n + 4k - 2 - 2k - \alpha + 1} dt
\]
\[
= \int_0^1 t^{2n + 2k - \alpha - 1} dt < \infty
\]
since \(2n + 2k - \alpha \geq n\). On the other hand we also have
\[
\int_1^\infty \psi(t)^2 t^{-2k - \alpha + 1} dt = \int_1^\infty t^{n + 2k + \beta - 4 - 2k - \alpha + 1} dt
\]
\[
= \int_1^\infty t^{n + \beta - \alpha - 3} dt,
\]
which is finite if \(n + \beta - \alpha - 3 < -1\) i.e. \(\beta < 2 + \alpha - n\).

Invoking (10) and (11) we conclude that
\[
r^{2k} |F(r)|^2 \leq Cr^{2 - \alpha - \beta} \int_0^\infty |T(t)|^2 t^{2k + \alpha - 1} dt, \quad r > 0,
\]
if \(2 - n - 2k \leq \beta \leq 1\) and \(\beta < 2 + \alpha - n\). Setting \(M = \{\beta; 2 - n - 2k \leq \beta \leq 1\} \text{ and } \beta < 2 + \alpha - n\) we obtain
\[
(12) \quad \beta(\alpha) \geq \sup_{\beta \in M} (\beta + n - 2) = n - 2 + \sup M.
\]

Then assume 0 < \(\alpha \leq n - 1\). We have 2 - n - 2k < 2 + \(\alpha - n\) \(\leq 1\), and it follows that \(\sup M = 2 + \alpha - n\) and thus \(\beta(\alpha) \geq n - 2 + 2 + \alpha - n = \alpha\) in this case.

Then assume \(n - 1 < \alpha \leq n\). In this case 2 + \(\alpha - n\) \(\geq 1\) and it follows that \(\sup M = 1\). Invoking (12) we obtain \(\beta(\alpha) \geq n - 1\).

Thus we have obtained lower bounds for \(\beta(\alpha)\). We shall now obtain upper bounds, and we first assume 0 < \(\alpha \leq n - 1\). Also assume that (6) holds for all \(F\) given by (5) with \(f \in \mathcal{I}\). We shall prove that then \(\beta \leq \alpha\). First choose \(f \in \mathcal{I}\) with \(f \neq 0\). Then there exists \(b > 0\) such that \(F(b) \neq 0\). Also set \(f_\alpha(s) = f(as), \quad a > 1\), and
\[
F_\alpha(r) = c_\alpha r^{1 - w^2 - k} \int_0^\infty f(as) J_{n^2 + k - 1}(rs) s^{w^2 + k} ds, \quad r > 0.
\]
Performing a change of variable as \( x = t \) we obtain

\[
F_a(r) = c_k r^{1 - n/2 - k} \int_0^\infty f(t) J_{n/2 + k - 1}(rt/a) t^{n/2 + k} dt a^{-n/2 - k - 1}
\]

\[
= a^{-n-2k} F(r/a),
\]

and (6) yields

\[
r^{2k} a^{-2n - 4k} |F(r/a)|^2 \leq Cr^{-\beta} \int_0^\infty |F(r/a)|^2 r^{2k + a - 1} dr a^{-2n - 4k}.
\]

Performing a change of variable we then get

\[
r^{2k} |F(r/a)|^2 \leq Cr^{-\beta} \int_0^\infty |F(s)|^2 s^{2k + a - 1} ds a^{2k + a}
\]

\[
= Cr^{-\beta} a^{2k + a}
\]

for all \( a > 1 \) and \( r > 1 \), where \( C \) depends on \( f \) but not on \( a \) or \( r \). We now choose \( a = r/b \), where \( r \) is large, and it follows from (13) that

\[
r^{2k} |F(b)|^2 \leq Cr^{-\beta} r^{2k + a} b^{-2k - a}.
\]

We conclude that \( r^\beta \leq Cr^\alpha \) and it follows that \( \beta \leq \alpha \). Hence \( \beta(\alpha) \leq \alpha \) for \( 0 < \alpha \leq n - 1 \) and we have proved that \( \beta(\alpha) = \alpha \) in this case.

It remains to study the case \( n - 1 < \alpha \leq n \). Assume as above that (6) holds for all \( f \in \mathcal{R} \).

Let \( \mathcal{L}_+ \) denote the class of all \( f \in L^1[0, \infty) \) with \( f \geq 0 \) and satisfying \( f(r) = 0 \) for \( r \geq 7/8 \) and \( f(r) = 0 \) for \( 0 \leq r \leq \epsilon \) for some \( \epsilon > 0 \). It is then easy to see that (6) holds also for all \( f \in \mathcal{L}_+ \). In fact, this follows from approximation of \( f \in \mathcal{L}_+ \) with \( f \ast w \), where the convolution is taken in \( \mathbb{R}^{n + 2k} \) and \( w \) is an approximate identity in \( \mathbb{R}^{n + 2k} \).

Then choose \( \varphi \in C^\infty_0(0, \infty) \) with \( \supp \varphi \subseteq (1/2, 7/8) \), \( \varphi \geq 0 \), and \( \varphi(3/4) = 1 \). Also set

\[
f(s) = f_{\varphi}(s) = e^{-ir\varphi(s)}, \quad s > 0,
\]

where \( R \) is large. Then

\[
f = f_1 - f_2 + \overline{f_2} - \overline{f_3},
\]

where \( f_j \in \mathcal{L}_+ \) and \( f_j \leq |f| \) for \( j = 1, 2, 3, 4 \). Let \( F_j \) correspond to \( f_j \) in the same way as \( F \) corresponds to \( f \) in (5). Then \( F_j = c_k r_i + 2k f_j \) and since (6)
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holds for $F_j$ we obtain

$$R^{2k} |F(R)|^2 \leq CR^{2k} \sum_j |F_j|^2 \leq CR^{-\beta} \sum_j \int_0^\infty |F_j(r)|^2 r^{2k+a-1} dr$$

$$= CR^{-\beta} \sum_j \int_{\mathbb{R}^{2k}} |F_j(r)|^2 \langle \xi \rangle^{a-n} d\xi .$$

In the case $n-1 < a < n$ we invoke Lemma 12.12 in Mattila [5], p. 162, and then get

$$R^{2k} |F(R)|^2 \leq CR^{-\beta} \sum_j \int_{\mathbb{R}^{2k}} \int_{\mathbb{R}^{2k}} |x-y|^{-a-2k} f_j(x) f_j(y) dx dy$$

$$\leq CR^{-\beta} \int_{\mathbb{R}^{2k}} \int_{\mathbb{R}^{2k}} |x-y|^{-a-2k} |f(x)| |f(y)| dx dy .$$

Hence

(14) $$R^{2k} |F(R)|^2 \leq CR^{-\beta},$$

where $C$ depends on $q$ but not on $R$. In the case $a = n$ (14) follows from an application of the Plancherel theorem.

We have

$$R^k F(R) = CR^{1-n/2} \int_0^\infty e^{-iRs} \varphi(s) J_{n/2+k-1}(Rs) s^{n/2+k} ds$$

and we shall use the asymptotic formula

$$J_{n/2+k-1}(t) = c_1 e^{-t} t^{-1/2} + c_2 e^{-t} t^{-1/2} + O(t^{-3/2}) , \quad t \to \infty$$

(see [8], p. 158). We obtain

$$R^k F(R) = CR^{1-n/2} \left[ c_1 e^{iRs} (Rs)^{1/2} + c_2 e^{-iRs} (Rs)^{1/2} + O((Rs)^{-3/2}) \right]$$

$$\cdot s^{n/2+k} e^{-iRs} \varphi(s) ds$$

$$= cc_1 R^{1/2-n/2} \int_0^1 s^{n/2+k-1/2} \varphi(s) ds$$
and hence
\[ R^{2k} |F(R)|^2 \geq cR^{1-n} \]
for large values of \( R \).

The formula (14) then yields
\[ R^1 \leq CR^{-\beta} \]
i.e.
\[ R^\beta \leq CR^{n-1} \]
and we conclude that \( \beta \leq n - 1 \). Hence \( \beta(\alpha) \leq n - 1 \) for \( n - 1 < \alpha \leq n \) and it follows that \( \beta(\alpha) = n - 1 \) in this case. The proof of the theorem is complete.

We shall finally discuss how results for radial functions (i.e. the case \( k = 0 \)) can be used to study the case \( k \geq 1 \). Therefore assume \( n \geq 2 \), \( k \geq 1 \) and \( 0 < \alpha \leq n \). If \( f \in \mathcal{R} \) and \( F \) is given by (5), then the estimate
\[ |F(R)|^2 \leq CR^{-\beta} \int_0^\infty |F(r)|^2 r^{\alpha + 2k - 1} dr \]
is equivalent to the estimate
\[ R^{2k} |F(R)|^2 \leq CR^{-(\beta - 2k)} \int_0^\infty |F(r)|^2 r^{\alpha + 2k - 1} dr. \]

It follows that
\[ \beta_{n,k}(\alpha) = \beta_{n+2k,0}(\alpha + 2k) - 2k. \]

Assume then that we know that \( \beta_{n,0}(\alpha) = \min(\alpha, n-1) \) for all \( n \geq 2 \) and \( 0 < \alpha \leq n \). For \( k \geq 1 \), \( n \geq 2 \) and \( 0 < \alpha \leq n \) (15) then yields
\[ \beta_{n,k}(\alpha) = \min(\alpha + 2k, n + 2k - 1) - 2k = \min(\alpha, n - 1), \]
which is the desired formula.
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