RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova, tome 106 (2001), p. 65-76

<http://www.numdam.org/item?id=RSMUP_2001__106__65_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 106 (2001)

Some Notes to the Transport Equation and to the Green Formula.

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ABSTRACT - In the paper [3] the authors claim that in order to obtain a particular form of the Green formula for $p = 2$ it is necessary to use the uniqueness result for the steady transport equation. Our aim is to generalize these results for $p \in (1, \infty)$ and show the Green formula without the use of the steady transport equation. As a consequence we get the density of smooth functions with compact support in the space $X_u^{p,p}(\Omega)$ defined below.

0. Introduction.

In the recent paper by V. Girault and L. R. Scott [3] the authors considered the system of equations describing the flow of the second grade fluid. Among others, they studied in detail the transport equation and showed the formula

$$\int_\Omega (\boldsymbol{u} \cdot \nabla \varrho)\, \varrho \, \mathrm{d}\mathbf{x} = 0 \tag{0.1}$$

under relatively weak assumptions — the bounded domain $\Omega \in C^{0,1}$,

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Supported by the Grant Agency of the Czech Republic (grant No. 201/00/0768) and by the Council of the Czech Government (project No. J14/98:153100011.)

$\boldsymbol{u} \in W^{1,2}(\Omega)$ with $\operatorname{div} \boldsymbol{u} = 0$, $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ at $\partial\Omega$ in the trace sense, and $\varrho \in L^2(\Omega)$ with $\boldsymbol{u} \cdot \nabla\varrho \in L^2(\Omega)$. Nonetheless, the technique used in [3] seems rather complicated. First, the authors proved a weak version of the Friedrichs lemma using the regularization technique presented in [5]; this technique enables to prove the Friedrichs lemma up to the boundary for Ω only Lipschitz-continuous domain. Then they established unique solvability of the transport equation

$$\lambda\varrho + \boldsymbol{u} \cdot \nabla\varrho = f \quad \text{in } \Omega \tag{0.2}$$

in $L^2(\Omega)$ for $\boldsymbol{u}$ and Ω as above. Finally, using this result, they proved the Green formula (0.1). It seems to be more natural to prove firstly the Green formula and then to get the unique solvability of problem (0.2). This is the first goal of this note. The second aim is to generalize formula (0.1) for $p \neq 2$, i.e. we will show that

$$\int_\Omega (\boldsymbol{u} \cdot \nabla\varrho) \, |\varrho|^{p-2} \varrho \, d\boldsymbol{x} = -\frac{1}{p} \int_\Omega \operatorname{div} \boldsymbol{u} \, |\varrho|^p \, d\boldsymbol{x} \tag{0.3}$$

for $\boldsymbol{u} \in W^{1,p'}(\Omega)$, $\operatorname{div} \boldsymbol{u} \in L^\infty(\Omega)$, $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ at $\partial\Omega$, ϱ and $\boldsymbol{u} \cdot \nabla\varrho \in L^p(\Omega)$, $1 < p < \infty$.

To this aim we first mention the general version of the Friedrichs lemma and then by means of it and by means of standard density argument we prove formula (0.3). Further, as an application, we easily obtain existence and uniqueness of the solution to the transport equation (0.2) and finally the main density result (see Theorem 2.2).

1. Friedrichs' lemma and its consequences.

The following part is only a slight generalization of the results presented in [3]. For the readers convenience we repeat here the main ideas; the proofs can be easily reproduced from [3] with [2]. First we recall the following useful property of Lipschitz-continuous domains

LEMMA 1.1. *Let $\Omega \in C^{0,1}$ be a bounded domain of $\mathbb{R}^N$; then Ω has a finite open covering*

$$\overline{\Omega} \subset \bigcup_{r=1}^{m} \mathcal{O}_r$$

with the following property. For all r, $1 \leq r \leq m$, there exists a nonzero *vector* $\boldsymbol{t}_r \in \mathbb{R}^N$ *and a number* $\delta_r > 0$ *such that for all* $0 < \varepsilon \leq 1$ *and all* $\boldsymbol{x} \in \overline{\Omega} \cap \mathcal{O}_r$

$$B(\boldsymbol{x}; \varepsilon\delta_r) + \varepsilon \boldsymbol{t}_r \subset \Omega ,$$

where $B(\boldsymbol{x}; \sigma)$ *denotes the ball centered at* $\boldsymbol{x}$ *with radius* σ.

PROOF. See [3]. ■

Next we want to generalize Lemma 3.2 from [3]; recall that the main idea is taken from that paper. By ω_r we denote the standard mollifier with support in $B(\boldsymbol{0}; \delta_r)$, i.e. $\omega_r \in \mathcal{D}(\mathbb{R}^N)$, $0 \leq \omega_r \leq 1$ in $\mathbb{R}^N$ and

$$\int_{\mathbb{R}^N} \omega_r \, \mathrm{d}\boldsymbol{x} = \int_{B(\boldsymbol{0}; \delta_r)} \omega_r \, \mathrm{d}\boldsymbol{x} = 1 .$$

Further, for any $\varepsilon \in (0, 1]$ we put

$$\omega_r^\varepsilon = \frac{1}{\varepsilon^N} \omega_r \left(\frac{\boldsymbol{x}}{\varepsilon} + \boldsymbol{t}_r \right).$$

Now we define for the function ϱ the mollification

$$(\varrho * \omega_r^\varepsilon)(\boldsymbol{x}) = \int_{B(\boldsymbol{0}; \delta_r)} \varrho(\boldsymbol{x} - \varepsilon(\boldsymbol{y} - \boldsymbol{t}_r)) \, \omega_r(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} .$$

Evidently, the mollification is well defined for any $\boldsymbol{x} \in \Omega_r = \overline{\Omega} \cap \mathcal{O}_r$ for ϱ locally integrable over Ω. Moreover, for any $p \in (1, \infty)$

$$\|\varrho * \omega_r^\varepsilon\|_{L^p(\Omega_r)} \leq C(p) \|\varrho\|_{L^p(\Omega)}.$$

Now we have

LEMMA 1.2. *Let* $\Omega \in C^{0,1}$ *be a bounded domain of* $\mathbb{R}^N$. *Further, let* $\boldsymbol{u} \in W^{1,q}(\Omega)$, $\varrho \in L^p(\Omega)$ *and let* $q \geq p'$, *where* p' *denotes the Hölder conjugate of* p. *Then there exists a constant* C *independent of* ϱ *and* $\boldsymbol{u}$ *such that for all* r, $1 \leq r \leq m$ *and all* $\varepsilon \in (0, 1]$ *we have*

$$\|\boldsymbol{u} \cdot \nabla(\varrho * \omega_r^\varepsilon) - (\boldsymbol{u} \cdot \nabla \varrho) * \omega_r^\varepsilon\|_{L^s(\Omega_r)} \leq C \|\nabla \boldsymbol{u}\|_{L^q(\Omega)} \|\varrho\|_{L^p(\Omega)},$$

where $\dfrac{1}{s} = \dfrac{1}{p} + \dfrac{1}{q}$.

PROOF. The proof can be done combining the ideas from [3] with the technique from [2]. ■

COROLLARY 1.1. *Under the assumptions of Lemma 1.2 we have for all* $1 \leq r \leq m$

$$\lim_{\varepsilon \to 0} \| \boldsymbol{u} \cdot \nabla(\varrho * \omega_r^\varepsilon) - (\boldsymbol{u} \cdot \nabla \varrho) * \omega_r^\varepsilon \|_{L^s(\Omega_r)} = 0 \, .$$

PROOF. The proof can be done by analogy with the proof of Corollary 3.3 in [3], using the standard density property. ∎

THEOREM 1.1. *Let* $\Omega \in C^{0,1}$ *be a bounded domain of* $\mathbb{R}^N$ *and let* $\boldsymbol{u} \in W^{1,q}(\Omega)$, $1 < q < \infty$. *Let* $\varrho \in L^p(\Omega)$, $1 < p < \infty$ *be such that the distribution* $\boldsymbol{u} \cdot \nabla \varrho \in L^s(\Omega)$, $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$. *Then there exists a sequence* $\{\varrho_k\}_{k=1}^\infty$ *of functions from* $C^\infty(\overline{\Omega})$ *such that*

$$\varrho_k \to \varrho \quad \text{in } L^p(\Omega), \qquad \boldsymbol{u} \cdot \nabla \varrho_k \to \boldsymbol{u} \cdot \nabla \varrho \quad \text{in } L^s(\Omega).$$

PROOF. Using the partition of unity, it is a direct consequence of the results stated above. ∎

Note that although we discussed up to now the situation when Ω is bounded, the same result (with $\varrho_k \in C_0^\infty(\overline{\Omega})$) holds also for Ω unbounded. We namely apply the procedure as above on $\varrho \eta_R$ with η_R being a properly chosen cut-off function with support in $B(\mathbf{0}; 2R)$.

REMARK 1.1. Theorem 1.1 claims that smooth functions up to the boundary with bounded support are dense in

$$X_{\boldsymbol{u}}^{p,s}(\Omega) = \{\varrho \in L^p(\Omega); \boldsymbol{u} \cdot \nabla \varrho \in L^s(\Omega)\}, \qquad \boldsymbol{u} \in W^{1,q}(\Omega).$$

Up to now, we did not have to use the assumption that $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial \Omega$ and any better regularity of $\operatorname{div} \boldsymbol{u}$. Next we come to the proof of Green's formula (0.3). Let $\varrho \in X_{\boldsymbol{u}}^{p,p}(\Omega)$ with $\boldsymbol{u} \in W^{1,p'}(\Omega)$, $\operatorname{div} \boldsymbol{u} \in L^\infty(\Omega)$ and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ at $\partial \Omega$. We want to show that

$$\int_\Omega (\boldsymbol{u} \cdot \nabla \varrho) \, |\varrho|^{p-2} \varrho \, d\boldsymbol{x} = -\frac{1}{p} \int_\Omega \operatorname{div} \boldsymbol{u} \, |\varrho|^p \, d\boldsymbol{x} \, . \tag{1.1}$$

Note that for $\eta \in W^{1,\infty}(\Omega)$ the formula

$$\int_\Omega (\boldsymbol{u} \cdot \nabla \varrho) \, \eta \, d\boldsymbol{x} = -\int_\Omega \operatorname{div} \boldsymbol{u} \varrho \eta \, d\boldsymbol{x} - \int_\Omega (\boldsymbol{u} \cdot \nabla \eta) \, \varrho \, d\boldsymbol{x}$$

holds true without the additional assumption on regularity of $\operatorname{div} \boldsymbol{u}$. The proof follows directly from Remark 1.1.

The main difficulty in showing Green' formula (1.1) is connected with the fact that the trace of $\boldsymbol{u}\cdot\boldsymbol{n}\,|\varrho|^p$ is generally not defined and Theorem 1.1 gives us the convergence of $\boldsymbol{u}\cdot\nabla\varrho_k$ only in $L^1(\Omega)$.

Nevertheless, we have

THEOREM 1.2. *Let $\Omega \in C^{0,1}$ be a bounded domain of $\mathbb{R}^N$ and $1 < p < \infty$. Let $\varrho \in X_{\boldsymbol{u}}^{p,p}(\Omega)$, where $\boldsymbol{u} \in W^{1,p'}(\Omega)$, $\operatorname{div} \boldsymbol{u} \in L^\infty(\Omega)$ and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$ in the trace sense. Then* (1.1) *holds true.*

PROOF. Let ϱ_k denotes the sequence constructed in Theorem 1.1, i.e. $\varrho_k \to \varrho$ in $L^p(\Omega)$ and $\boldsymbol{u}\cdot\nabla\varrho_k \to \boldsymbol{u}\cdot\nabla\varrho$ in $L^1(\Omega)$. Let $\delta > 0$. Then we have

$$
\begin{aligned}
(1.2)\qquad \int_\Omega \boldsymbol{u}\cdot\nabla\varrho_k \frac{|\varrho_k|^{p-2}\varrho_k}{1+\delta|\varrho_k|^{p-1}}\,d\boldsymbol{x} &= \int_{\partial\Omega} \boldsymbol{u}\cdot\boldsymbol{n} \frac{|\varrho_k|^p}{1+\delta|\varrho_k|^{p-1}}\,d\boldsymbol{\sigma} \\
&- \int_\Omega \operatorname{div} \boldsymbol{u} \frac{|\varrho_k|^p}{1+\delta|\varrho_k|^{p-1}}\,d\boldsymbol{x} - (p-1)\int_\Omega \boldsymbol{u}\cdot\nabla\varrho_k \frac{|\varrho_k|^{p-2}\varrho_k}{1+\delta|\varrho_k|^{p-1}}\,d\boldsymbol{x} + \\
&+ (p-1)\int_\Omega \delta\boldsymbol{u}\cdot\nabla\varrho_k \frac{|\varrho_k|^{2p-3}\varrho_k}{(1+\delta|\varrho_k|^{p-1})^2}\,d\boldsymbol{x}\,.
\end{aligned}
$$

Then, since ϱ_k is smooth, the boundary term is zero and our aim is to pass with k to ∞ and with δ to zero. Formally we get the desired equality (1.1). Nevertheless, we must do the limiting passages in the right order, first with k then with δ.

Now

$$
\begin{aligned}
\lim_{k\to\infty} \int_\Omega \boldsymbol{u}\cdot\nabla\varrho_k \frac{|\varrho_k|^{p-2}\varrho_k}{1+\delta|\varrho_k|^{p-1}}\,d\boldsymbol{x} &= \lim_{k\to\infty} \int_\Omega (\boldsymbol{u}\cdot\nabla\varrho_k - \boldsymbol{u}\cdot\nabla\varrho) \frac{|\varrho_k|^{p-2}\varrho_k}{1+\delta|\varrho_k|^{p-1}}\,d\boldsymbol{x} \\
+ \lim_{k\to\infty} \int_\Omega \boldsymbol{u}\cdot\nabla\varrho \left[\frac{|\varrho_k|^{p-2}\varrho_k}{1+\delta|\varrho_k|^{p-1}} - \frac{|\varrho|^{p-2}\varrho}{1+\delta|\varrho|^{p-1}}\right] d\boldsymbol{x} &+ \int_\Omega \boldsymbol{u}\cdot\nabla\varrho \frac{|\varrho|^{p-2}\varrho}{1+\delta|\varrho|^{p-1}}\,d\boldsymbol{x}\,.
\end{aligned}
$$

The first term tends to zero since $\boldsymbol{u}\cdot\nabla\varrho_k \to \boldsymbol{u}\cdot\nabla\varrho$ in $L^1(\Omega)$ and $\frac{|\varrho_k|^{p-2}\varrho_k}{1+\delta|\varrho_k|^{p-1}}$ is bounded in $L^\infty(\Omega)$, the second term tends to zero since $\frac{|\varrho_k|^{p-2}\varrho_k}{1+\delta|\varrho_k|^{p-1}}$ tends to $\frac{|\varrho|^{p-2}\varrho}{1+\delta|\varrho|^{p-1}}$ a.e. in Ω and $\boldsymbol{u}\cdot\nabla\varrho$ is integrable.

Further, the term $\boldsymbol{u}\cdot\nabla\varrho|\varrho|^{p-2}\varrho$ is integrable over Ω and we may easily pass with δ to zero and get the desired term $\int\limits_\Omega \boldsymbol{u}\cdot\nabla\varrho|\varrho|^{p-2}\varrho\,\mathrm{d}\boldsymbol{x}$.

Exactly in the same way we may treat the fourth term. Easily we also get

$$\lim_{\delta\to 0^+}\left(\lim_{k\to\infty}\int\limits_\Omega \frac{\operatorname{div}\boldsymbol{u}|\varrho_k|^p}{1+\delta|\varrho_k|^{p-1}}\,\mathrm{d}\boldsymbol{x}\right)=\int\limits_\Omega \operatorname{div}\boldsymbol{u}|\varrho|^p\,\mathrm{d}\boldsymbol{x}\,.$$

Finally

$$\lim_{k\to\infty}\int\limits_\Omega \delta\boldsymbol{u}\cdot\nabla\varrho_k\frac{|\varrho_k|^{2p-3}\varrho_k}{(1+\delta|\varrho_k|^{p-1})^2}\,\mathrm{d}\boldsymbol{x}=\int\limits_\Omega \boldsymbol{u}\cdot\nabla\varrho\frac{|\varrho|^{2p-3}\varrho}{(1+\delta|\varrho|^{p-1})^2}\,\mathrm{d}\boldsymbol{x}\,.$$

Now $|\boldsymbol{u}\cdot\nabla\varrho|\,|\varrho|^{p-1}$ is integrable and $\frac{\delta|\varrho|^{p-1}}{(1+\delta|\varrho|^{p-1})^2}$ is bounded, independently of δ. Thus the Lebesgue dominated convergence theorem finishes the proof. ■

Similarly we can also show the following lemma.

LEMMA 1.3. *Let* $\Omega\in C^{0,1}$ *be a bounded domain of* $\mathbb{R}^N$, $\boldsymbol{u}\in W^{1,p}(\Omega)\cap W^{1,p'}(\Omega)$, $1<p<\infty$, $\operatorname{div}\boldsymbol{u}\in L^\infty(\Omega)$ *with* $\boldsymbol{u}\cdot\boldsymbol{n}=0$ *at* $\partial\Omega$ *in the trace sense. Let* $\varrho\in X_{\boldsymbol{u}}^{p,p}(\Omega)$ *and* $\omega\in X_{\boldsymbol{u}}^{p',p'}(\Omega)$. *Then*

$$\int\limits_\Omega \varrho\boldsymbol{u}\cdot\nabla\omega\,\mathrm{d}\boldsymbol{x}+\int\limits_\Omega \omega\boldsymbol{u}\cdot\nabla\varrho\,\mathrm{d}\boldsymbol{x}=-\int\limits_\Omega \varrho\omega\operatorname{div}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,.$$

PROOF. We take $\omega_k\to\omega$ in $L^{p'}(\Omega)$ such that $\boldsymbol{u}\cdot\nabla\omega_k\to\boldsymbol{u}\cdot\nabla\omega$ in $L^1(\Omega)$ and $\varrho_k\to\varrho$ in $L^p(\Omega)$ such that $\boldsymbol{u}\cdot\nabla\varrho\to\boldsymbol{u}\cdot\nabla\varrho$ in $L^1(\Omega)$, apply the Green theorem on

$$\int\limits_\Omega \frac{\varrho_k\boldsymbol{u}\cdot\nabla\omega_k}{1+\delta|\varrho_k|+\delta|\omega_k|}\,\mathrm{d}\boldsymbol{x}$$

and pass with k to ∞ and with δ to zero. ■

2. Spaces $X_u^{p,p}(\Omega)$ and their properties.

In this section we would like to show some properties of the space

$$X_u^{p,p}(\Omega) = \{\varrho \in L^p(\Omega);\ \boldsymbol{u}\cdot\nabla\varrho \in L^p(\Omega)\}$$
$$\text{with } \boldsymbol{u} \in W^{1,p}(\Omega) \cap W^{1,p'}(\Omega),\ \operatorname{div} \boldsymbol{u} \in L^\infty(\Omega),\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 \ \text{ at } \partial\Omega\,. \tag{2.1}$$

Note that for $p > 2$ we have to require $\boldsymbol{u} \in W^{1,p}(\Omega)$, i.e. more regularity than it was necessary in order to get the Green formula. Our main goal is to show that under the assumptions (2.1) on $\boldsymbol{u}$ and for $\Omega \in C^{0,1}$ the space $C_0^\infty(\Omega)$ is dense in $X_u^{p,p}(\Omega)$. The same result for $p = 2$ was shown in [3]; note that in this case $p = p' = 2$.

Unlike their approach we were able to establish the Green formula without using the transport equation. Nonetheless, in what follows, we will need

THEOREM 2.1. *Let $1 < p < \infty$, $f \in L^p(\Omega)$ and let $\Omega \in C^{0,1}$ be a bounded domain of $\mathbb{R}^N$. Let $\boldsymbol{u} \in W^{1,p'}(\Omega)$, $\operatorname{div} \boldsymbol{u} \in L^\infty(\Omega)$ and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ at $\partial\Omega$. Let $\|\operatorname{div}\boldsymbol{u}\|_\infty < p'$. Then there exists exactly one weak solution to the transport equation*

$$\varrho + \boldsymbol{u}\cdot\nabla\varrho + \varrho \operatorname{div} \boldsymbol{u} = f \tag{2.2}$$

in $X_u^{p,p}(\Omega)$. Moreover

$$\|\varrho\|_p \leq \frac{p'\|f\|_p}{p' - \|\operatorname{div} \boldsymbol{u}\|_\infty}\,.$$

PROOF. Although this result is well known (see [1], [4]), the authors usually assume the Green formula to hold; as shown above and in [3], this is for weak solution far from being evident. Now, having the Green formula, the theorem can be shown in a standard way; we e.g. add to (2.2) the strongly elliptic term $-\varepsilon\Delta\varrho$, establish for any $\varepsilon > 0$ the existence of a solution ϱ_ε to this perturbed problem by means of the Lax-Milgram lemma and get an ε independent estimate of ϱ_ε. Thus a weak limit ϱ of ϱ_ε is a weak solution to (2.2). The uniqueness now follows easily; let ϱ_1 and ϱ_2 be the two possibly different solutions. Then testing the difference of equations for ϱ_1 and ϱ_2 by $|\varrho_1 - \varrho_2|^{p-2}(\varrho_1 - \varrho_2)$ yields

$$\|\varrho_1 - \varrho_2\|_p^p \left(1 - \frac{\|\operatorname{div} \boldsymbol{u}\|_\infty}{p'}\right) \leqslant 0\,,$$

i.e. $\varrho_1 = \varrho_2$ a.e. in Ω. ■

Next we would like to show the following density result

THEOREM 2.2. *Let $\Omega \in C^{0,1}$ be a bounded domain of $\mathbb{R}^N$. Let $1 < p < \infty$ and let $\boldsymbol{u}$ satisfy* (2.1). *Then the space $C_0^\infty(\Omega)$ is dense in the space $X_{\boldsymbol{u}}^{p,p}(\Omega)$.*

In order to prove Theorem 2.2, we first have to characterize the continuous linear functionals on $X_{\boldsymbol{u}}^{p,p}(\Omega)$. We have the following representation formula

LEMMA 2.1. *Under the assumptions of Theorem 2.2 we have that the space $X_{\boldsymbol{u}}^{p,p}(\Omega)$ is a separable and reflexive space. Moreover, let $\mathcal{F}$ be a continuous linear functional on $X_{\boldsymbol{u}}^{p,p}(\Omega)$. Then there exists at least one $g \in X_{\boldsymbol{u}}^{p',p'}(\Omega)$ such that*

$$\mathcal{F}(\varrho) = \int_\Omega g\varrho \,\mathrm{d}\boldsymbol{x} + \int_\Omega (\boldsymbol{u}\cdot\nabla g)(\boldsymbol{u}\cdot\nabla\varrho)\,\mathrm{d}\boldsymbol{x}\,.$$

PROOF. First let us note that we can without loss of generality assume that $\|\operatorname{div} \boldsymbol{u}\|_\infty$ is sufficiently small. Now let us define an operator T on $X_{\boldsymbol{u}}^{p,p}(\Omega)$ into $L^p(\Omega) \times L^p(\Omega)$ by $\mathrm{T}\varrho = (\varrho, \boldsymbol{u}\cdot\nabla\varrho)$. Thus $X_{\boldsymbol{u}}^{p,p}(\Omega)$ is isometrically isomorph to R(T) which is a closed subspace of $L^p(\Omega) \times L^p(\Omega)$ and we can conclude that the first part of Lemma 2.1 holds true.

Let $\mathcal{F} \in (X_{\boldsymbol{u}}^{p,p}(\Omega))'$. We define Φ a continuous linear functional on R(T) by

$$\Phi(\mathrm{T}\varrho) = \mathcal{F}(\varrho)\,.$$

We may, due to the Hahn-Banach theorem, extend Φ onto $L^p(\Omega) \times \times L^p(\Omega)$ and we get the following representation

$$\mathcal{F}(\varrho) = \int_\Omega f_1 \varrho \,\mathrm{d}\boldsymbol{x} + \int_\Omega f_2(\boldsymbol{u}\cdot\nabla\varrho)\,\mathrm{d}\boldsymbol{x}\,, \tag{2.3}$$

where $f_i \in L^{p'}(\Omega)$, $i = 1, 2$.

Now under the assumptions (2.1) there exists a unique solution $(G_1, G_2) \in X_{\boldsymbol{u}}^{p',p'}(\Omega) \times X_{\boldsymbol{u}}^{p',p'}(\Omega)$ of the following system

$$\begin{aligned} &\frac{1}{2}(f_1 + f_2) = G_1 + \boldsymbol{u} \cdot \nabla G_1 + \frac{1}{2}(\operatorname{div} \boldsymbol{u})(G_1 - G_2) \\ &\frac{1}{2}(f_1 - f_2) = G_2 - \boldsymbol{u} \cdot \nabla G_2 + \frac{1}{2}(\operatorname{div} \boldsymbol{u})(G_1 - G_2). \end{aligned} \tag{2.4}$$

To show this, it is enough to define an operator $\mathrm{S}: L^{p'}(\Omega) \to L^{p'}(\Omega)$ such that $\mathrm{S}(\omega) = G_2$ where

$$\frac{1}{2}(f_1 + f_2) = G_1 + \boldsymbol{u} \cdot \nabla G_1 + \frac{1}{2}(\operatorname{div} \boldsymbol{u})(G_1 - \omega)$$

$$\frac{1}{2}(f_1 - f_2) = G_2 - \boldsymbol{u} \cdot \nabla G_2 + \frac{1}{2}(\operatorname{div} \boldsymbol{u})(G_1 - G_2).$$

Since $\|\operatorname{div} \boldsymbol{u}\|_\infty$ is small, we easily have that the operator S is a contraction and the existence of the unique solution to (2.4) can be established by the Banach fixed point theorem. Thus we can write

$$f_1 = g + \boldsymbol{u} \cdot \nabla h + (\operatorname{div} \boldsymbol{u})\, h$$

$$f_2 = h + \boldsymbol{u} \cdot \nabla g\,,$$

where $g = G_1 + G_2 \in X_{\boldsymbol{u}}^{p',p'}(\Omega)$ and $h = G_1 - G_2 \in X_{\boldsymbol{u}}^{p',p'}(\Omega)$. Inserting this into (2.3) we have

$$\mathcal{F}(\varrho) = \int_\Omega (g + \boldsymbol{u} \cdot \nabla h + (\operatorname{div} \boldsymbol{u})\, h)\, \varrho \,\mathrm{d}\boldsymbol{x} + \int_\Omega (h + \boldsymbol{u} \cdot \nabla g)(\boldsymbol{u} \cdot \nabla \varrho)\,\mathrm{d}\boldsymbol{x}\,.$$

Finally, applying Lemma 1.3, we obtain the following representation formula

$$\mathcal{F}(\varrho) = \int_\Omega g \varrho \,\mathrm{d}\boldsymbol{x} + \int_\Omega (\boldsymbol{u} \cdot \nabla g)(\boldsymbol{u} \cdot \nabla \varrho)\,\mathrm{d}\boldsymbol{x}\,.$$

Thus to show Theorem 2.2 let us take $g \in X_{\boldsymbol{u}}^{p',p'}(\Omega)$. Suppose that

$$\left[\int_\Omega g \varrho \,\mathrm{d}\boldsymbol{x} + \int_\Omega (\boldsymbol{u} \cdot \nabla g)(\boldsymbol{u} \cdot \nabla \varrho)\,\mathrm{d}\boldsymbol{x} = 0,\ \forall \varrho \in C_0^\infty(\Omega)\right] \Rightarrow g = 0\,, \tag{2.5}$$

then Theorem 2.2 is shown. Assume that the left-hand side of implication (2.5) holds. Denote $\omega = \boldsymbol{u} \cdot \nabla g$. Then evidently $\omega \in L^{p'}(\Omega)$ and we have in the sense of distributions

$$g = \omega \operatorname{div} \boldsymbol{u} + \boldsymbol{u} \cdot \nabla\omega . \tag{2.6}$$

Therefore $\boldsymbol{u} \cdot \nabla\omega \in L^{p'}(\Omega)$ and thus $\omega \in X_{\boldsymbol{u}}^{p', p'}(\Omega)$.

We conclude the proof of Theorem 2.2 by the following lemma

LEMMA 2.2. *Under the assumptions of Theorem* 2.2 *implication* (2.5) *holds true.*

PROOF. We take $\delta > 0$ and we multiply equality (2.6) by $\frac{g}{(1 + |g| + \delta|\omega|)^{2-p'}}$ if $p' < 2$ and by g if $p' > 2$. In the last case, since evidently g and ω belong to $X_{\boldsymbol{u}}^{p', p'}(\Omega)$, we can apply Lemma 1.3 to get

$$\|g\|_2^2 = \int_\Omega (\operatorname{div} \boldsymbol{u})\, \omega g \, \mathrm{d}\boldsymbol{x} + \int_\Omega (\boldsymbol{u} \cdot \nabla\omega)\, g \, \mathrm{d}\boldsymbol{x}$$

$$= - \int_\Omega (\boldsymbol{u} \cdot \nabla g)\, \omega \, \mathrm{d}\boldsymbol{x} = - \int_\Omega |\boldsymbol{u} \cdot \nabla g|^2 \, \mathrm{d}\boldsymbol{x} ,$$

and thus $g = 0$. Now if $p' < 2$ we must proceed more carrefully. We have

$$\int_\Omega \frac{g^2}{(1 + |g| + \delta|\omega|)^{2-p'}} \mathrm{d}\boldsymbol{x} = \int_\Omega \frac{(\operatorname{div} \boldsymbol{u})\, \omega g}{(1 + |g| + \delta|\omega|)^{2-p'}} \mathrm{d}\boldsymbol{x}$$

$$+ \int_\Omega \frac{(\boldsymbol{u} \cdot \nabla\omega)\, g}{(1 + |g| + \delta|\omega|)^{2-p'}} \mathrm{d}\boldsymbol{x} .$$

We want to apply the Green formula in the last term of the right hand side. Since g and $\boldsymbol{u} \cdot \nabla\omega$ belong to $L^{p'}(\Omega)$, $p' < 2$, we must show the Green formula separately for this case, see Lemma 2.3 below. Applying Lemma 2.3 we get

$$\int_\Omega \frac{g^2}{(1 + |g| + \delta|\omega|)^{2-p'}} \mathrm{d}\boldsymbol{x} = - \int_\Omega \frac{(\boldsymbol{u} \cdot \nabla g)\, \omega}{(1 + |g| + \delta|\omega|)^{2-p'}} \mathrm{d}\boldsymbol{x}$$

$$+ (2 - p') \int_\Omega \frac{(\boldsymbol{u} \cdot \nabla g)\, |g| \omega}{(1 + |g| + \delta|\omega|)^{3-p'}} \mathrm{d}\boldsymbol{x}$$

$$+ (2 - p') \int_\Omega \frac{\delta(\boldsymbol{u} \cdot \nabla\omega)\, g |\omega|}{(1 + |g| + \delta|\omega|)^{3-p'}} \mathrm{d}\boldsymbol{x} .$$

Evidently, we have

$$\int_{\Omega} \frac{g^2}{(1+|g|+\delta|\omega|)^{2-p'}}\,dx + (p'-1)\int_{\Omega} \frac{(u\cdot\nabla g)^2}{(1+|g|+\delta|\omega|)^{2-p'}}\,dx$$

$$\leqslant (2-p')\int_{\Omega} \frac{\delta|u\cdot\nabla\omega|\,|g|\,|\omega|}{(1+|g|+\delta|\omega|)^{3-p'}}\,dx\,.$$

Since

$$\int_{\Omega} \frac{\delta|u\cdot\nabla\omega|\,|g|\,|\omega|}{(1+|g|+\delta|\omega|)^{3-p'}}\,dx \leqslant \int_{\Omega} |u\cdot\nabla\omega|(1+|g|+\delta|\omega|)^{p'-1}\,dx \leqslant C\,,$$

where C is a constant independent of δ, we can pass with δ to zero by means of the Lebesgue dominated convergence theorem to obtain

$$\int_{\Omega} \frac{g^2}{(1+|g|+|\omega|)^{2-p'}}\,dx \leqslant 0\,,$$

which finshes the proof of Lemma 2.2 and thus the proof of Theorem 2.2. ■

LEMMA 2.3. *Under the assumptions of Theorem 2.2 and for* $\omega, g \in X_u^{p',p'}(\Omega)$ *we have the Green formula*

$$\int_{\Omega} \frac{(u\cdot\nabla\omega)\,g}{(1+|g|+\delta|\omega|)^{2-p'}}\,dx = -\int_{\Omega} \frac{(\operatorname{div} u)\,\omega g}{(1+|g|+\delta|\omega|)^{2-p'}}\,dx$$

$$-\int_{\Omega} \frac{(u\cdot\nabla g)\,\omega}{(1+|g|+\delta|\omega|)^{2-p'}}\,dx + (2-p')\int_{\Omega} \frac{(u\cdot\nabla g)|g|\omega + \delta(u\cdot\nabla\omega)\,g|\omega|}{(1+|g|+\delta|\omega|)^{3-p'}}\,dx\,.$$

PROOF. We take ω_k and g_k such that $\omega_k \to \omega$, $g_k \to g$ in $L^{p'}(\Omega)$ and $u\cdot\nabla\omega_k \to u\cdot\nabla\omega$, $u\cdot\nabla g_k \to u\cdot\nabla g$ in $L^1(\Omega)$ (see Theorem 1.1). Let $\varepsilon > 0$. Then we apply the Green formula on the term

$$\int_{\Omega} \frac{(u\cdot\nabla\omega_k)\,g_k}{(1+|g_k|+\delta|\omega_k|)^{2-p'}}\,\frac{1}{(1+\varepsilon|g_k|^{p'-1}+\varepsilon|\omega_k|^{p'-1})}\,dx\,,$$

and pass first with k to infinity and finally with ε to zero. Since the technique is very similar to this one used in the proof of Theorem 1.2, we omit the details here. ■

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Manoscritto pervenuto in redazione il 25 luglio 2000.