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On Quasi-Projective Uniserial Modules.

DMITRI ALEXEEV (*)

ABSTRACT - Let R be a valuation domain with maximal ideal P . We study quasi-projective uniserial modules over R . By making use of the absence of «shrinkable» uniserial modules over R we prove our main result: a characterization of quasi-projectivity of a uniserial module U over R in terms of lifting of endomorphisms of factors of U . Using this characterization allows us to describe quasi-projective ideals of R in terms of completeness of certain localizations of factor-rings of R . Archimedean ideals of R admit the best possible description from this point of view. We show that a non-principal archimedean ideal of R is quasi-projective if and only if R/K is complete in the R/K -topology for each archimedean ideal $K \neq P$. Finally, we show that taking tensor products with archimedean ideals preserves quasi-projectivity.

1. Uniserial modules.

Let us begin with necessary definitions.

DEFINITIONS. A module over a ring is called *uniserial* if its submodules form a chain under inclusion. A commutative integral domain R is a *valuation domain* if it is a uniserial module over itself.

From now on, the letter R will denote a valuation domain with maximal ideal P and quotient field Q , unless stated otherwise. We refer the reader to Fuchs and Salce [6] for a treatment of modules over valuation domains.

Submodules and factor modules of uniserial modules are likewise

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uniserial. The simplest examples of uniserial modules are besides the valuation rings themselves, their rings of quotients, their cyclic modules, and more generally, the R -modules of the form I/J where $J < I$ are R -submodules of Q . Uniserial modules of the latter kind are called *standard uniserial*. The existence of non-standard uniserial modules has been first established by Shelah in [10].

Standard uniserial modules are completely classified by the following proposition.

PROPOSITION 1.1 (Shores and Lewis [11]). *Two standard uniserial modules I/J and I'/J' over a valuation domain R are isomorphic if and only if there exists an element $0 \neq q \in Q$ such that $I = qI'$ and $J = qJ'$.*

Several properties of a uniserial module depend on its type.

DEFINITIONS. Let R be a valuation domain with the maximal ideal P . We use the following notation from [6]. If U is a torsion uniserial and $0 \neq u \in U$, we set

$$I = H(u) = \{r^{-1} \mid u \in rU\} \quad \text{and} \quad J = \text{Ann } u = \{r \in R \mid ru = 0\}.$$

The fractional ideal I is called the height ideal of u , and we say that U is of type $[I/J]$ (the isomorphy class of I/J), $t(U) = [I/J]$. The type $t(U)$ does not depend on the choice of $0 \neq u \in U$. For a uniserial module U we define ideals

$$U^\# = \{r \in R \mid rU < U\} \quad \text{and} \quad U_\# = \{r \in R \mid ra = 0 \text{ for some } 0 \neq a \in U\}.$$

It is easy to verify that both $U^\# = I^\#$ and $U_\# = J_\#$ are prime ideals of R containing the annihilator $\text{Ann } U$. An ideal I of R is called *archimedean* if $I^\# = P$.

It is a good time to introduce the main objects of our study: the quasi-projective modules. In the following definition, R may be an arbitrary ring.

DEFINITION. An R -module U is called quasi-projective if it is projective relative to all exact sequences of the form $0 \rightarrow V \rightarrow U \xrightarrow{\pi} U/V \rightarrow 0$, where V is a submodule of U and π is the canonical projection. That is, for every homomorphism $f: U \rightarrow U/V$ there exists a map $f': U \rightarrow U$ such

that the following diagram commutes:

$$\begin{array}{ccccccc}
 & & & & U & & \\
 & & & f' \swarrow & \downarrow f & & \\
 0 & \longrightarrow & V & \longrightarrow & U & \xrightarrow{\pi} & U/V \longrightarrow 0
 \end{array}$$

For more on quasi-projective modules via relative projectivity see [7].

As the following proposition states, possibilities of endomorphisms of quasi-projective uniserials are limited.

PROPOSITION 1.2. *A surjective endomorphism of a quasi-projective uniserial module U over any ring R is an automorphism.*

PROOF. Uniserial modules are indecomposable. Fuchs and Rangaswamy show in [5] that if a factor U/V of a quasi-projective module U is isomorphic to a summand of U , then V is also isomorphic to a summand of U . Thus, if $\varphi : U \rightarrow U$ is a surjective endomorphism with non-trivial kernel V , then $U/V \cong U$ and V must be isomorphic to U . Since V is a proper submodule of U , it has to be standard uniserial. Thus, U is standard uniserial either. Suppose that $U = I/J$ for some $J < I \leq Q$. Then $V = I_1/J$ for some I_1 such that $J < I_1 < I$. By Proposition 1, there are non-zero elements p, q in Q such that

$$I_1 = pI, \quad J = pJ, \quad I = qI, \quad J = qI_1.$$

Thus, we have $pqI = J = pqI_1$, which implies $I = I_1$. This renders $U/V \cong \cong U$ impossible, a contradiction. ■

The following corollary is an immediate consequence of Proposition 1.2.

COROLLARY 1.3. *Let R be a valuation ring. If U is a quasi-projective uniserial module over R , then $U_{\#} \leq U^{\#}$. Moreover, if U is non-standard, then $U_{\#} = U^{\#}$.*

PROOF. Suppose that U is a quasi-projective uniserial R -module. If $r \in R$ then multiplication by r is a surjective endomorphism of U if and only if $r \notin U^{\#}$ and an injective endomorphism if and only if $r \notin U_{\#}$. The first statement follows now from Proposition 1.2.

If U is non-standard quasi-projective, then every monic endomorphi-

sm of U has to be an isomorphism because all proper submodules of U are standard uniserials. This implies $U^\sharp \leq U_\sharp$ and, consequently, $U^\sharp = U_\sharp$. ■

The following Lemma uses a result by Facchini and Salce [3]. They call a uniserial module U over an arbitrary ring *shrinkable* if $U \cong V/W$ for some proper submodules $0 < W < V < U$. It is proved in [3] that there are no shrinkable uniserial modules over commutative or Noetherian rings. We use the result to prove a more general statement.

LEMMA 1.4. *Let U and V be two uniserial modules over a valuation domain R . If there exist both a monomorphism f and an epimorphism g from U to V then at least one of f and g is an isomorphism. Moreover, if both U, V are standard and*

- (a) $V_\sharp < V^\sharp$, then g is an isomorphism;
- (b) $V^\sharp < V_\sharp$, then f is an isomorphism.

PROOF. There are four possibilities depending on whether U and V are standard or not. If both U and V are non-standard, then every monomorphism $f : U \rightarrow V$ has to be onto. If U is standard and V is not, then no epimorphism $g : U \rightarrow V$ exists. Similarly, if U is non-standard and V is standard then no monomorphism f exists. It remains to consider the case of standard U and V .

There is nothing to prove if f or g is an isomorphism. Suppose that f maps monomorphically onto a proper submodule of V and g has a non-trivial kernel. We claim that this is impossible. To prove the claim, we write $U = I/J$ and $V = I'/J'$, with submodules $J < I, J' < I'$ of Q . Then $\text{Im } f \cong I_1'/J'$ and $\text{Ker } g \cong J_1/J$ for appropriate J_1 and I_1' in Q . We have isomorphisms

$$V = I'/J' \cong I/J_1 \cong I_1'/J_1',$$

implying that V is shrinkable, which is impossible by [3]. This proves the first statement.

We conclude that whenever there is a pair f, g satisfying the hypotheses, then either $J_1' = J'$ or $I' = I_1'$. These possibilities are defined by the inclusions of the «sharps» of V and imply that either g is monic or f is epic, accordingly. ■

We are going to show that the quasi-projective property of uniserial modules is closely related to the property of lifting endomorphisms. The following definition does not require R to be a valuation domain.

DEFINITION. Let M be an R -module and N its submodule. We say that an endomorphism h of M/N can be lifted or lifts to M , if there exists an endomorphism h' of M making the following diagram commutative:

$$\begin{array}{ccc} M & \xrightarrow{\quad h' \quad} & M \\ \pi \downarrow & & \downarrow \pi \\ M/N & \xrightarrow{\quad h \quad} & M/N \end{array}$$

Here π denotes the canonical projection. If all the endomorphisms of each factor of a module M lift, then M will be called *weakly quasi-projective*. This term was introduced by Rangaswamy and Vanaja in [9].

Trivially, a quasi-projective module is weakly quasi-projective. The converse is not true even for uniserial modules: the abelian group $\mathbb{Z}(p^\infty)$ is an example of a weakly quasi-projective but not quasi-projective uniserial module over $\mathbb{Z}$. Interestingly, in case of valuation domains, the necessary condition of weak quasi-projectivity together with $U_\# \leq U^\#$ is also sufficient for U to become quasi-projective.

THEOREM 1.5. *Let R be a valuation domain and U a uniserial R -module. The following conditions are equivalent:*

- (a) U is weakly quasi-projective and $U_\# \leq U^\#$.
- (b) U is quasi-projective.

PROOF OF (a) $\Rightarrow$ (b). Suppose that every $h \in \text{End}_R U/V$, for each $V < U$, can be lifted to an element of $\text{End}_R U$ and $U_\# \leq U^\#$. Let V be a submodule of U and $f : U \rightarrow U/V$ a homomorphism. Thus, we are given the solid part of the following diagram, where π_1 is the canonical projection:

$$\begin{array}{ccccccc} & & & & U & & \\ & & & & \downarrow f & & \\ 0 & \longrightarrow & V & \longrightarrow & U & \xrightarrow{\pi_1} & U/V \longrightarrow 0 \end{array}$$

If we denote the kernel of f by W , then f factors through the canonical projection $\pi_2 : U \rightarrow U/W$ and an inclusion $i : U/W \rightarrow U/V$. Since U is uniserial, either $V = W$, $V < W$, or $W < V$.

If $V = W$, then $i \in \text{End}_R U/V$. It lifts to U by assumption.

If $V < W$, then there is canonical projection $\pi_3: U/V \rightarrow U/W$. Embedding it in the previous diagram results in the following:

$$\begin{array}{ccccccc}
 & & & & U/V & & \\
 & & & \nearrow \pi_1 & \downarrow \pi_3 & & \\
 & & U & \xrightarrow{\pi_2} & U/W & & \\
 & \nwarrow f' & \downarrow f & \nearrow i & & & \\
 0 & \longrightarrow & V & \longrightarrow & U & \xrightarrow{\pi_1} & U/V \longrightarrow 0
 \end{array}$$

The composition $i \circ \pi_3$ is an endomorphism of U/V and can be lifted to a map $f': U \rightarrow U$ by the assumption.

If $W < V$, then we have the canonical epimorphism $\pi: U/W \rightarrow U/V$, and so by Lemma 1, i is an isomorphism. Fixing submodules I, J of Q such that $t(U) = [I/J]$, we can write $V \cong I_1/J$ and $W \cong I_2/J$, where $J < I_2 < I_1 < I$ and the latter isomorphism is the restriction of the former. Since $U/V \cong U/W$, we must have $I/I_1 \cong I/I_2$. By Proposition 1, there exists a $q \in R$ such that $qI = I$ and $qI_1 = I_2$. This means $q \notin I^\# = U^\# \geq U_\#$. Therefore, multiplication $\dot{q}$ by q is an automorphism of U such that $\dot{q}V = W$. It naturally induces an isomorphism $\bar{q}: U/V \rightarrow U/W$. In this case we have the following diagram:

$$\begin{array}{ccccccc}
 & & U & \xrightarrow{\pi_2} & U/W & & \\
 & & \downarrow g & \searrow f & \downarrow i & & \\
 & & U & \xrightarrow{\pi_1} & U/V & \longrightarrow & 0 \\
 & \nwarrow f' & \downarrow \dot{q} & \nearrow \bar{q} & & & \\
 0 & \longrightarrow & V & \longrightarrow & U & \xrightarrow{\pi_2} & U/W
 \end{array}$$

Here, g is a lifting of $\bar{q} \circ i$. The map $f' \equiv \dot{q}^{-1} \circ g$ is a desired lifting of f .

(b) $\Rightarrow$ (a). This implication is trivial. $\blacksquare$

In particular, for ideals of R , weak quasi-projectivity always implies quasi-projectivity.

2. Endomorphisms of uniserial modules.

Since quasi-projectivity depends on the lifting property of endomorphisms, knowing the structure of endomorphism rings of uniserial modules will enhance our understanding of their quasi-projectivity.

If U is a uniserial module and $\text{Ann } U$ is its annihilator ideal, then we use the following notation from Fuchs and Salce [6]: we write $\text{Ann } U = I$ if there is a $u \in U$ such that $\text{Ann } u = \text{Ann } U$, and we write $\text{Ann } U = I^+$ otherwise. The module U is called *finitely annihilated* or *non-finitely annihilated*, respectively. It is known that a uniserial U of type $[I/J]$ is non-finitely annihilated exactly if $J^\# I = I$. See Bazzoni, Fuchs and Salce [2] for details.

Thus, there are two types of uniserial modules distinguished by the structure of their annihilator ideals. Each type has a specific kind of the endomorphism ring, described by the following proposition.

THEOREM 2.1 (Shores and Lewis [11]). *Let R be a valuation ring and U a uniserial R -module.*

(a) *If $\text{Ann } U = I$, then $U_\# \leq U^\#$ and U carries the natural structure of an S -module, where S is the ring R/I localized at $U^\#/I$, and $\text{End}_R U \cong S$.*

(b) *If $\text{Ann } U = I^+$, then $U^\# \leq U_\#$ and U carries the natural structure of a T -module, where T is the ring R/I localized at $U_\#/I$, and $\text{End}_R U$ is isomorphic to the completion $\widehat{T}$ of T in the T -topology.*

The endomorphisms of U can be viewed as multiplications by appropriate elements from either S or $\widehat{T}$.

COROLLARY 2.2. *A valuation domain R is maximal if and only if all uniserial R -modules U with $U_\# \leq U^\#$ are quasi-projective.*

PROOF. Theorem 3.5 of Herrmann [8] states that R is maximal if all submodules of Q are quasi-projective. This proves sufficiency of the condition.

Conversely, if R is maximal, then R and all its factors are linearly compact in the discrete topology. Thus all uniserial R -modules are standard. Hence, for every uniserial module $U = I/J$, $J < I \leq Q$, with $U_\# \leq U^\#$, the endomorphism ring $\text{End}_R(U)$ is the localization $(R/\text{Ann } U)_{(U^\#/\text{Ann } U)}$, which is also maximal. Therefore, endomorphisms U and its factors are induced by multiplications by appropriate elements of $R_{U^\#}$. Thus, every

such U is weakly quasi-projective. The statement follows from Theorem 1.5. ■

Corollary 2.2 for ideals is a particular case of a more general result by Fuchs [4]. He proves that for a cardinal κ a valuation domain R is κ -maximal if and only if every κ -generated ideal of R is quasi-projective.

We are going to show that quasi-projectivity of a uniserial module is completely determined by its type.

THEOREM 2.3. *Let R be a valuation domain and U be a uniserial R -module. Suppose that $t(U) = [I/J]$ ($J < I \leq Q$). Then U is quasi-projective if and only if I/J is.*

PROOF. By Theorem 1.5, the quasi-projectivity of (non-) standard uniserial U satisfying $U_{\sharp} \leq U^{\sharp}$ ($U_{\sharp} = U^{\sharp}$) is equivalent to its weak quasi-projectivity. The «sharp» ideals associated to U and I/J are necessarily the same. Thus, we have to show that the two modules are weakly quasi-projective at the same time. Suppose that V is a (proper) submodule of U , $V \leq U$. We can choose $I_1 \leq I$ such that $t(V) = [I_1/J]$ and $t(U/V) = [I/I_1]$. On the other hand, for any given ideal I_1 ($J \leq I_1 \leq I$) there is a submodule V of U with type as above. Since uniserial modules of the same type have naturally isomorphic endomorphism rings, the endomorphisms of U/V lift to U if and only if those of I/I_1 lift to I/J . The proof is finished. ■

COROLLARY 2.4. *Let U be a uniserial R -module. If $\text{Tor}_R^1(U, K/L)$ ($L < K \leq Q$) is quasi-projective, then $\text{Tor}_R^1(U, V)$ is quasi-projective for each V of type $[K/L]$.*

PROOF. It is known (see [2]) that, for uniserial R -modules U and V with types $[I/J]$ and $[K/L]$ respectively, the Tor_R^1 product of U and V is uniserial and

$$t(\text{Tor}_R^1(U, V)) = \left[\frac{IL \cap JK}{JL} \right].$$

Thus, $\text{Tor}_R^1(U, K/L)$ and $\text{Tor}_R^1(U, V)$ have the same type. A reference to Theorem 2.3 finishes the proof. ■

Bazzoni, Fuchs, and Salce prove in [2] that the isomorphism classes of torsion uniserial R -modules form a commutative semigroup under opera-

tion Tor_R^1 . Corollary 2 implies that the orbit of a module U under the Tor_R^1 operation with uniserials of the same type consists entirely of quasi-projective modules if it contains at least one quasi-projective.

3. Ideals of valuation domains.

In this section we use Theorems 1.5 and 2.1 to study quasi-projectivity of ideals of a valuation domain R . We will show that quasi-projectivity of an archimedean ideal depends on the completeness of certain localizations of factor-rings of R .

THEOREM 3.1. *Let R be a valuation domain. An ideal I of R is quasi-projective if and only if for each ideal $J < I$ with $\text{Ann } I/J = (J : I)^+$ the ring*

$$S = (R/(J : I))_{(J^\#/J : I)}$$

is complete in its S -topology.

PROOF. This is a direct consequence of the Theorem 1 and Theorem 2. Since for an ideal I , $0 = I^\# \leq I^\#$ always, the quasi-projectivity of I is equivalent to lifting of each endomorphism of every factor I/J to I . When $\text{Ann } I/J = (J : I)^+$ and the ring S is not complete, the ring $\text{End}_R I/J = \hat{S}$ contains elements which are not induced by elements of $\text{End}_R I = R_{I^\#}$. The absence of such elements is equivalent to the completeness of S (for every such J), and the quasi-projectivity of I . ■

It is possible to give a more explicit characterization of quasi-projectivity of archimedean ideals. There are two cases, according as P is principal or not. If P is principal, then only the principal ideals are archimedean. Thus, they all are projective and, therefore, quasi-projective. To consider the second alternative, we need a special case of Lemma 1.4 from [2]. The proof is provided for the sake of completeness.

LEMMA 3.2. *Let $J < I$ be archimedean ideals of a valuation domain R . Then $\text{Ann } I/J = (J : I)^+$ if and only if I is not principal.*

PROOF. The «only if» part is trivial. To prove the converse, let I be an infinitely generated archimedean ideal. Assume that there is an element $i \in I$ such that $\text{Ann } iR/J = \text{Ann } I/J$. Since I is not principal, we can choose $i' \in I$ such that $iR < i'R < I$. That is, $i = si'$ for a non-unit s of R .

Then $\text{Ann } iR/J = \text{Ann } i'R/J$, hence $i^{-1}J = i'^{-1}J$, and i, i' are associates. A contradiction. Hence, no such element $i \in I$ exists and $\text{Ann } I/J = (J : I)^+$. ■

THEOREM 3.3. *Let R be a valuation domain with infinitely generated maximal ideal P . A non-principal archimedean ideal I is quasi-projective if and only if R/K is complete in the R/K -topology for each archimedean ideal $K \neq P$.*

PROOF. By Theorem 3.1, I is quasi-projective if and only if R/K is complete whenever $\text{Ann } (I/J) = K^+$ for some ideal J . The ideal J has to be archimedean by Theorem 2.1. It remains to prove that for different J , $\text{Ann } (I/J)$ can be any archimedean ideal $\neq P$, but nothing else.

If $K \neq P$ then $\text{Ann } I/IK = K^+$.

If $K \cong P$ then $K = sP$ for some $s \in R$. Assume that $\text{Ann } I/J = K^+$ for some (archimedean) ideal J . We have $IK = IsP = sI \leq J$. But then $(sR)I \leq J$, which means that I/J has annihilator larger than K . Contradiction. Hence, ideals isomorphic to P cannot be annihilators of I/J . This completes the proof. ■

The following corollary is an immediate consequence of Theorem 3.3.

COROLLARY 3.4. *Let R be a valuation domain with infinitely generated maximal ideal P . If there exists a non-principal quasi-projective archimedean ideal, then all archimedean ideals of R are quasi-projective.* ■

It is difficult to give a more explicit characterization of the quasi-projectivity of non-archimedean ideals beyond Theorem 3.1. However, we show that one has a way of «producing» new quasi-projective ideals starting with a quasi-projective ideal I and taking tensor products of I with archimedean ideals. This result is in a way analogous to Theorem 3.3 from [1].

First, we need the following lemma.

LEMMA 3.5. *Let R be a valuation domain with maximal ideal P . Suppose that I is a non-principal ideal of R . If $0 \neq r \in R$, then the endomorphism rings of I/rR and I/rP are (naturally) isomorphic.*

PROOF. This is a consequence of Theorem 2.1 and the following two observations. Firstly, I/rR and I/rP have the same annihilator A^+ ,

where $A = rR : I = rP : I$. For, if $sI \subseteq rR$ ($0 \neq s \in A$), then $sI \subseteq rP$ since I is not principal and rP is the maximal submodule of rR . Secondly, both modules have the same «sharp» ideals because rR and rP are archimedean. By Theorem 2.1, the endomorphism rings of I/rR and I/rP are isomorphic to the completion of R/A in the R/A -topology. This isomorphism is natural. ■

We have the following theorem.

THEOREM 3.6. *Let R be a valuation domain with non-principal maximal ideal P . Suppose that I is a non-principal ideal of R and J is an archimedean ideal. Then I is quasi-projective if and only if $I \otimes_R J$ is.*

PROOF. Over valuation domains $I \otimes_R J$ is naturally isomorphic to IJ . This allows us to consider submodules of $I \otimes_R J$ as ideals contained in IJ and vice-versa. For ideals of valuation domains the quasi-projectivity is equivalent to the weak quasi-projectivity. Thus we need to prove that I is weakly quasi-projective if and only if $I \otimes_R J$ is. Since the arguments in each way are similar, we give the proof of the «if» part only. The case of principal J is trivial.

Suppose that $I \otimes_R J$ is weakly quasi-projective, $K < I$ and $\varphi \in \text{End}_R(I/K)$. If K is not principal, tensoring with J is «reversible», that is, $K \otimes_R J \otimes_R (R : J) \cong K \otimes_R P \cong K$. The isomorphisms are natural. Consider the following diagrams.

$$\begin{array}{ccc}
 \begin{array}{ccc}
 I & \xrightarrow{\quad f' \quad} & I \\
 \pi \downarrow & & \downarrow \pi \\
 \frac{I}{K} & \xrightarrow{\quad \varphi \quad} & \frac{I}{K}
 \end{array} & \xrightarrow[\otimes_R J]{} & \begin{array}{ccc}
 I \otimes_R J & \xrightarrow{\quad f \quad} & I \otimes_R J \\
 \pi \otimes_R 1 \downarrow & & \downarrow \pi \otimes_R 1 \\
 \frac{I \otimes_R J}{K \otimes_R J} & \xrightarrow{\quad \varphi \otimes_R 1 \quad} & \frac{I \otimes_R J}{K \otimes_R J}
 \end{array}
 \end{array}$$

Taking the tensor product of the solid part of the diagram on the left with J , one obtains the diagram on the right. Here, map f is a lifting of $\varphi \otimes_R 1$, which exists by assumption. Maps π and $\pi \otimes_R 1$ are canonical projections. Taking the tensor product with $R : J$, one returns to the left diagram. Map $f' \cong f \otimes_R 1_{R : J}$ is a lifting of φ , which makes the left square commute. It remains to consider the special case of principal K .

If $K = rR$ ($0 \neq r \in R$), then $K \otimes_R J \otimes_R (R : J) \cong rP$. By Lemma 3.5, the endomorphism $\varphi : I/rR \rightarrow I/rR$ is an element of the $R/\text{Ann}(I/rR)$ -completion of the ring $R/\text{Ann}(I/rR)$. Since this ring is also the endomor-

phism ring of I/rP , φ can be considered as an endomorphism of I/rP . As such, it can be lifted to I using technique of the previous paragraph. Clearly, this lifting is also a lifting of the original $\varphi : I/rR \rightarrow I/rR$. The proof is finished. ■

We conclude with the following observation. Since isomorphy classes of archimedean ideals form a group under the tensor product operation, application of Theorem 3.6 delivers an alternative proof of Corollary 3.4.

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