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Homogenization of Some Contact Problems for the System of Elasticity in Perforated Domains.

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ABSTRACT - Some unilateral problems for the system of linear elasticity are considered in a perforated domain with an ε-periodic structure. Boundary conditions characterizing friction are imposed on the surface of the cavities (or channels), while the body is clamped along the outer portion of its boundary. We investigate the asymptotic behavior of solutions to such boundary value problems for variational inequalities as $\varepsilon \to 0$ and construct the limit problem, according to the form of the friction forces and their dependence on the parameter ε. In some cases, this dependence results in additional restrictions on the set of admissible displacements in the homogenized problem which has the form of a variational inequality over a certain closed convex cone in a Sobolev space. This cone is described in terms of the functions involved in the nonlinear boundary conditions on the perforation and may depend on its geometry. A homogenization theorem is also proved for some unilateral problems with boundary conditions of Signorini type for the system of elasticity in a partially perforated domain.

1. A problem with boundary conditions of friction type.

Let Ω^ε be a perforated domain in $\mathbb{R}^n$ with an ε-periodic structure: $\Omega^\varepsilon = \Omega \cap \varepsilon\omega$, where Ω is a fixed bounded domain, ε is a small positive parameter, ω is an unbounded 1-periodic domain. Thus, ω is invariant under the shifts by all vectors with integer components; $\varepsilon\omega$ is its homothetic contraction with ratio ε, and $\omega_\square = \square \cap \omega$ is the cell of periodicity, where $\square =]0, 1[^n$. It is assumed that Ω, Ω^ε and $\omega_\square$ are

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domains (i.e., connected open sets) with Lipschitz continuous boundaries.

The set $\Omega\setminus\varepsilon\omega$ represents a perforation of Ω. This perforation may be formed by small cavities cut out of Ω, or by small channels penetrating Ω.

For an elastic body occupying Ω^ε, the *stress tensor* at a point $x = (x_1, \dots, x_n)$ is an $(n \times n)$ matrix $\boldsymbol{\sigma}(\boldsymbol{u}^\varepsilon) = \mathfrak{A}^\varepsilon(x)\, \boldsymbol{e}(\boldsymbol{u}^\varepsilon)$. Here $\boldsymbol{u}^\varepsilon = \boldsymbol{u}^\varepsilon(x)$ is the *displacement* identified with the column vector ${}^t(u_1^\varepsilon, \dots, u_n^\varepsilon)$, $\boldsymbol{e}(\boldsymbol{u}^\varepsilon)$ is the *linearized strain tensor*, i.e., the matrix with the elements $(\boldsymbol{e}(\boldsymbol{u}^\varepsilon))_{ij} = 2^{-1}(\partial u_i^\varepsilon/\partial x_j + \partial u_j^\varepsilon/\partial x_i)$; $\mathfrak{A}^\varepsilon(x)$ is the *elasticity tensor* identified with a linear transformation of the space $\mathbb{M}^n$ of real $(n \times n)$ matrices. This tensor has the form

$$\mathfrak{A}^\varepsilon(x) = \mathfrak{A}(\varepsilon^{-1}x) = \{a_{ij}^{hk}(\varepsilon^{-1}x)\}, \quad (\mathfrak{A}(\xi)\boldsymbol{p})_{jk} = a_{ij}^{hk}(\xi)\, p_{ih}, \quad \forall \boldsymbol{p} = \{p_{ih}\} \in \mathbb{M}^n.$$

The coefficients $a_{ij}^{hk}(\xi)$ are 1-periodic functions of $\xi \in \mathbb{R}^n$ which satisfy the usual conditions of symmetry and positive definiteness:

$$a_{ij}^{hk}(\xi) = a_{ji}^{kh}(\xi) = a_{hj}^{ik}(\xi), \qquad \kappa_1 b_{ih} b_{ih} \leqslant a_{ij}^{hk}(\xi)\, b_{ih} b_{jk} \leqslant \kappa_2 b_{ih} b_{ih},$$

for all real symmetric matrices $\{b_{ij}\} \in \mathbb{S}^n$ and all ξ, where $\kappa_1, \kappa_2 = \mathrm{const} > 0$. Here and in what follows, we assume summation over repeated indices from 1 to n, unless indicated otherwise; boldface letters denote matrices and column vectors.

By $\boldsymbol{p} : \boldsymbol{q} = p_{ij} q_{ij}$ we denote the scalar product of two matrices $\boldsymbol{p}$, $\boldsymbol{q} \in \mathbb{M}^n$, and by $\boldsymbol{\zeta}\cdot\boldsymbol{\eta} = \zeta_i \eta_i$ the scalar product of two column vectors $\boldsymbol{\zeta}$, $\boldsymbol{\eta} \in \mathbb{R}^n$. For a matrix valued function $\boldsymbol{p}(x) = \{p_{ij}(x)\}$, $\mathbf{div}\boldsymbol{p}$ is the column vector with the components $(\mathbf{div}\boldsymbol{p})_i = \partial p_{ij}/\partial x_j$.

For a bounded Lipschitz domain Q, the Sobolev space $H^1(Q)$ is the completion of $C^\infty(\overline{Q})$ with respect to the norm $\|u\|_{1,Q} = \|u\|_{L^2(Q)} + \|\nabla_x u\|_{L^2(Q)}$, and $H_0^1(Q)$ is its subspace formed by all $u \in H^1(Q)$ with zero trace on ∂Q. As usual, $\overline{A}$ is the closure of a set $A \subset \mathbb{R}^n$, ∂A is its boundary.

Inside Ω^ε, the displacements are supposed to satisfy the usual equations of static equilibrium $\mathbf{div}\, \boldsymbol{\sigma}(\boldsymbol{u}^\varepsilon) = -\boldsymbol{F}(x)$ with external body forces $\boldsymbol{F} \in (L^2(\Omega))^n$. On the *outer* portion of the boundary of Ω^ε, denoted by $\Gamma^\varepsilon = \partial\Omega \cap \partial\Omega^\varepsilon$, the body is clamped: $\boldsymbol{u}^\varepsilon|_{\Gamma^\varepsilon} = \mathbf{0}$, whereas on the *surface of the cavities* inside Ω, denoted by $S^\varepsilon = (\partial\Omega^\varepsilon) \cap \Omega$, we impose nonlinear boundary conditions, such as those expressing the *Coulomb law of contact with friction*, or the conditions of *normal displacement with friction* (see [DL] and Examples 1 and 2 below). In accordance with [DL], we formulate such boundary value problems in terms of variational inequalities involving convex nondifferentiable functionals, in other words, we consider weak solutions of these problems. Formal (i.e., regardless of regularity) equivalence of weak and «classical» solutions of problems with friction is discussed in [DL]. The existence and the uniqueness of weak solutions to variational inequalities considered in this section follow from the general results to be found, for instance, in [ET] and [L] (see also [DL]).

The present paper continues the studies started in [Y1], [Y2], where some other homogenization problems with nonlinear boundary conditions have been considered. In order to avoid too many auxiliary propositions, we will use readily available ones from [OSY], [Y1], [Y2], which makes us to assume the coefficients and the boundaries $\partial\Omega$, $\partial\omega$ sufficiently smooth, and to impose an additional assumption on the structure of the perforation: the set $\square\backslash\overline{\omega}$, as well as the intersection of $\square\backslash\overline{\omega}$ with a δ-neighborhood of $\partial\square$, consists of finitely many Lipschitz domains separated from one another and from the $(n-2)$-dimensional edges of the cube $\square$ by a positive distance. However, with the help of the results from [Y3], it would not be very difficult to reduce these assumptions to the case of bounded measurable coefficients and Ω, Ω^ε, $\omega_\square$ being Lipschitz domains.

We are going to study the asymptotic behavior (as $\varepsilon \to 0$) of solutions of the following problem for a variational inequality:

Find the displacement $\boldsymbol{u}^\varepsilon \in (H_0^1(\Omega^\varepsilon, \Gamma^\varepsilon))^n$ *such that for any* $\boldsymbol{v} \in (H_0^1(\Omega^\varepsilon, \Gamma^\varepsilon))^n$

$$(1) \qquad \int_{\Omega^\varepsilon} e(\boldsymbol{v}-\boldsymbol{u}^\varepsilon) : \mathfrak{A}(\varepsilon^{-1}x)\, e(\boldsymbol{u}^\varepsilon)\,\mathrm{d}x + J^\varepsilon(\boldsymbol{v}) - J^\varepsilon(\boldsymbol{u}^\varepsilon) \geqslant \int_{\Omega^\varepsilon} \boldsymbol{F}\cdot(\boldsymbol{v}-\boldsymbol{u}^\varepsilon)\,\mathrm{d}x\,.$$

Here $H_0^1(\Omega^\varepsilon, \Gamma^\varepsilon) = \{u \in H^1(\Omega^\varepsilon) : u|_{\Gamma^\varepsilon} = 0$ is the closure in $H^1(\Omega^\varepsilon)$ of its subspace consisting of the functions that vanish in a neighborhood of Γ^ε; $J^\varepsilon(\boldsymbol{v})$ are convex continuous (nondifferentiable, in general) functionals on $(H^1(\Omega^\varepsilon))^n$ specifying the boundary conditions on S^ε and defined in terms of the following two classes of functions:

Class $\mathfrak{B}_1$ consists of real valued functions $\psi(\boldsymbol{\eta}, \xi)$ on $\mathbb{R}^n \times \partial\omega$, 1-periodic and measurable in ξ for each $\boldsymbol{\eta} \in \mathbb{R}^n$, and satisfying the conditions:

(i) $|\psi(\boldsymbol{\eta}^1, \xi) - \psi(\boldsymbol{\eta}^2, \xi)| \leqslant c_0 |\boldsymbol{\eta}^1 - \boldsymbol{\eta}^2|$, $\forall \xi \in \partial\omega$, $\forall \boldsymbol{\eta}^1, \boldsymbol{\eta}^2 \in \mathbb{R}^n$;

(ii) $\psi(\boldsymbol{\eta}, \xi)$ is convex in $\boldsymbol{\eta}$, $\forall \xi \in \partial\omega$.

Class $\mathcal{B}_0$ consists of $\psi(\boldsymbol{\eta}, \xi) \in \mathcal{B}_1$ with the additional properties:

(iii) $0 \leqslant \psi(t\boldsymbol{\eta}, \xi) = t\psi(\boldsymbol{\eta}, \xi)$, $\forall \boldsymbol{\eta} \in \mathbb{R}^n$, $\forall \xi \in \partial\omega$, $\forall t \geqslant 0$;

(iv) $\psi(\boldsymbol{\eta}^0, \xi^0) > 0 \Rightarrow \operatorname{meas}_{n-1}\{\xi \in \partial\omega : \psi(\boldsymbol{\eta}^0, \xi) > 0\} > 0$.

Consider two fixed Lipschitz subdomains $\Omega_0, \Omega_1 \subset \Omega$ and denote the surface of the cavities inside these by $S_0^\varepsilon = S^\varepsilon \cap \Omega_0$, $S_1^\varepsilon = S^\varepsilon \cap \Omega_1$. The convex continuous functional $J^\varepsilon(\boldsymbol{v})$ in (1) may be chosen to specify different boundary conditions on $S_0^\varepsilon, S_1^\varepsilon$ (see Examples 1 and 2 below) and has the form

(2) $$J^\varepsilon(\boldsymbol{v}) = \mu_0(\varepsilon)\, j_0^\varepsilon(\boldsymbol{v}) + \mu_1(\varepsilon)\, j_1^\varepsilon(\boldsymbol{v}),$$

$$j_\alpha^\varepsilon(\boldsymbol{v}) = \int_{S_\alpha^\varepsilon} \Psi_\alpha(\boldsymbol{v}, \varepsilon^{-1}x)\, \mathrm{d}S, \qquad \Psi_\alpha(\boldsymbol{\eta}, \xi) \in \mathcal{B}_\alpha, \qquad \alpha = 0, 1,$$

where $\mu_0(\varepsilon) \geqslant 0$, $\mu_1(\varepsilon) \geqslant 0$ are real parameters such that

(3) $$\varepsilon^{-1}\mu_0(\varepsilon) \to \infty, \qquad \varepsilon^{-1}\mu_1(\varepsilon) \to a < \infty \quad \text{as} \quad \varepsilon \to 0.$$

Suppose also that one of the following two conditions holds:

(4) either $\mu_0(\varepsilon) \to 0$ as $\varepsilon \to 0$ or $\operatorname{int}\{\boldsymbol{\eta} \in \mathbb{R}^n : \overline{\Psi}_0(\boldsymbol{\eta}) = 0\} \neq \emptyset$,

where $\overline{\Psi}(\boldsymbol{\eta}) = |\partial_\square \omega|^{-1} \int_{\partial_\square \omega} \Psi(\boldsymbol{\eta}, \xi)\, \mathrm{d}S_\xi$ is the mean value of $\Psi(\boldsymbol{\eta}, \xi)$ over $\partial\omega$, $\partial_\square \omega = \partial\omega \cap [0, 1[^n$, and $\operatorname{int} A$ is the set of interior points of $A \subset \subset \mathbb{R}^n$ in the topology of $\mathbb{R}^n$.

In the trivial case of $J^\varepsilon(\boldsymbol{v}) \equiv 0$, the variational inequality (1) reduces to the usual integral identity for the solution of the following problem for the system of elasticity with zero Neumann conditions on S^ε:

(5) $$\left.\begin{aligned} &-\operatorname{\mathbf{div}}(\mathfrak{A}^\varepsilon(x)\, \boldsymbol{e}(\boldsymbol{u}^\varepsilon)) = \boldsymbol{F}(x) \ \text{in} \ \Omega^\varepsilon, \quad \boldsymbol{u}^\varepsilon = \boldsymbol{0} \ \text{on} \ \Gamma^\varepsilon, \\ &(\mathfrak{A}^\varepsilon \boldsymbol{e}(\boldsymbol{u}^\varepsilon))\, \boldsymbol{\nu}^\varepsilon = \boldsymbol{0} \ \text{on} \ S^\varepsilon, \end{aligned}\right\}$$

where $\boldsymbol{\nu}^\varepsilon = {}^t(\nu_1^\varepsilon, \ldots, \nu_n^\varepsilon)$ is the outward unit normal to $\partial\Omega^\varepsilon$. A detailed examination of this problem can be found, for instance, in [OSY], where it is shown that for small ε the solution $\boldsymbol{u}^\varepsilon$ is in some sense close to the solution $\boldsymbol{u}^0$ of the Dirichlet problem

(6) $$-\operatorname{\mathbf{div}}(\widehat{\mathfrak{A}}\boldsymbol{e}(\boldsymbol{u}^0)) = \boldsymbol{F} \ \text{in} \ \Omega, \quad \boldsymbol{u}^0 = \boldsymbol{0} \ \text{on} \ \partial\Omega,$$

where $\widehat{\mathfrak{A}} = \{\widehat{a}_{ij}^{kh}\}$ is the so-called *homogenized elasticity tensor* with constant coefficients $\widehat{a}_{ij}^{kh}$, which can be expressed in terms of solutions of certain periodic boundary value problems in ω (see, for instance, [OSY], Ch. II, § 1). Equivalently, this tensor can be defined as follows (see [JKO], [Y2]).

For a vector field $\boldsymbol{v}(\xi) = {}^t(v_1(\xi), \ldots, v_n(\xi))$, by $\boldsymbol{e}_\xi(\boldsymbol{v})$ we denote the matrix with the elements $(\boldsymbol{e}_\xi(\boldsymbol{v}))_{ij} = 2^{-1}(\partial v_i/\partial \xi_j + \partial v_j/\partial \xi_i)$.

Let $C^\infty_{\mathrm{per}}(\overline{\omega})$ be the space of smooth 1-periodic functions in $\overline{\omega}$. Denote by $H^1_{\mathrm{per}}(\omega)$ the completion of $C^\infty_{\mathrm{per}}(\overline{\omega})$ with respect to the norm $\|\cdot\|_{1,\,\omega_\square}$ in $H^1(\omega_\square)$, and let $\widehat{H}^1_{\mathrm{per}}(\omega) = \{v \in H^1_{\mathrm{per}}(\omega) : \langle v \rangle_\omega = 0\}$, where $\langle v \rangle_\omega = |\omega_\square|^{-1} \int_{\omega_\square} v(\xi)\, d\xi$ is the mean value over the domain ω. Consider the following periodic problem on the cell $\omega_\square$:

Given a constant matrix $\boldsymbol{p} \in \mathbb{M}^n$, *find* $\boldsymbol{V}_{\boldsymbol{p}}(\xi) \in (\widehat{H}^1_{\mathrm{per}}(\omega))^n$ *such that*

$$\langle \boldsymbol{e}_\xi(\boldsymbol{\varphi}) : \mathfrak{a}(\xi)(\boldsymbol{p} + \boldsymbol{e}_\xi(\boldsymbol{V}_{\boldsymbol{p}})) \rangle_\omega = 0\,, \qquad \forall\, \boldsymbol{\varphi} \in (\widehat{H}^1_{\mathrm{per}}(\omega))^n. \tag{7}$$

This problem has a unique solution by the Riesz theorem, since the bilinear form $\langle \boldsymbol{e}_\xi(\boldsymbol{v}) : \mathfrak{a}(\xi)\, \boldsymbol{e}_\xi(\boldsymbol{w}) \rangle_\omega$ may be regarded as a scalar product in $(\widehat{H}^1_{\mathrm{per}}(\omega))^n$ by virtue of the Korn inequality for the elements of $(\widehat{H}^1_{\mathrm{per}}(\omega))^n$ ([OSY], Ch. I, Theorem 2.8).

Clearly, the solution $\boldsymbol{V}_p$ of problem (7) linearly depends on $\boldsymbol{p} \in \mathbb{M}^n$. Let us define a linear mapping $\widehat{\mathfrak{a}} : \mathbb{M}^n \to \mathbb{M}^n$ by

$$\widehat{\mathfrak{a}}\boldsymbol{p} = \langle \mathfrak{a}(\xi)(\boldsymbol{p} + \boldsymbol{e}_\xi(\boldsymbol{V}_p)) \rangle_\omega, \qquad \forall \boldsymbol{p} \in \mathbb{M}^n. \tag{8}$$

It can be verified directly that $\widehat{\mathfrak{a}}$ coincides with the homogenized tensor from (6), as defined in [OSY], and has similar properties of symmetry and positive definiteness on symmetric matrices as the tensor $\mathfrak{a}(\xi)$, namely,

$$\boldsymbol{q} : \widehat{\mathfrak{a}}\boldsymbol{p} = \boldsymbol{p} : \widehat{\mathfrak{a}}\boldsymbol{q}\,, \qquad \forall \boldsymbol{p}, \boldsymbol{q} \in \mathbb{M}^n,$$

$$\widehat{\mathfrak{a}}\boldsymbol{p} = 0 \Leftrightarrow {}^t\boldsymbol{p} = -\boldsymbol{p}\,,$$

$$\tilde{\kappa}_1 |\boldsymbol{p}|^2 \leq \boldsymbol{p} : \widehat{\mathfrak{a}}\boldsymbol{p} \leq \tilde{\kappa}_2 |\boldsymbol{p}|^2, \qquad \forall \boldsymbol{p} \in \mathbb{S}^n \;\; (\tilde{\kappa}_1, \tilde{\kappa}_2 = \mathrm{const} > 0)\,,$$

where ${}^t\boldsymbol{p}$ is the transpose matrix of $\boldsymbol{p} \in \mathbb{M}^n$. Moreover, this tensor admits the representation:

$$\boldsymbol{p} : \widehat{\mathfrak{a}}\boldsymbol{p} = \inf_{\boldsymbol{v} \in \boldsymbol{\mathcal{V}}} \langle (\boldsymbol{p} + \boldsymbol{e}_\xi(\boldsymbol{v})) : \mathfrak{a}(\xi)(\boldsymbol{p} + \boldsymbol{e}_\xi(\boldsymbol{v})) \rangle_\omega, \qquad \forall \boldsymbol{p} \in \mathbb{M}^n, \tag{9}$$

where $\boldsymbol{\mathcal{V}}$ can be any of the spaces $(\widehat{H}^1_{\mathrm{per}}(\omega))^n$, $(H^1_{\mathrm{per}}(\omega))^n$, $(H^1_{\mathrm{per}}(\mathbb{R}^n))^n$.

Our aim is to show that under the assumptions (3) and (4) on $J^\varepsilon(\boldsymbol{v})$, the homogenized problem for (1) has the form of a variational inequality with the tensor $\widehat{\mathcal{A}}$ defined by (8) and, in general, additional restrictions on the set of admissible displacements:

Find the displacement $\boldsymbol{u}^0 \in \mathcal{W}_0 = \{\boldsymbol{u} \in (H_0^1(\Omega))^n \colon \overline{\Psi}_0(\boldsymbol{u}(x)) = 0$ a.e. in $\Omega_0\}$ *such that for any* $\boldsymbol{v} \in \mathcal{W}_0$

$$(10) \qquad \int_\Omega \boldsymbol{e}(\boldsymbol{u}^0) : \widehat{\mathcal{A}}\boldsymbol{e}(\boldsymbol{v} - \boldsymbol{u}^0)\, dx + \widehat{J}(\boldsymbol{v}) - \widehat{J}(\boldsymbol{u}^0) \geqslant \int_\Omega \boldsymbol{F} \cdot (\boldsymbol{v} - \boldsymbol{u}^0)\, dx\,,$$

where the functional $\widehat{J}$ *has the form*

$$(11) \qquad \widehat{J}(\boldsymbol{v}) = a\gamma \int_{\Omega_1} \overline{\Psi}_1(\boldsymbol{v})\, dx\,, \qquad \gamma = |\partial_\square \omega| / |\omega_\square|\,.$$

Thus, the term $\mu_0(\varepsilon)\, j_0^\varepsilon(\boldsymbol{v})$ of the functional $J^\varepsilon(\boldsymbol{v})$ determines the set of admissible displacements $\mathcal{W}_0$ of the limit problem, whereas the term $\mu_1(\varepsilon)\, j_1^\varepsilon(\boldsymbol{v})$ makes a contribution to the variational inequality.

Let us formulate the main homogenization theorem for problem (1). As usual, for the displacements $\boldsymbol{u}^\varepsilon$ the *generalized gradients* are defined by

$$\boldsymbol{\Gamma}^\varepsilon(x) = \begin{cases} \mathcal{A}(\varepsilon^{-1}x)\, \boldsymbol{e}(\boldsymbol{u}^\varepsilon) & \text{for } x \in \Omega^\varepsilon, \\ 0 & \text{for } x \notin \Omega^\varepsilon. \end{cases}$$

By $P_\varepsilon \colon (H_0^1(\Omega^\varepsilon, \Gamma^\varepsilon))^n \to (H_0^1(\widetilde{\Omega}))^n$ we denote continuous linear extension operators from the domain Ω^ε to a fixed domain $\widetilde{\Omega} \supset \overline{\Omega}$. According to Theorem 4.3, Ch. I, [OSY], these operators can be constructed in such a way that $\sup_\varepsilon \|P_\varepsilon\| < \infty$ and $P_\varepsilon \boldsymbol{u}(x) = \mathbf{0}$ for almost all $x \in \widetilde{\Omega} \setminus \Omega$ such that $\operatorname{dist}(x, \partial\Omega) \geqslant 4\sqrt{n}\,\varepsilon$. The symbols «$\to$» and «$\rightharpoonup$» denote, respectively, strong and weak convergence in a specified Hilbert space.

THEOREM 1. *Under the assumptions* (3) *and* (4), *let* $\boldsymbol{u}^\varepsilon$ *and* $\boldsymbol{u}^0$ *be the solutions of problems* (1) *and* (10), *respectively. Then, as* $\varepsilon \to 0$, *we have*

$$(12) \qquad \begin{gathered} P_\varepsilon \boldsymbol{u}^\varepsilon \to \boldsymbol{u}^0 \text{ in } (L^2(\Omega))^n, \qquad \nabla P_\varepsilon \boldsymbol{u}^\varepsilon \rightharpoonup \nabla \boldsymbol{u}^0 \text{ in } (L^2(\Omega))^{n^2}, \\ \boldsymbol{\Gamma}^\varepsilon(x) \rightharpoonup |\omega_\square|\, \widehat{\mathcal{A}}\boldsymbol{e}(\boldsymbol{u}^0) \text{ in } (L^2(\Omega))^{n^2}, \end{gathered}$$

$$(13) \qquad \int_{\Omega^\varepsilon} \boldsymbol{e}(\boldsymbol{u}^\varepsilon) : \mathcal{A}(\varepsilon^{-1}x)\, \boldsymbol{e}(\boldsymbol{u}^\varepsilon)\, dx \;\to\; |\omega_\square| \int_\Omega \boldsymbol{e}(\boldsymbol{u}^0) : \widehat{\mathcal{A}}\boldsymbol{e}(\boldsymbol{u}^0)\, dx\,.$$

The proof of this homogenization theorem is based on the following lemmas.

First of all we need the following estimate for the $L^2(S^\varepsilon)$ norm, denoted by $\|\cdot\|_{0,S^\varepsilon}$, of traces of functions in $H^1(\Omega)$ on the surface of cavities S^ε.

LEMMA 1. *For any $v \in H^1(\Omega)$ the inequality $\|v\|_{0,S^\varepsilon} \leqslant C\varepsilon^{-1/2}\|v\|_{1,\Omega}$ holds with a constant C independent of ε.*

This estimate is proved in Lemma 2 of [Y1].

In order to pass to the limit in the variational inequality (1) as $\varepsilon \to 0$ and construct the set of admissible displacements for the homogenized problem, we need the following result about asymptotic behavior of traces of nonlinear functions of the displacements.

LEMMA 2. *Let $\Psi(\boldsymbol{\eta}, \xi) \in \mathcal{B}_1$ and let Ω' be a subdomain of Ω with Lipschitz continuous boundary. Then for any sequence $\boldsymbol{w}^\varepsilon \in (H^1(\Omega))^n$ such that $\boldsymbol{w}^\varepsilon \rightharpoonup \boldsymbol{w}^0$ in $(H^1(\Omega))^n$, we have*

$$\varepsilon \int_{S^\varepsilon \cap \Omega'} \Psi(\boldsymbol{w}^\varepsilon, \varepsilon^{-1}x)\,\mathrm{d}S \to |\partial_\square \omega| \int_{\Omega'} \overline{\Psi}(\boldsymbol{w}^0)\,\mathrm{d}x \quad \text{as} \quad \varepsilon \to 0 .$$

This result is obtained by an obvious modification of the proof of Lemma 4 in [Y1].

LEMMA 3. *If ψ_ε, $\varphi_\varepsilon \in L^2(\Omega^\varepsilon)$ and $\|\psi_\varepsilon - \psi_0\|_{L^2(\Omega^\varepsilon)} \to 0$, $\|\varphi_\varepsilon - \varphi_0\|_{L^2(\Omega^\varepsilon)} \to 0$ as $\varepsilon \to 0$ for some ψ_0, $\varphi_0 \in L^2(\Omega)$, then for any 1-periodic $F(\xi) \in L^\infty(\omega)$, we have*

$$\int_{\Omega^\varepsilon} F(\varepsilon^{-1}x)\,\psi_\varepsilon \varphi_\varepsilon\,\mathrm{d}x \to \left[\int_{\omega_\square} F(\xi)\,\mathrm{d}\xi\right] \int_{\Omega} \psi_0 \varphi_0\,\mathrm{d}x \quad \text{as} \quad \varepsilon \to 0 .$$

This convergence is established in Corollary 1.7, Ch. I, [OSY].

The next result is a modification of a standard argument used in the homogenization theory. Its proof is given in [Y2] (Lemma 9).

LEMMA 4. *Let $\boldsymbol{w}^\varepsilon \in (H^1(\Omega))^n$ be a sequence such that $\boldsymbol{w}^\varepsilon \rightharpoonup \boldsymbol{w}^0$ in $(H^1(\Omega))^n$ as $\varepsilon \to 0$. Set $\boldsymbol{\Gamma}^\varepsilon(x) = \mathfrak{a}(\varepsilon^{-1}x)\,\boldsymbol{e}(\boldsymbol{w}^\varepsilon(x))$ in Ω^ε, $\boldsymbol{\Gamma}^\varepsilon(x) = \mathbf{0}$ outside Ω^ε, and let Ω' be an arbitrary Lipschitz subdomain of Ω. Then the* *condition*

$$\varepsilon \int_{\Omega' \cap \Omega^\varepsilon} \psi(x)\, \boldsymbol{e}(\boldsymbol{V}(\varepsilon^{-1}x)) : \boldsymbol{\Gamma}^\varepsilon(x)\, \mathrm{d}x \to 0 \quad \text{as} \quad \varepsilon \to 0 \tag{14}$$

valid for any $\boldsymbol{V}(\xi) \in (C^\infty_{\mathrm{per}}(\overline{\omega}))^n$ *and any* $\psi \in C_0^\infty(\Omega')$, $\psi \geqslant 0$, *implies the convergence*

$$\boldsymbol{\Gamma}^\varepsilon(x) \xrightarrow{(L^2(\Omega'))^{n^2}} |\omega_\square|\, \widehat{\mathfrak{A}}\boldsymbol{e}(\boldsymbol{w}^0) \quad \text{as} \quad \varepsilon \to 0, \tag{15}$$

where $\widehat{\mathfrak{A}}$ *is the elasticity tensor* (8).

The following lemma has an important role in the homogenization theory and generalizes a well-known property of Γ-convergence to the case of elasticity tensors in perforated domains. Its proof is given in [Y2] (Lemma 8) and utilizes some ideas from Sects. 3.1 and 5.1 of [JKO].

LEMMA 5. *Let* $\boldsymbol{w}^\varepsilon \in (H^1(\Omega))^n$ *be a sequence such that* $\boldsymbol{w}^\varepsilon \rightharpoonup \boldsymbol{w}^0$ *in* $(H^1(\Omega))^n$ *as* $\varepsilon \to 0$. *Then*

$$\liminf_{\varepsilon \to 0} \int_{\Omega' \cap \Omega^\varepsilon} \boldsymbol{e}(\boldsymbol{w}^\varepsilon) : \mathfrak{A}(\varepsilon^{-1}x)\, \boldsymbol{e}(\boldsymbol{w}^\varepsilon)\, \mathrm{d}x \geqslant |\omega_\square| \int_{\Omega'} \boldsymbol{e}(\boldsymbol{w}^0) : \widehat{\mathfrak{A}}\boldsymbol{e}(\boldsymbol{w}^0)\, \mathrm{d}x, \tag{16}$$

for any Lipschitz subdomain $\Omega' \subset \Omega$, *where* $\widehat{\mathfrak{A}}$ *is the elasticity tensor* (8).

Our homogenization result for problem (1) and the description of the set of admissiblee displacements for the limit problem as $\varepsilon \to 0$ rely on the following properties of class $\mathfrak{B}_0$, which can be easily deduced from the conditions (i)-(iv) in the definition of that class (cf. Lemma 5 in [Y2]). As above, $\overline{\psi}(\boldsymbol{\eta})$ stands for the mean value of $\psi(\boldsymbol{\eta}, \xi)$ over $\xi \in \partial\omega$.

LEMMA 6. *Let* $\psi(\boldsymbol{\eta}, \xi) \in \mathfrak{B}_0$, *i.e., the conditions* (i)-(iv) *hold. Then:*

(v) $\psi(\boldsymbol{0}, \xi) = \overline{\psi}(\boldsymbol{0}) = 0$, $\forall \xi \in \partial\omega$;

(vi) $\overline{\psi}(\boldsymbol{\eta}^0) = 0 \Leftrightarrow \psi(\boldsymbol{\eta}^0, \xi) = 0$, $\forall \xi \in \partial\omega$;

(vii) $\mathfrak{M} = \{\boldsymbol{\eta} \in \mathbb{R}^n : \overline{\psi}(\boldsymbol{\eta}) = 0$ *and* $\mathfrak{M}_\xi = \{\boldsymbol{\eta} \in \mathbb{R}^n : \psi(\boldsymbol{\eta}, \xi) = 0\}$ *are closed convex cones in* $\mathbb{R}^n$;

(viii) $\mathcal{U}_0 = \{\boldsymbol{u} \in (H^1(\Omega))^n : \overline{\psi}(\boldsymbol{u}(x)) = 0$ a.e. in $\Omega_0\}$ *is a closed* *convex cone in* $(H^1(\Omega))^n$, *which coincides with*

$$\mathcal{U}_1 = \left\{ \boldsymbol{u} \in (H^1(\Omega))^n \colon \int_{\Omega_0} \overline{\psi}(\boldsymbol{u}(x))\,\mathrm{d}x = 0 \right\},$$

$$\mathcal{U}_2 = \{ \boldsymbol{u} \in (H^1(\Omega))^n \colon \forall\, \xi \in \partial\omega \ \ \psi(\boldsymbol{u}(x),\, \xi) = 0 \text{ a.e. in } x \in S_0^\varepsilon \};$$

(ix) $\boldsymbol{u} \in \mathcal{U}_0 \Rightarrow \psi(\boldsymbol{u}(x),\, \varepsilon^{-1}x) = 0$ a.e. on S_0^ε.

Proof of Theorem 1. By C, C_j we denote positive constants that do not depend on ε; the same symbol may be used to denote different constants in different places.

First of all, for the solution $\boldsymbol{u}^\varepsilon$ of problem (1) we establish an estimate, which is uniform with respect to ε. Setting $\boldsymbol{v} = \boldsymbol{0}$ in (1), we get

$$\text{(17)} \qquad \int_{\Omega^\varepsilon} \boldsymbol{e}(\boldsymbol{u}^\varepsilon) \colon \mathcal{A}(\varepsilon^{-1}x)\, \boldsymbol{e}(\boldsymbol{u}^\varepsilon)\,\mathrm{d}x + \mu_0(\varepsilon)\, j_0^\varepsilon(\boldsymbol{u}^\varepsilon) \leqslant$$

$$\leqslant \mu_1(\varepsilon)\, j_1^\varepsilon(\boldsymbol{0}) - \mu_1(\varepsilon)\, j_1^\varepsilon(\boldsymbol{u}^\varepsilon) + \int_{\Omega^\varepsilon} \boldsymbol{F}\cdot\boldsymbol{u}^\varepsilon\,\mathrm{d}x\,.$$

Since $\mu_1(\varepsilon) = a\varepsilon + o(\varepsilon)$, $\operatorname{meas}_{n-1} S^\varepsilon \sim \varepsilon^{-1}$ and $\|\boldsymbol{u}^\varepsilon\|_{0,\,S^\varepsilon} \leqslant C\varepsilon^{-1/2}\|\boldsymbol{u}^\varepsilon\|_{1,\,\Omega^\varepsilon}$ (by Lemma 1 applied to the extension $P_\varepsilon \boldsymbol{u}^\varepsilon$), we find that

$$|\mu_1(\varepsilon)\, j_1^\varepsilon(\boldsymbol{0}) - \mu_1(\varepsilon)\, j_1^\varepsilon(\boldsymbol{u}^\varepsilon)| \leqslant C_1 \mu_1(\varepsilon) \int_{S_1^\varepsilon} |\boldsymbol{u}^\varepsilon|\, dS \leqslant$$

$$\leqslant C_1 \mu_1(\varepsilon) \|\boldsymbol{u}^\varepsilon\|_{0,\,S^\varepsilon} \|1\|_{0,\,S^\varepsilon} \leqslant C_2 \|\boldsymbol{u}^\varepsilon\|_{1,\,\Omega^\varepsilon}\,,$$

where we have also used the property (i) of $\Psi_1(\boldsymbol{\eta},\, \xi)$ in the definition of class $\mathcal{B}_1$. Thus, from (17) and the Korn inequality for the elements of $(H_0^1(\Omega^\varepsilon,\, \Gamma^\varepsilon))^n$ (i.e., $\|\boldsymbol{e}(\boldsymbol{u}^\varepsilon)\|_{0,\,\Omega^\varepsilon} \geqslant C\|\boldsymbol{u}^\varepsilon\|_{1,\,\Omega^\varepsilon}$; see [OSY], Ch. I, Theorem 4.5) we obtain the estimate

$$\text{(18)} \qquad \|\boldsymbol{u}^\varepsilon\|^2_{1,\,\Omega^\varepsilon} + \mu_0(\varepsilon) \int_{S_0^\varepsilon} \Psi_0(\boldsymbol{u}^\varepsilon,\, \varepsilon^{-1}x)\,\mathrm{d}S \leqslant C_0.$$

Therefore, the norms $\|P_\varepsilon \boldsymbol{u}^\varepsilon\|_{1,\,\widetilde{\Omega}}$ are bounded uniformly with respect to ε. Due to the weak compactness of a ball in Hilbert space, the compactness of the imbedding $H^1(\widetilde{\Omega}) \subset L^2(\widetilde{\Omega})$, and the properties of the operators P_ε, there exist $\boldsymbol{u}^0 \in (H^1(\widetilde{\Omega}))^n$ and $\boldsymbol{\Gamma}^0(x) \in (L^2(\widetilde{\Omega}))^{n^2}$ such that $\boldsymbol{u}^0\,|_\Omega \in$

$\in (H_0^1(\Omega))^n$ and

$$P_\varepsilon \boldsymbol{u}^\varepsilon \xrightarrow{(L^2(\Omega))^n} \boldsymbol{u}^0, \qquad \boldsymbol{\Gamma}^\varepsilon \xrightarrow{(L^2(\Omega))^{n^2}} \boldsymbol{\Gamma}^0, \qquad P_\varepsilon \boldsymbol{u}^\varepsilon \xrightarrow{(H^1(\Omega))^n} \boldsymbol{u}^0 \tag{19}$$

for a subsequence of $\varepsilon \to 0$. We are going to show that $\boldsymbol{u}^0$ is a solution of problem (10) and $\boldsymbol{\Gamma}^0 = |\omega_\square| \widehat{\mathfrak{A}}\boldsymbol{e}(\boldsymbol{u}^0)$.

Let us multiply (18) by $\varepsilon/\mu_0(\varepsilon)$ and pass to the limit as $\varepsilon \to 0$ over the said subsequence. Since $\varepsilon/\mu_0(\varepsilon) \to 0$ by (3), Lemma 2 yields $\int_{\Omega_0} \overline{\Psi}_0(\boldsymbol{u}^0)\,\mathrm{d}x = 0$, and therefore, $\boldsymbol{u}^0 \in \mathcal{W}_0$ by Lemma 6 (viii).

In order to show that $\boldsymbol{\Gamma}^0 = |\omega_\square| \widehat{\mathfrak{A}}\boldsymbol{e}(\boldsymbol{u}^0)$, let us apply Lemma 4. Obviously, it suffices to verify the convergence (14) for $\Omega' = \Omega$, $\boldsymbol{w}^\varepsilon = P_\varepsilon \boldsymbol{u}^\varepsilon$.

Consider the first alternative in (4), i.e., the case $\mu_0(\varepsilon) \to 0$. Fixing an arbitrary $\boldsymbol{V}(\xi) \in (C_{\mathrm{per}}^\infty(\overline{\omega}))^n$ and taking $\boldsymbol{v}(x) = \boldsymbol{u}^\varepsilon \pm \varepsilon\psi(x)\,\boldsymbol{V}(\varepsilon^{-1}x)$ in (1) with $\psi \in C_0^\infty(\Omega)$, $\psi \geqslant 0$, we get

$$\pm\varepsilon \int_{\Omega^\varepsilon} \psi \nabla_x \boldsymbol{V} : \mathfrak{A}^\varepsilon \nabla_x \boldsymbol{u}^\varepsilon\,\mathrm{d}x \pm \varepsilon \int_{\Omega^\varepsilon} (\boldsymbol{V} \otimes \nabla_x \psi) : \mathfrak{A}^\varepsilon \nabla_x \boldsymbol{u}^\varepsilon\,\mathrm{d}x \geqslant \tag{20}$$
$$\geqslant \pm\varepsilon \int_{\Omega^\varepsilon} \psi \boldsymbol{V}\cdot\boldsymbol{F}\,\mathrm{d}x + \mu_0(\varepsilon)(j_0^\varepsilon(\boldsymbol{u}^\varepsilon) - j_0^\varepsilon(\boldsymbol{u}^\varepsilon \pm \varepsilon\psi\boldsymbol{V})) +$$
$$+\mu_1(\varepsilon)(j_1^\varepsilon(\boldsymbol{u}^\varepsilon) - j_1^\varepsilon(\boldsymbol{u}^\varepsilon \pm \varepsilon\psi\boldsymbol{V}))$$

where $\boldsymbol{V} = \boldsymbol{V}(\varepsilon^{-1}x)$, $\mathfrak{A}^\varepsilon = \mathfrak{A}(\varepsilon^{-1}x)$; the product $\boldsymbol{u} \otimes \nabla_x \psi$ for a vector valued function $\boldsymbol{u}(x)$ is the matrix with the elements $(\boldsymbol{u} \otimes \nabla_x \psi)_{ij} = u_i\,\partial\psi/\partial x_j$, in which case $\nabla_x(\psi\boldsymbol{u}) = \psi\nabla_x \boldsymbol{u} + \boldsymbol{u} \otimes \nabla_x \psi$.

Clearly, the second term on the left-hand side of (20) and the first term on its right-hand side tend to zero as $\varepsilon \to 0$. By the Lipschitz condition (i) in the definition of class $\mathcal{B}_1$, we have

$$\mu_\alpha(\varepsilon)\,|j_\alpha^\varepsilon(\boldsymbol{u}^\varepsilon \pm \varepsilon\psi\boldsymbol{V}) - j_\alpha^\varepsilon(\boldsymbol{u}^\varepsilon)| \leqslant \tag{21}$$
$$\leqslant \mu_\alpha(\varepsilon) \int_{S_\alpha^\varepsilon} |\Psi_\alpha(\boldsymbol{u}^\varepsilon \pm \varepsilon\psi\boldsymbol{V}, \varepsilon^{-1}x) - \Psi_\alpha(\boldsymbol{u}^\varepsilon, \varepsilon^{-1}x)\,\mathrm{d}S \leqslant$$
$$\leqslant C\mu_\alpha(\varepsilon)\,\varepsilon(\mathrm{meas}_{n-1} S^\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0,\ \alpha = 0, 1\,.$$

Therefore, the first term on the left-hand side of (20) also tends to zero, and thus the convergence (14) is proved.

Consider the case $\mu_0(\varepsilon) \not\to 0$. Then the cone $\mathfrak{M} = \{\boldsymbol{\eta} \in \mathbb{R}^n \colon \overline{\Psi}_0(\boldsymbol{\eta}) = 0\}$ has interior points in the topology of $\mathbb{R}^n$, by assumption. Therefore, for any fixed $V(\xi) \in (C^\infty_{\mathrm{per}}(\overline{\omega}))^n$, there is $\boldsymbol{\eta}^0 \in \mathfrak{M}$ such that $\boldsymbol{\eta}^0 \pm V(\xi) \in \mathfrak{M}$ for any $\xi \in \partial\omega$, since the values of $V(\xi)$ range within a bounded set. Consequently, by Lemma 6 (vi),

$$\Psi_0((\boldsymbol{\eta}^0 \pm V(\xi)), \xi) = 0\,, \qquad \forall\, \xi \in \partial\omega\,. \tag{22}$$

Let us take $\boldsymbol{v}(x) = \boldsymbol{u}^\varepsilon + \varepsilon\psi(x)(\boldsymbol{\eta}^0 \pm V(\varepsilon^{-1}x))$ in (1), with $\psi \in C_0^\infty(\Omega)$, $\psi \geqslant 0$. Then

$$\pm\,\varepsilon \int_{\Omega^\varepsilon} \psi \nabla_x V : \mathfrak{A}^\varepsilon \nabla_x \boldsymbol{u}^\varepsilon \,\mathrm{d}x + \varepsilon \int_{\Omega^\varepsilon} ((\boldsymbol{\eta}^0 \pm V) \otimes \nabla_x \psi) : \mathfrak{A}^\varepsilon \nabla_x \boldsymbol{u}^\varepsilon \,\mathrm{d}x \geqslant \tag{23}$$

$$\geqslant \varepsilon \int_{\Omega^\varepsilon} \psi(\boldsymbol{\eta}^0 \pm V)\cdot \boldsymbol{F} \,\mathrm{d}x + \mu_0(\varepsilon)(j_0^\varepsilon(\boldsymbol{u}^\varepsilon) - j_0^\varepsilon(\boldsymbol{u}^\varepsilon + \varepsilon\psi(\boldsymbol{\eta}^0 \pm V))) +$$

$$+\mu_1(\varepsilon)(j_1^\varepsilon(\boldsymbol{u}^\varepsilon) - j_1^\varepsilon(\boldsymbol{u}^\varepsilon + \varepsilon\psi(\boldsymbol{\eta}^0 \pm V)))\,,$$

where $V = V(\varepsilon^{-1}x)$, $\mathfrak{A}^\varepsilon = \mathfrak{A}(\varepsilon^{-1}x)$. As above, we find that the second term on the left-hand side of this inequality, as well as the first and the third terms on its right-hand side, tends to zero as $\varepsilon \to 0$.

Consider the second term on the right-hand side of (23). The properties (ii) and (iii) of class $\mathfrak{B}_0$ guarantee that

$$\Psi_0(t_1\boldsymbol{\eta}^2 + t_2\boldsymbol{\eta}^2, \xi) \leqslant t_1\Psi_0(\boldsymbol{\eta}^1, \xi) + t_2\Psi_0(\boldsymbol{\eta}^2, \xi)\,,$$

$$\forall \xi \in \partial\omega\,, \qquad \forall\, \boldsymbol{\eta}^1, \boldsymbol{\eta}^2 \in \mathbb{R}^n, \qquad \forall\, t_1, t_2 \geqslant 0\,.$$

Therefore,

$$j_0^\varepsilon(\boldsymbol{u}^\varepsilon) - j_0^\varepsilon(\boldsymbol{u}^\varepsilon + \varepsilon\psi(\boldsymbol{\eta}^0 \pm V)) \geqslant -j_0^\varepsilon(\varepsilon\psi(\boldsymbol{\eta}^0 \pm V)) = 0\,,$$

with the right-hand side being equal to zero, because of (22) and the property (iii) of $\Psi_0(\boldsymbol{\eta}, \xi)$. Consequently, the second term on the right-hand side of (23) is always non-negative. Thus, the first term on the left-hand side of (23) tends to zero and we again have the convergence (14). Therefore, $\boldsymbol{\Gamma}^0 = |\omega_\square|\, \widehat{\mathfrak{A}}\boldsymbol{e}(\boldsymbol{u}^0)$ by Lemma 4.

Finally, let us show that $\boldsymbol{u}^0$ is a solution of problem (10). We introduce the bilinear forms

$$\widehat{\mathcal{A}}_\Omega(\boldsymbol{v}, \boldsymbol{w}) = \int_\Omega \boldsymbol{e}(\boldsymbol{v}) : \widehat{\mathfrak{A}}\boldsymbol{e}(\boldsymbol{w}) \,\mathrm{d}x\,,$$

$$\mathcal{A}^\varepsilon_{\Omega^\varepsilon}(\boldsymbol{v}, \boldsymbol{w}) = \int_{\Omega^\varepsilon} \boldsymbol{e}(\boldsymbol{v}) : \mathfrak{A}(\varepsilon^{-1}x)\, \boldsymbol{e}(\boldsymbol{w}) \,\mathrm{d}x\,.$$

Let $\boldsymbol{v}$ be an arbitrary element of $\mathcal{W}_0$. By Lemmas 2, 5 and 6, we have

$$-|\omega_\square|\widehat{\mathcal{A}}_\Omega(\boldsymbol{u}^0, \boldsymbol{u}^0) \geqslant \limsup_{\varepsilon\to 0}(-\mathcal{A}^\varepsilon_{\Omega^\varepsilon}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon)),$$

$$j_0^\varepsilon(\boldsymbol{v}) = 0, \qquad j_0^\varepsilon(\boldsymbol{u}^\varepsilon) \geqslant 0,$$

$$\lim_{\varepsilon\to 0}\mu_1(\varepsilon)\, j_1^\varepsilon(\boldsymbol{v}) = |\omega_\square|\widehat{J}(\boldsymbol{v}), \quad \lim_{\varepsilon\to 0}\mu_1(\varepsilon)\, j_1^\varepsilon(\boldsymbol{u}^\varepsilon) = |\omega_\square|\widehat{J}(\boldsymbol{u}^0),$$

$$\lim_{\varepsilon\to 0}\mathcal{A}^\varepsilon_{\Omega^\varepsilon}(\boldsymbol{u}^\varepsilon, \boldsymbol{v}) = |\omega_\square|\widehat{\mathcal{A}}_\Omega(\boldsymbol{u}^0, \boldsymbol{v}),$$

where $\widehat{J}$ is the functional (11). Therefore,

$$\widehat{\mathcal{A}}_\Omega(\boldsymbol{u}^0, \boldsymbol{v}-\boldsymbol{u}^0) + \widehat{J}(\boldsymbol{v}) - \widehat{J}(\boldsymbol{u}^0) =$$

$$= -\widehat{\mathcal{A}}_\Omega(\boldsymbol{u}^0, \boldsymbol{u}^0) + \widehat{\mathcal{A}}_\Omega(\boldsymbol{u}^0, \boldsymbol{v}) + |\omega_\square|^{-1}\lim_{\varepsilon\to 0}\mu_1(\varepsilon)(j_1^\varepsilon(\boldsymbol{v}) - j_1^\varepsilon(\boldsymbol{u}^\varepsilon)) \geqslant$$

$$\geqslant |\omega_\square|^{-1}\limsup_{\varepsilon\to 0}(-\mathcal{A}^\varepsilon_{\Omega^\varepsilon}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon) + \mathcal{A}^\varepsilon_{\Omega^\varepsilon}(\boldsymbol{u}^\varepsilon, \boldsymbol{v})) +$$

$$+ |\omega_\square|^{-1}\limsup_{\varepsilon\to 0}(\mu_1(\varepsilon)(j_1^\varepsilon(\boldsymbol{v}) - j_1^\varepsilon(\boldsymbol{u}^\varepsilon)) + \mu_0(\varepsilon)(j_0^\varepsilon(\boldsymbol{v}) - j_0^\varepsilon(\boldsymbol{u}^\varepsilon)) \geqslant$$

$$\geqslant |\omega_\square|^{-1}\limsup_{\varepsilon\to 0}(\mathcal{A}^\varepsilon_{\Omega^\varepsilon}(\boldsymbol{u}^\varepsilon, \boldsymbol{v}-\boldsymbol{u}^\varepsilon) + J^\varepsilon(\boldsymbol{v}) - J^\varepsilon(\boldsymbol{u}^\varepsilon)) \geqslant$$

$$\geqslant |\omega_\square|^{-1}\limsup_{\varepsilon\to 0}\int_{\Omega^\varepsilon}\boldsymbol{F}\cdot(\boldsymbol{v}-\boldsymbol{u}^\varepsilon)\,dx = \int_\Omega \boldsymbol{F}\cdot(\boldsymbol{v}-\boldsymbol{u}^0)\,dx,$$

which shows that $\boldsymbol{u}^0$ is indeed a solution of problem (10). Since this problem can have only one solution, the above arguments applied to any subsequence of $\boldsymbol{u}^\varepsilon$ again bring us to $\boldsymbol{u}^0$. ∎

REMARK 1. It should be observed that condition (4) is essential in our proof of Theorem 1. The case of $\mu_0(\varepsilon) \not\to 0$ and at the same time the cone $\{\boldsymbol{\eta}\in\mathbb{R}^n\colon \overline{\Psi}_0(\boldsymbol{\eta}) = 0\}$ having no interior points in $\mathbb{R}^n$ (i.e., its dimension being $< n$) requires further investigation and will be treated elsewhere. As suggested by examples in [Y2], in this case, too, the homogenized tensor is likely to depend on the boundary conditions on S^ε.

REMARK 2. It can be easily seen from the proof of Theorem 1 that instead of the functional $J^\varepsilon(\boldsymbol{v})$ of the form (2) in problem (1) we can take a more general functional, in particular,

$$J^{\varepsilon}(\boldsymbol{v}) = \sum_{m=1}^{K} (\mu_{0,m}(\varepsilon)\, j^{\varepsilon}_{0,m}(\boldsymbol{v}) + \mu_{1,m}(\varepsilon)\, j^{\varepsilon}_{1,m}(\boldsymbol{v}))\,,$$

where

$$0 \leqslant \varepsilon^{-1}\mu_{0,m}(\varepsilon) \to \infty\,, \qquad 0 \leqslant \varepsilon^{-1}\mu_{1,m}(\varepsilon) \to a_m < \infty \quad \text{as} \quad \varepsilon \to 0\,,$$

$$j^{\varepsilon}_{0,m}(\boldsymbol{v}) = \int_{S^{\varepsilon}_0} \Psi_{0,m}(\boldsymbol{v}, \varepsilon^{-1}x)\,\mathrm{d}S\,, \qquad \Psi_{0,m}(\boldsymbol{\eta}, \xi) \in \mathcal{B}_0,$$

$$j^{\varepsilon}_{1,m}(\boldsymbol{v}) = \int_{S^{\varepsilon}_1} \Psi_{1,m}(\boldsymbol{v}, \varepsilon^{-1}x)\,\mathrm{d}S\,, \qquad \Psi_{1,m}(\boldsymbol{\eta}, \xi) \in \mathcal{B}_1.$$

Suppose also that (cf. (4))

$$\text{either } \mu_{0,m} \to 0\,, \qquad m = 1, \dots, K\,,$$

$$\text{or } \operatorname{int}\{\boldsymbol{\eta} \in \mathbb{R}^n \colon \overline{\Psi}_{0,m}(\boldsymbol{\eta}) = 0\,,\ m = 1, \dots, K\} \neq \emptyset\,.$$

Then the homogenized problem for (1) has the form (10) with the functional

$$\hat{J}(\boldsymbol{v}) = \gamma \sum_{m=1}^{K} a_m \int_{\Omega_1} \overline{\Psi}_{1,m}(\boldsymbol{v})\,\mathrm{d}x$$

(cf. (11)) and the set of admissible displacements

$$\mathcal{W}_0 = \{\boldsymbol{v} \in (H^1_0(\Omega))^n \colon \overline{\Psi}_{0,m}(\boldsymbol{v})\,|_{\Omega_0} = 0,\ m = 1, \dots, K\}\,.$$

2. Examples of problems with friction.

In this section we apply Theorem 1 to concrete contact problems of elasticity and, in some cases, indicate how the limit set of admissible displacements depends on the original boundary conditions and the geometry of the contact region.

Let $\partial_0\omega$ be a non-empty 1-periodic subset of $\partial\omega$, open in the topology induced from $\mathbb{R}^n$. The corresponding subset of S^{ε} is denoted by $\Sigma^{\varepsilon}_{\text{cont}} = (\varepsilon\partial_0\omega) \cap \Omega$, and it is on this set $\Sigma^{\varepsilon}_{\text{cont}}$ that the body occupying the domain Ω^{ε} may be subject to *contact with friction.* We are going to consider some boundary conditions of friction on $\Sigma^{\varepsilon}_{\text{cont}}$ described in [DL] in terms of certain convex continuous functionals J^ε in problem (1).

For the normal and the tangential components of the displacements and stresses on $\partial\Omega^\varepsilon$ we use the following formal (i.e., regardless of regularity) notations:

$$\sigma_{\mathrm{N}}(\boldsymbol{u}) = \boldsymbol{\nu}^\varepsilon \cdot (\boldsymbol{\sigma}(\boldsymbol{u})\, \boldsymbol{\nu}^\varepsilon), \qquad \boldsymbol{\sigma}_{\mathrm{T}}(\boldsymbol{u}) = \boldsymbol{\sigma}(\boldsymbol{u})\, \boldsymbol{\nu}^\varepsilon - \sigma_{\mathrm{N}}(\boldsymbol{u})\, \boldsymbol{\nu}^\varepsilon,$$

$$u_{\mathrm{N}} = \boldsymbol{u} \cdot \boldsymbol{\nu}^\varepsilon, \qquad \boldsymbol{u}_{\mathrm{T}} = \boldsymbol{u} - u_{\mathrm{N}}\, \boldsymbol{\nu}^\varepsilon,$$

where $\boldsymbol{\nu}^\varepsilon$ is the unit outward normal to $\partial\Omega^\varepsilon$, $\boldsymbol{\sigma}(\boldsymbol{u}) = \mathbf{\mathfrak{a}}(\varepsilon^{-1}x)\, \boldsymbol{e}(\boldsymbol{u})$.

EXAMPLE 1. *Coulomb's law of two-sided contact with friction.* Consider the problem:

$$(24)\qquad \left.\begin{aligned}
&-\mathbf{div}\,(\mathbf{\mathfrak{a}}(\varepsilon^{-1}x)\, \boldsymbol{e}(\boldsymbol{u}^\varepsilon)) = \boldsymbol{F} \quad \text{in} \quad \Omega^\varepsilon,\\
&\boldsymbol{u}^\varepsilon = \mathbf{0} \ \text{ on } \Gamma^\varepsilon, \quad \boldsymbol{\sigma}(\boldsymbol{u}^\varepsilon)\, \boldsymbol{\nu}^\varepsilon = \mathbf{0} \quad \text{on} \quad \partial\Omega^\varepsilon \setminus (\Gamma^\varepsilon \cup \Sigma^\varepsilon_{\mathrm{cont}}),\\
&\sigma_{\mathrm{N}}(\boldsymbol{u}^\varepsilon) = \mu_1(\varepsilon)\, \phi(\varepsilon^{-1}x) \quad \text{on} \quad \Sigma^\varepsilon_{\mathrm{cont}},\\
&\text{for } x \in \Gamma^\varepsilon_{\mathrm{cont}} \text{ the following implications hold:}\\
&|\boldsymbol{\sigma}_{\mathrm{T}}(\boldsymbol{u}^\varepsilon)| < \mu(\varepsilon)\, \psi(\varepsilon^{-1}x) \ \Rightarrow\ \boldsymbol{u}^\varepsilon_{\mathrm{T}} = 0\,,\\
&|\boldsymbol{\sigma}_{\mathrm{T}}(\boldsymbol{u}^\varepsilon)| = \mu(\varepsilon)\, \psi(\varepsilon^{-1}x) \ \Rightarrow\ \exists\, \lambda \geqslant 0 : \boldsymbol{u}^\varepsilon_{\mathrm{T}} = -\lambda \boldsymbol{\sigma}_{\mathrm{T}}(\boldsymbol{u}^\varepsilon)\,.
\end{aligned}\right\}$$

Here $\mu_1(\varepsilon)$, $\mu(\varepsilon)$ are nonnegative parameters specified below; the scalar 1-periodic functions $\phi(\xi) \in L^\infty_{\mathrm{per}}(\partial\omega)$ characterize normal stresses, while $\psi(\xi) \in L^\infty_{\mathrm{per}}(\partial\omega)$ describe friction forces, and we assume that

$$\phi(\xi)\,|_{\partial\omega\setminus\partial_0\omega} = \psi(\xi)\,|_{\partial\omega\setminus\partial_0\omega} = 0\,, \qquad \psi(\xi)\,|_{\partial_0\omega} \geqslant \kappa_0 = \text{const} > 0\,.$$

A mechanical interpretation of such problems in terms of two-sided contact with friction on $\Sigma^\varepsilon_{\mathrm{cont}}$ described by the Coulomb law is given in [DL], together with the justification and the definition of weak solutions considered here.

Let us introduce a convex continuous functional $J^\varepsilon(\boldsymbol{v})$ on $(H^1(\Omega^\varepsilon))^n$ (in general, nondifferentiable), setting

$$(25)\qquad J^\varepsilon(\boldsymbol{v}) = \mu(\varepsilon) \int\limits_{\Sigma^\varepsilon_{\mathrm{cont}}} \psi(\varepsilon^{-1}x)\, |\boldsymbol{v}_{\mathrm{T}}|\, \mathrm{d}S - \mu_1(\varepsilon) \int\limits_{\Sigma^\varepsilon_{\mathrm{cont}}} \phi(\varepsilon^{-1}x)\, v_{\mathrm{N}}\, \mathrm{d}S =$$

$$= \mu(\varepsilon) \int\limits_{\Sigma^\varepsilon_{\mathrm{cont}}} \psi(\varepsilon^{-1}x)\, |\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{\nu}^\varepsilon)\, \boldsymbol{\nu}^\varepsilon|\, \mathrm{d}S - \mu_1(\varepsilon) \int\limits_{\Sigma^\varepsilon_{\mathrm{cont}}} \phi(\varepsilon^{-1}x)\, \boldsymbol{v} \cdot \boldsymbol{\nu}^\varepsilon\, \mathrm{d}S\,.$$

According to [DL], *the weak solution of problem* (24) is the displacement

$\boldsymbol{u}^\varepsilon$ in $(H_0^1(\Omega^\varepsilon, \Gamma^\varepsilon))^n$ that satisfies the variational inequality (1) with the functional (25).

In order to apply Theorem 1, let us consider more closely the functional (25). Clearly,

$$J^\varepsilon(\boldsymbol{v}) = \mu(\varepsilon) \int_{S^\varepsilon} \Theta(\boldsymbol{v}, \varepsilon^{-1}x)\, \mathrm{d}S + \mu_1(\varepsilon) \int_{S^\varepsilon} \Lambda(\boldsymbol{v}, \varepsilon^{-1}x)\, \mathrm{d}S\,,$$

with

$$\Theta(\boldsymbol{\eta}, \xi) = \begin{cases} \psi(\xi)\, |\boldsymbol{\eta} - (\boldsymbol{\eta}\cdot\boldsymbol{v}(\xi))\, \boldsymbol{v}(\xi)\,|\,, & \xi \in \partial_0\omega\,, \\ 0\,, & \xi \in \partial\omega \backslash \partial_0\omega\,, \end{cases}$$

$$\Lambda(\boldsymbol{\eta}, \xi) = \begin{cases} -\phi(\xi)(\boldsymbol{\eta}\cdot\boldsymbol{v}(\xi))\,, & \xi \in \partial_0\omega\,, \\ 0\,, & \xi \in \partial\omega \backslash \partial_0\omega\,, \end{cases}$$

where $\boldsymbol{v}(\xi)$ is the unit outward normal to $\partial\omega$. Then the functions

$$\Theta^0(\boldsymbol{\eta}) = \frac{1}{|\partial_\square\omega|} \int_{\partial_\square\omega \cap \partial_0\omega} \psi(\xi)\, |\boldsymbol{\eta} - (\boldsymbol{\eta}\cdot\boldsymbol{v}(\xi))\, \boldsymbol{v}(\xi)\,|\, \mathrm{d}S\,,$$

$$\Lambda^0(\boldsymbol{\eta}) = \frac{-1}{|\partial_\square\omega|} \int_{\partial_\square\omega \cap \partial_0\omega} \phi(\xi)(\boldsymbol{\eta}\cdot\boldsymbol{v}(\xi))\, \mathrm{d}S$$

are, respectively, the mean values of $\Theta(\boldsymbol{\eta}, \xi)$, $\Lambda(\boldsymbol{\eta}, \xi)$ in $\xi \in \partial\omega$. Obviously,

$$\Theta(\boldsymbol{\eta}, \xi) \in \mathcal{B}_0,\ \Lambda(\boldsymbol{\eta}, \xi) \in \mathcal{B}_1,\ b\Theta(\boldsymbol{\eta}, \xi) + a\Lambda(\boldsymbol{\eta}, \xi) \in \mathcal{B}_1,\ \forall\, a, b = \mathrm{const} \geq 0\,.$$

Consider two qualitatively different cases:

1) $\varepsilon^{-1}\mu(\varepsilon) \to b$, $\varepsilon^{-1}\mu_1(\varepsilon) \to a$. Then, according to Remark 2, the homogenized problem for (1), (25) coincides with (10), where

$$\mathcal{W}_0 = (H_0^1(\Omega))^n, \qquad \hat{J}(\boldsymbol{v}) = \int_\Omega (b\Theta^0(\boldsymbol{v}) + a\Lambda^0(\boldsymbol{v}))\, \mathrm{d}x\,.$$

2) $\varepsilon^{-1}\mu(\varepsilon) \to \infty$, $\varepsilon^{-1}\mu_1(\varepsilon) \to a$. Then by Theorem 1, the homogenized problem has the form (10) with

$$\mathcal{W}_0 = \{\boldsymbol{v} \in (H_0^1(\Omega))^n\colon \Theta^0(\boldsymbol{v})\,|_\Omega = 0\}\,, \quad \hat{J}(\boldsymbol{v}) = a \int_\Omega \Lambda^0(\boldsymbol{v})\, \mathrm{d}x\,,$$

provided that either $\mu(\varepsilon) \to 0$ or the set $\{\boldsymbol{\eta} \in \mathbb{R}^n\colon \Theta^0(\boldsymbol{\eta}) = 0\}$ has interior points in $\mathbb{R}^n$. In order to illustrate the influence of the geometry of $\partial_0 \omega$ on the set of admissible displacements $\mathcal{W}_0$, consider the case $n = 3$. Since $\psi(\xi)\,|_{\partial_0 \omega} \geqslant \kappa_0 > 0$ by assumption, the relation $\Theta^0(\boldsymbol{\eta}) = 0$ means that

$$\boldsymbol{\eta} = (\boldsymbol{\eta} \cdot \boldsymbol{\nu}(\xi))\, \boldsymbol{\nu}(\xi) \qquad \text{a.e. in } \xi \in \partial_0 \omega .$$

If the set $\partial_0 \omega \cap \partial_\square \omega$ contains three planar regions on which $\boldsymbol{\nu}(\xi)$ takes three linearly independent values $\boldsymbol{\alpha}^1$, $\boldsymbol{\alpha}^2$, $\boldsymbol{\alpha}^3$, then, obviously, the implication holds $\Theta^0(\boldsymbol{\eta}) = 0 \Rightarrow \boldsymbol{\eta} = \mathbf{0}$, and therefore, $\mathcal{W}_0 = \{\mathbf{0}\}$. If $\partial_0 \omega \cap \partial_\square \omega$ consists of two planar regions with linearly independent normals $\boldsymbol{\alpha}^1$, $\boldsymbol{\alpha}^2$, then

$$\mathcal{W}_0 = \{\boldsymbol{v} \in (H_0^1(\Omega))^3 \colon \boldsymbol{v}\,|_{\Omega} \cdot \boldsymbol{\alpha}^1 = \boldsymbol{v}\,|_{\Omega} \cdot \boldsymbol{\alpha}^2 = 0\} .$$

If $\partial_0 \omega \cap \partial_\square \omega$ consists of parallel planar regions on which $\boldsymbol{\nu}(\xi)$ takes only two values $\boldsymbol{\alpha}^1$ and $-\boldsymbol{\alpha}^1$, then

$$\boldsymbol{\eta} = (\boldsymbol{\eta} \cdot \boldsymbol{\alpha}^1)\, \boldsymbol{\alpha}^1 \Leftrightarrow \boldsymbol{\eta} \cdot \boldsymbol{\alpha}^2 = \boldsymbol{\eta} \cdot \boldsymbol{\alpha}^3 = 0 ,$$

where $\boldsymbol{\alpha}^1$, $\boldsymbol{\alpha}^2$, $\boldsymbol{\alpha}^3$ form an orthonormal basis in $\mathbb{R}^3$. In this case

$$\mathcal{W}_0 = \{\boldsymbol{v} \in (H_0^1(\Omega))^3 \colon \boldsymbol{v}\,|_{\Omega} \cdot \boldsymbol{\alpha}^2 = \boldsymbol{v}\,|_{\Omega} \cdot \boldsymbol{\alpha}^3 = 0\} .$$

EXAMPLE 2. *Normal displacement with friction.* Consider the problem

$$\left.\begin{aligned}
&-\mathbf{div}\,(\mathcal{A}(\varepsilon^{-1}x)\, \boldsymbol{e}(\boldsymbol{u}^\varepsilon)) = \boldsymbol{F} \quad \text{in} \quad \Omega^\varepsilon, \\
&\boldsymbol{u}^\varepsilon = \mathbf{0} \text{ on } \Gamma^\varepsilon, \quad \boldsymbol{\sigma}(\boldsymbol{u}^\varepsilon)\, \boldsymbol{\nu}^\varepsilon = \mathbf{0} \text{ on } \partial\Omega^\varepsilon \backslash (\Gamma^\varepsilon \cup \Sigma^\varepsilon_{\mathrm{cont}}), \\
&\boldsymbol{\sigma}_{\mathrm{T}}(\boldsymbol{u}^\varepsilon) = \mathbf{0} \quad \text{on} \quad \Sigma^\varepsilon_{\mathrm{cont}}, \\
&\quad \text{for } x \in \Sigma^\varepsilon_{\mathrm{cont}}, \text{ the following implications hold:} \\
&-\mu_1(\varepsilon)\, g_1(\varepsilon^{-1}x) < \sigma_{\mathrm{N}}(\boldsymbol{u}^\varepsilon) < \mu_2(\varepsilon)\, g_2(\varepsilon^{-1}x) \Rightarrow u^\varepsilon_{\mathrm{N}} = 0 , \\
&\quad \sigma_{\mathrm{N}}(\boldsymbol{u}^\varepsilon) = -\mu_1(\varepsilon) g_1(\varepsilon^{-1}x) \Rightarrow u^\varepsilon_{\mathrm{N}} \geqslant 0 , \\
&\quad \sigma_{\mathrm{N}}(\boldsymbol{u}^\varepsilon) = \quad \mu_2(\varepsilon)\, g_2(\varepsilon^{-1}x) \Rightarrow u^\varepsilon_{\mathrm{N}} \leqslant 0 .
\end{aligned}\right\} \tag{26}$$

Here $\mu_j(\varepsilon) \geqslant 0$ are real parameters, $g_j(\xi) \geqslant 0$ $(j = 1, 2)$ are 1-periodic functions in $L^\infty(\partial\omega)$ characterizing friction forces. We are going to study weak solutions of this problem, as defined in [DL].

Let us introduce a convex continuous functional $J^{\varepsilon}(\boldsymbol{v})$ on $(H^1(\Omega^{\varepsilon}))^n$, setting

$$(27)\qquad J^{\varepsilon}(\boldsymbol{v}) = \int\limits_{\Sigma^{\varepsilon}_{\text{cont}}} (\mu_1(\varepsilon)\, g_1(\varepsilon^{-1}x)\, v_{\mathrm{N}}^{+} + \mu_2(\varepsilon)\, g_2(\varepsilon^{-1}x)\, v_{\mathrm{N}}^{-})\, \mathrm{d}S =$$

$$= \int\limits_{\Sigma^{\varepsilon}_{\text{cont}}} (\mu_1(\varepsilon)\, g_1(\varepsilon^{-1}x)(\boldsymbol{v}\cdot\boldsymbol{\nu}^{\varepsilon})^{+} + \mu_2(\varepsilon)\, g_2(\varepsilon^{-1}x)(\boldsymbol{v}\cdot\boldsymbol{\nu}^{\varepsilon})^{-})\, \mathrm{d}S\,,$$

where $a^{+} = \max\{a, 0\}$, $a^{-} = \max\{-a, 0\}$, $\boldsymbol{\nu}^{\varepsilon}(x) = \boldsymbol{\nu}(\varepsilon^{-1}x)$.

According to [DL], *the weak solution of problem* (26) is the displacement $\boldsymbol{u}^{\varepsilon} \in (H_0^1(\Omega^{\varepsilon}, \Gamma^{\varepsilon}))^n$ that satisfies the inequality (1) with the functional (27).

The functional (27) has the form

$$(28)\qquad J^{\varepsilon}(\boldsymbol{v}) = \mu_1(\varepsilon) \int\limits_{S^{\varepsilon}} G_1(\boldsymbol{v}, \varepsilon^{-1}x)\, \mathrm{d}S + \mu_2(\varepsilon) \int\limits_{S^{\varepsilon}} G_2(\boldsymbol{v}, \varepsilon^{-1}x)\, \mathrm{d}S\,,$$

where

$$G_1(\boldsymbol{\eta}, \xi) = \begin{cases} g_1(\xi)(\boldsymbol{\eta}\cdot\boldsymbol{\nu}(\xi))^{+}, & \xi \in \partial_0\omega\,, \\ 0\,, \quad \xi \in \partial\omega \backslash \partial_0\omega\,, \end{cases}$$

$$G_2(\boldsymbol{\eta}, \xi) = \begin{cases} g_2(\xi)(\boldsymbol{\eta}\cdot\boldsymbol{\nu}(\xi))^{-}, & \xi \in \partial_0\omega\,, \\ 0\,, \quad \xi \in \partial\omega \backslash \partial_0\omega \end{cases}$$

are functions of class $\mathcal{B}_0$ with the mean values

$$G_1^0(\boldsymbol{\eta}) = \frac{1}{|\partial_{\square}\omega|} \int\limits_{\partial_{\square}\omega \cap \partial_0\omega} g_1(\xi)(\boldsymbol{\eta}\cdot\boldsymbol{\nu}(\xi))^{+}\, \mathrm{d}S\,,$$

$$G_2^0(\boldsymbol{\eta}) = \frac{1}{|\partial_{\square}\omega|} \int\limits_{\partial_{\square}\omega \cap \partial_0\omega} g_2(\xi)(\boldsymbol{\eta}\cdot\boldsymbol{\nu}(\xi))^{-}\, \mathrm{d}S\,.$$

Consider three qualitatively different cases:

1) If $\varepsilon^{-1}\mu_1 \to a_1 < \infty$, $\varepsilon^{-1}\mu_2(\varepsilon) \to a_2 < \infty$, then by Theorem 1 and Remark 2, the homogenized problem for (26) (in terms of (1), (28)) has the form of the variational inequality (10) with

$$\mathcal{W}_0 = (H_0^1(\Omega))^n, \quad \widehat{J}(\boldsymbol{v}) = \gamma \int\limits_{\Omega} (a_1 G_1^0(\boldsymbol{v}) + a_2 G_2^0(\boldsymbol{v}))\, \mathrm{d}x\,.$$

2) If $\varepsilon^{-1}\mu_1 \to a_1 < \infty$, $\varepsilon^{-1}\mu_2(\varepsilon) \to \infty$, then the homogenized problem is (10) with

$$\mathfrak{W}_0 = \{v \in (H_0^1(\Omega))^n\colon G_2^0(v)\,|_\Omega = 0\}, \qquad \hat{J}(v) = a_1\gamma \int_\Omega G_1^0(v)\,\mathrm{d}x\,,$$

provided that either the set $\{\boldsymbol{\eta} \in \mathbb{R}^n\colon G_2^0(\boldsymbol{\eta}) = 0\}$ has interior points in $\mathbb{R}^n$ or $\mu_2(\varepsilon) \to 0$.

3) If $\varepsilon^{-1}\mu_1 \to \infty$, $\varepsilon^{-1}\mu_2(\varepsilon) \to \infty$, then the homogenized problem is (10) with

$$\mathfrak{W}_0 = \{v \in (H_0^1(\Omega))^n\colon G_1^0(v)\,|_\Omega = G_2^0(v)\,|_\Omega = 0\}, \qquad \hat{J}(v) \equiv 0\,,$$

provided that either the set $\{\boldsymbol{\eta} \in \mathbb{R}^n\colon G_1^0(\boldsymbol{\eta}) = G_2^0(\boldsymbol{\eta}) = 0\}$ has interior points in $\mathbb{R}^n$ or $\mu_1(\varepsilon) \to 0$, $\mu_2(\varepsilon) \to 0$.

Note that *the limit set of admissible displacements depends on the geometry of* $\Sigma^\varepsilon_{\mathrm{cont}}$. Consider, for instance, $\mathfrak{W}_0 = \{v \in (H_0^1(\Omega))^n\colon G_2^0(v)\,|_\Omega = 0\}$ and assume that $g_2(\xi) > 0$ on $\partial_0\omega$. Then the condition $G_2^0(\boldsymbol{\eta}) = 0$ means that $\boldsymbol{\eta}\cdot\boldsymbol{\nu}(\xi)\,|_{\partial_0\omega} \geqslant 0$ a.e. Thus, if $\partial_0\omega$ consists of plane mutually parallel regions on which $\boldsymbol{\nu}(\xi)$ does not change direction, say $\nu_1 = 1$, then $\mathfrak{W}_0 = \{v\colon v_1\,|_\Omega \geqslant 0\}$; if $\boldsymbol{\nu}(\xi)$ changes direction, say $\nu_1 = \pm 1$, then $\mathfrak{W}_0 = \{v : v_1\,|_\Omega = 0\}$.

It is easy to indicate various other instances of the dependence of $\mathfrak{W}_0$ on the geometry of $\partial_0\omega$, and also to give other examples of mechanical problems that fit into the framework of Theorem 1.

3. Problems with boundary conditions of Signorini type in partially perforated domains.

For perforated domains $\Omega^\varepsilon \cap \varepsilon\omega$ as in the previous sections, homogenization of some unilateral elasticity problems with nonlinear boundary conditions, in particular, of Signorini type, has been considered in [Y2], Sect. 4. Here we extend these results to the case of partially perforated domains and somewhat more general boundary conditions.

Let $\Omega_0 \subset \Omega \subset \mathbb{R}^n$ be bounded Lipschitz domains, and let $\omega \subset \mathbb{R}^n$ be a 1-periodic domain of the type considered in Sect. 1, with the cell of periodicity $\omega_\square = \omega \cap \square$. In the perforated domain $\Omega \cap \varepsilon\omega$ let us fill up all cells of the form $\varepsilon(z + \omega_\square)$, $z \in \mathbb{Z}^n$, which lie outside Ω_0 or have a nonempty intersection with the layer of width $c\varepsilon$ near $\partial\Omega_0$. As a result, from $\Omega \cap \cap\, \varepsilon\omega$ we obtain a domain Ω^ε with a perforated part in Ω_0. Formally, such a domain Ω^ε can be defined as follows. Let I_ε be the subset of $\mathbb{Z}^n$ consisting of all z such that $\varepsilon(z+\square)\subset\Omega_0$, $\operatorname{dist}(\varepsilon(z+\square),\partial\Omega_0)\geqslant c\varepsilon$. Let

$$Q^\varepsilon=\operatorname{int}\bigcup_{z\in I_\varepsilon}\varepsilon(z+\overline{\square}),\qquad \Omega^\varepsilon=\operatorname{int}((\Omega\backslash Q^\varepsilon)\cup\overline{Q^\varepsilon\cap\varepsilon\omega}).$$

It is assumed that Q^ε, $Q^\varepsilon\cap\varepsilon\omega$, and Ω^ε are Lipschitz domains. The boundary $\partial\Omega^\varepsilon$ is the union of $\partial\Omega$ and the surface of cavities $S_0^\varepsilon=(\partial\Omega^\varepsilon)\cap\Omega\subset\subset\Omega_0$.

Consider an elastic body occupying the domain Ω^ε with the elasticity tensor

$$\mathfrak{a}^\varepsilon(x)=\begin{cases}\mathfrak{a}_0(\varepsilon^{-1}x), & x\in Q^\varepsilon\cap\varepsilon\omega\,,\\ \mathfrak{a}_1(x), & x\in\Omega^\varepsilon\backslash(Q^\varepsilon\cap\varepsilon\omega),\end{cases}$$

where $\mathfrak{a}_0(\xi)$ is 1-periodic in ξ and has the same structure as the tensor $\mathfrak{a}(\xi)$ in Sect. 1, whereas $\mathfrak{a}_1(x)$ is another elasticity tensor independent of ε, defined for all $x\in\mathbb{R}^n$, and satisfying the usual conditions of symmetry and positive definiteness on $\mathbb{S}^n$ with constants $\tilde\kappa_1,\tilde\kappa_2>0$.

The class of problems studied in this section includes the following model problem of *one-sided contact without friction* (see [DL], [F]) on a subset $\Sigma^\varepsilon_{\mathrm{cont}}\subset\partial\Omega^\varepsilon$, namely,

$$-\mathbf{div}(\mathfrak{a}^\varepsilon(x)\,\boldsymbol{e}(\boldsymbol{u}^\varepsilon))=\boldsymbol{f}^\varepsilon\ \text{ in }\Omega^\varepsilon,\qquad \boldsymbol{\sigma}(\boldsymbol{u}^\varepsilon)\,\boldsymbol{\nu}^\varepsilon=\boldsymbol{0}\ \text{ on }(\partial\Omega^\varepsilon)\backslash\Sigma^\varepsilon_{\mathrm{cont}},$$

$$\boldsymbol{\sigma}_{\mathrm{T}}(\boldsymbol{u}^\varepsilon)=\boldsymbol{0},\qquad u^\varepsilon_{\mathrm{N}}\leqslant0\,,\qquad \sigma_{\mathrm{N}}(\boldsymbol{u}^\varepsilon)\geqslant0\,,\qquad \sigma_{\mathrm{N}}(\boldsymbol{u}^\varepsilon)\,u^\varepsilon_{\mathrm{N}}=0\ \text{ on }\Sigma^\varepsilon_{\mathrm{cont}},$$

where $\Sigma^\varepsilon_{\mathrm{cont}}=\Gamma\cup(S_0^\varepsilon\cap\varepsilon\partial_0\omega)$, Γ is a measurable subset of $\partial\Omega$ and $\partial_0\omega$ is a nonempty 1-periodic open subset on $\partial\omega$. The above relations on $\Sigma^\varepsilon_{\mathrm{cont}}$ are called the *Signorini boundary conditions.*

It is well known (see, for instance, [F], [DL]) that problems of this type can be formulated in terms of variational inequalities on certain closed convex sets of admissible displacements in $(H^1(\Omega^\varepsilon))^n$. In particular, the *weak solution* of the above problem is the displacement $\boldsymbol{u}^\varepsilon\in\mathfrak{W}_\varepsilon=\{\boldsymbol{v}\in(H^1(\Omega^\varepsilon))^n:(\boldsymbol{v}\cdot\boldsymbol{\nu}^\varepsilon)^+|_{\Sigma^\varepsilon_{\mathrm{cont}}}=0\}$ satisfying the inequality

$$\text{(29)}\qquad \int_{\Omega^\varepsilon}\boldsymbol{e}(\boldsymbol{v}-\boldsymbol{u}^\varepsilon):\mathfrak{a}^\varepsilon(x)\,\boldsymbol{e}(\boldsymbol{u}^\varepsilon)\,\mathrm{d}x\geqslant\int_{\Omega^\varepsilon}(\boldsymbol{v}-\boldsymbol{u}^\varepsilon)\cdot\boldsymbol{f}^\varepsilon\,\mathrm{d}x\,,\qquad \forall\boldsymbol{v}\in\mathfrak{W}_\varepsilon,$$

which means that the boundary conditions on $\Sigma^\varepsilon_{\mathrm{cont}}$ have the form $\boldsymbol{v}\cdot\boldsymbol{\nu}^\varepsilon\leqslant\leqslant0$. We are going to consider somewhat more general sets of admissible displacements, namely,

(30) $$\mathfrak{W}_\varepsilon = \{ \boldsymbol{v} \in (H^1(\Omega^\varepsilon))^n :\ \Psi(\boldsymbol{v}(x), \varepsilon^{-1}x)\,|_{S_0^\varepsilon} = 0,\ g(\boldsymbol{v}(x), x)\,|_{\partial\Omega} = 0 \},$$

where $\Psi(\boldsymbol{\eta}, \xi)$ is a function of class $\mathfrak{B}_0$ defined in Sect. 1, while $g(\boldsymbol{\eta}, x)$ is a function on $\mathbb{R}^n \times \partial\Omega$, measurable in $x \in \partial\Omega$ for any $\boldsymbol{\eta}$, and satisfying the conditions:

(i) $|g(\boldsymbol{\eta}^1, x) - g(\boldsymbol{\eta}^2, x)| \leqslant c\,|\boldsymbol{\eta}^1 - \boldsymbol{\eta}^2|$, $\forall x \in \partial\Omega$, $\forall \boldsymbol{\eta}^1, \boldsymbol{\eta}^2 \in \mathbb{R}^n$;

(ii) $g(\boldsymbol{\eta}, x)$ is convex in $\boldsymbol{\eta}$, $\forall x \in \partial\Omega$.

(iii) $0 \leqslant g(t\boldsymbol{\eta}, x) = tg(\boldsymbol{\eta}, x)$, $\forall \boldsymbol{\eta} \in \mathbb{R}^n$, $\forall x \in \partial\Omega$, $\forall t \geqslant 0$.

Obviously, the boundary condition $\boldsymbol{v} \cdot \boldsymbol{\nu}^\varepsilon |_{\Sigma^\varepsilon_{\text{cont}}} \leqslant 0$ can be defined in terms of (30) with $g(\boldsymbol{\eta}, x) = \chi_\Gamma(x)(\boldsymbol{\eta} \cdot \boldsymbol{\nu}(x))^+$, $\Psi(\boldsymbol{\eta}, \xi) = \chi_{\partial_0\omega}(\xi)(\boldsymbol{\eta} \cdot \boldsymbol{\nu}(\xi))^+$, where $\boldsymbol{\nu}(x)$ and $\boldsymbol{\nu}(\xi)$ are the unit outward normals to $\partial\Omega$ and $\partial\omega$, respectively; and $\chi_\Gamma(x)$, $\chi_{\partial_0\omega}(\xi)$ are the characteristic functions of the sets Γ and $\partial_0\omega$.

In order to formulate the homogenized problem for (29), (30), we define in the domain Ω the elasticity tensor

(31) $$\widehat{\mathfrak{A}}(x) = |\omega_\square|^{-1} \chi_{\Omega\setminus\Omega_0}(x)\, \mathfrak{A}_1(x) + \chi_{\Omega_0}(x)\, \widehat{\mathfrak{A}}_0,$$

where $\widehat{\mathfrak{A}}_0$ is the homogenized tensor (8) corresponding to $\mathfrak{A}(\xi) \equiv \mathfrak{A}_0(\xi)$, and $\chi_G(x)$ is the characteristic function of the set G. We also introduce the set of admissible displacements

(32) $$\mathfrak{W}_0 = \{ \boldsymbol{v} \in (H^1(\Omega))^n :\ \overline{\Psi}(\boldsymbol{v}(x))\,|_{\Omega_0} = 0,\ g(\boldsymbol{v}(x), x)\,|_{\partial\Omega} = 0 \},$$

where $\overline{\Psi}(\boldsymbol{\eta})$ is the mean value of $\Psi(\boldsymbol{\eta}, \xi)$ in $\xi \in \partial\omega$. With the help of Lemma 6 it is easy to show that $\mathfrak{W}_\varepsilon$ and $\mathfrak{W}_0$ defined by (30) and (32) are closed convex cones in $(H^1(\Omega^\varepsilon))^n$ and $(H^1(\Omega))^n$, respectively.

Our aim is to establish closeness of the solutions $\boldsymbol{u}^\varepsilon \in \mathfrak{W}_\varepsilon$ of problem (29), (30) to the solution $\boldsymbol{u}^0 \in \mathfrak{W}_0$ of the homogenized problem

(33) $$\int_\Omega \boldsymbol{e}(\boldsymbol{v} - \boldsymbol{u}^0) : \widehat{\mathfrak{A}}(x)\, \boldsymbol{e}(\boldsymbol{u}^0)\, \mathrm{d}x \geqslant$$

$$\geqslant \int_\Omega (\boldsymbol{v} - \boldsymbol{u}^0) \cdot (|\omega_\square|^{-1} \chi_{\Omega\setminus\Omega_0} \boldsymbol{f}^0 + \chi_{\Omega_0} \boldsymbol{f}^0)\, \mathrm{d}x, \quad \forall \boldsymbol{v} \in \mathfrak{W}_0,$$

with the tensor (31), provided that

$$\sup_{\varepsilon} \|\boldsymbol{f}^{\varepsilon}\|_{0,\,\Omega^{\varepsilon}} < \infty, \qquad \|\boldsymbol{f}^{\varepsilon} - \boldsymbol{f}^{0}\|_{0,\,\Omega^{\varepsilon}} \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

where $\|\cdot\|_{0,\,\Omega^{\varepsilon}} = \|\cdot\|_{L^2(\Omega^{\varepsilon})}$.

We also impose some additional conditions which ensure the solvability of problems (29), (30) and (33). In particular, under the *coerciveness conditions*, which in our case amount to the inequalities of Korn's type

$$\|\boldsymbol{e}(\boldsymbol{v})\|_{0,\,\Omega^{\varepsilon}} \geqslant C_1 \|\boldsymbol{v}\|_{1,\,\Omega^{\varepsilon}}, \qquad \forall \boldsymbol{v} \in \mathcal{W}_{\varepsilon}, \tag{34}$$

$$\|\boldsymbol{e}(\boldsymbol{v})\|_{0,\,\Omega} \geqslant C_2 \|\boldsymbol{v}\|_{1,\,\Omega}, \qquad \forall \boldsymbol{v} \in \mathcal{W}_{0}, \tag{35}$$

C_1, $C_2 = \text{const} > 0$, each of the said problems admits one and only one solution (see [F], [DL], [L]). Note that the inequalities (34), (35) hold if there is a nonempty open (in the induced topology) set $\gamma \subset \partial\Omega$ and $g(\boldsymbol{\eta}, x)\,|_{x \in \gamma} = |\boldsymbol{\eta}|$. Then the condition $g(\boldsymbol{v}, x)\,|_{\partial\Omega} = 0$ implies that $\boldsymbol{v}\,|_{\gamma} = 0$ and the Korn inequality holds for $\boldsymbol{v}$ (see [OSY], Ch. I). Note that the boundary conditions we want to consider on $\partial\Omega$ may be other than those of the Dirichlet type: for $g(\boldsymbol{\eta}, x) \equiv 0$ we have zero Neumann condition on $\partial\Omega$, whereas for $g(\boldsymbol{\eta}, x) \equiv \chi_{\gamma}(x)(\boldsymbol{\eta}\cdot\boldsymbol{\nu}(x))^{+}$ the boundary condition on γ is of Signorini type (here $\boldsymbol{\nu}(x)$ is the unit outward normal to $\partial\Omega$). Thus, there may be no coerciveness in the sense of (34), (35). To ensure the solvability of the above problems in the absence of coerciveness, it suffices to make the following assumptions.

In the space of rigid displacements $\mathcal{R}$ consider the subspaces

$$\mathcal{R}_{\varepsilon} = \{\boldsymbol{\varrho} \in \mathcal{R}:\ \pm\boldsymbol{\varrho} \in \mathcal{W}_{\varepsilon}\},\ \mathcal{R}_{f^{\varepsilon}} = \{\boldsymbol{\varrho} \in \mathcal{R}: (\boldsymbol{\varrho}, \boldsymbol{f}^{\varepsilon})_{0,\,\Omega^{\varepsilon}} = 0\},$$

$$\mathcal{R}_{0} = \{\boldsymbol{\varrho} \in \mathcal{R}:\ \pm\boldsymbol{\varrho} \in \mathcal{W}_{0}\}, \quad \mathcal{R}_{f^{0}} = \{\boldsymbol{\varrho} \in \mathcal{R}: [\boldsymbol{\varrho}, \boldsymbol{f}^{0}]_{0,\,\Omega} = 0\},$$

where

$$(\boldsymbol{f}, \boldsymbol{h})_{0,\,G} = \int_G \boldsymbol{f}\cdot\boldsymbol{h}\,\mathrm{d}x, \quad [\boldsymbol{f}, \boldsymbol{h}]_{0,\,\Omega} = |\omega_{\square}|^{-1}(\boldsymbol{f}, \boldsymbol{h})_{0,\,\Omega\setminus\Omega_0} + (\boldsymbol{f}, \boldsymbol{h})_{0,\,\Omega_0}.$$

Then, according to the results of [F], Part II, the conditions

$$\mathcal{R}_{\varepsilon} \subset \mathcal{R}_{f^{\varepsilon}}, \quad (\boldsymbol{f}^{\varepsilon}, \boldsymbol{\varrho})_{0,\,\Omega^{\varepsilon}} < 0, \quad \forall \boldsymbol{\varrho} \in (\mathcal{R} \cap \mathcal{W}_{\varepsilon}) \setminus \mathcal{R}_{\varepsilon}, \tag{36}$$

$$\mathcal{R}_{0} \subset \mathcal{R}_{f^{0}}, \quad [\boldsymbol{f}^{0}, \boldsymbol{\varrho}]_{0,\,\Omega} < 0, \quad \forall \boldsymbol{\varrho} \in (\mathcal{R} \cap \mathcal{W}_{0}) \setminus \mathcal{R}_{0} \tag{37}$$

guarantee the solvability of problems (29), (30) and (33), respectively.

Moreover, the solution is unique to within rigid displacements: if $\boldsymbol{u}^\varepsilon$ is a solution of problem (29), (30), then any other its solution has the form $\boldsymbol{u}^\varepsilon + \boldsymbol{\varrho}^\varepsilon$, where $\boldsymbol{\varrho}^\varepsilon \in \mathcal{R}$, $(\boldsymbol{f}^\varepsilon, \boldsymbol{\varrho}^\varepsilon)_{0,\Omega^\varepsilon} = 0$, $\boldsymbol{u}^\varepsilon + \boldsymbol{\varrho}^\varepsilon \in \mathcal{W}_\varepsilon$. A similar uniqueness result holds for solutions of problem (33).

In what follows, we need extension operators and Korn's inequalities in partially perforated domains Ω^ε.

LEMMA 7. *There exist linear extension operators* $P_\varepsilon : (H^1(\Omega^\varepsilon))^n \to (H^1(\Omega))^n$ *such that* $\sup_\varepsilon \|P_\varepsilon\| < \infty$ *and* $P_\varepsilon \boldsymbol{\eta} = \boldsymbol{\eta}$ *for all* $\boldsymbol{\eta} \in \mathcal{R}$. *The second Korn inequality*

$$\|\boldsymbol{u} - \Pi^\varepsilon_{\mathcal{R}} \boldsymbol{u}\|_{1,\Omega^\varepsilon} \leqslant C \|e(\boldsymbol{u})\|_{0,\Omega^\varepsilon}, \quad \forall \boldsymbol{u} \in (H^1(\Omega^\varepsilon))^n$$

holds with a constant C *independent of* ε, *where* $\Pi^\varepsilon_{\mathcal{R}} : (H^1(\Omega^\varepsilon))^n \to \mathcal{R}$ *is the operator of orthogonal projection.*

This lemma follows from Theorems 3.9 and 3.11 of [Y3]. It can also be obtained by slightly modifying the corresponding results of [OSY], Ch. I, § 4.

Let $\boldsymbol{\Gamma}^\varepsilon(x) \in (L^2(\Omega))^{n^2}$ be the generalized gradients of the solutions $\boldsymbol{u}^\varepsilon$:

$$\boldsymbol{\Gamma}^\varepsilon(x) = \begin{cases} \mathfrak{A}^\varepsilon(x)\, \boldsymbol{e}(\boldsymbol{u}^\varepsilon) & \text{for } x \in \Omega^\varepsilon, \\ 0 & \text{for } x \notin \Omega^\varepsilon, \end{cases}$$

and let P_ε be the extension operators from Lemma 7. The following homogenization theorem establishes closeness of the solutions of problems (29), (30) and (33).

THEOREM 2. *Suppose that the set* $\{\boldsymbol{\eta} \in \mathbb{R}^n : \overline{\Psi}(\boldsymbol{\eta}) = 0\}$ *has internal points in the topology of* $\mathbb{R}^n$.

(i) *If the solvability conditions* (36), (37) *hold and* $\dim \mathcal{R}_\varepsilon = \dim \mathcal{R}_0$, *then for any sequence of solutions* $\boldsymbol{u}^\varepsilon$ *of problem* (29), (30) *there exist rigid displacements* $\boldsymbol{\zeta}^\varepsilon$ *such that for* $\varepsilon \to 0$ *we have*

$$P_\varepsilon \boldsymbol{u}^\varepsilon - \boldsymbol{\zeta}^\varepsilon \to \boldsymbol{u}^0 \text{ in } (L^2(\Omega))^n, \quad \boldsymbol{e}(P_\varepsilon \boldsymbol{u}^\varepsilon) \rightharpoonup \boldsymbol{e}(\boldsymbol{u}^0) \text{ in } (L^2(\Omega))^{n^2},$$

$$\nabla(P_\varepsilon \boldsymbol{u}^\varepsilon - \boldsymbol{\zeta}^\varepsilon) \rightharpoonup \nabla \boldsymbol{u}^0 \text{ in } (L^2(\Omega))^{n^2}, \tag{38}$$

$$\boldsymbol{\Gamma}^\varepsilon(x) \rightharpoonup |\omega_\square|\, \widehat{\mathfrak{A}}(x)\, \boldsymbol{e}(\boldsymbol{u}^0) \text{ in } (L^2(\Omega))^{n^2},$$

(39) $$\int_{\Omega^\varepsilon} \boldsymbol{e}(\boldsymbol{u}^\varepsilon) : \mathfrak{A}^\varepsilon(x)\, \boldsymbol{e}(\boldsymbol{u}^\varepsilon)\, dx \to |\omega_\square| \int_{\Omega} \boldsymbol{e}(\boldsymbol{u}^0) : \widehat{\mathfrak{A}}(x)\, \boldsymbol{e}(\boldsymbol{u}^0)\, dx ,$$

where $\boldsymbol{u}^0$ *is a solution of problem* (33) *and* $\widehat{\mathfrak{A}}(x)$ *is the elasticity tensor* (31).

(ii) *Under the coerciveness conditions* (34), (35), *with* C_1 *independent of* ε, *the relations* (38), (39) *hold with* $\boldsymbol{\zeta}^\varepsilon = \boldsymbol{0}$.

PROOF. An analogue of this result has been established in Theorem 1 for problems with friction in a perforated domain $\Omega^\varepsilon = \Omega \cap \varepsilon\omega$. In that case we have used uniform (with respect to ε) estimates of the solutions in $(H^1(\Omega^\varepsilon))^n$ ensured by the Dirichlet conditions on the outer part of the boundary Γ^ε. In the present case of a partially perforated domain Ω^ε, the boundary conditions $g(\boldsymbol{u}^\varepsilon, x)\,|_{\partial\Omega} = 0$ may not guarantee such estimates, in general, and we will use another approach based on the ideas from [F].

For $\boldsymbol{u} \in (H^1(\Omega^\varepsilon))^n$, by $\Pi^\varepsilon_{\mathcal{R}} \boldsymbol{u}$ (resp., $\Pi_{\mathcal{R}_\varepsilon} \boldsymbol{u}$) we denote the orthogonal projection of $\boldsymbol{u}$ to $\mathcal{R}$ (resp., $\mathcal{R}_\varepsilon$) with respect to the scalar product $(\cdot,\cdot)_{1,\Omega^\varepsilon}$ in $(H^1(\Omega^\varepsilon))^n$. By $\mathcal{A}^\varepsilon_{\Omega^\varepsilon}(\boldsymbol{u}, \boldsymbol{v})$ we denote the bilinear energy form on $(H^1(\Omega^\varepsilon))^n$ corresponding to the tensor $\mathfrak{A}^\varepsilon(x)$, namely,

$$\mathcal{A}^\varepsilon_{\Omega^\varepsilon}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega^\varepsilon} \boldsymbol{e}(\boldsymbol{u}) : \mathfrak{A}^\varepsilon(x)\, \boldsymbol{e}(\boldsymbol{v})\, dx .$$

Consider the case (i). First of all, let us show that

(40) $$\sup_\varepsilon \|\boldsymbol{u}^\varepsilon - \Pi_{\mathcal{R}_\varepsilon} \boldsymbol{u}^\varepsilon\|_{1,\Omega^\varepsilon} < \infty .$$

Assume the contrary. Then there is a subsequence (still denoted by $\boldsymbol{u}^\varepsilon$) such that

(41) $$s_\varepsilon \overset{\text{def}}{=} \|(I - \Pi_{\mathcal{R}_\varepsilon})\, \boldsymbol{u}^\varepsilon\|_{1,\Omega^\varepsilon} \to \infty \quad \text{as} \quad \varepsilon \to 0 ,$$

where I is the identity operator.

It follows from (29) with $\boldsymbol{v} = \boldsymbol{0}$ that $\mathcal{A}^\varepsilon_{\Omega^\varepsilon}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon) \leq (\boldsymbol{f}^\varepsilon, \boldsymbol{u}^\varepsilon)_{0,\Omega^\varepsilon}$. Therefore, using the Korn inequality from Lemma 7 and the assumption $\mathcal{R}_\varepsilon \subset \mathcal{R}_{f^\varepsilon}$, we find that

(42) $$c\|(I - \Pi^\varepsilon_{\mathcal{R}})\, \boldsymbol{u}^\varepsilon\|^2_{1,\Omega^\varepsilon} \leq \mathcal{A}^\varepsilon_{\Omega^\varepsilon}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon) \leq (\boldsymbol{f}^\varepsilon, \boldsymbol{u}^\varepsilon - \Pi_{\mathcal{R}_\varepsilon} \boldsymbol{u}^\varepsilon)_{0,\Omega^\varepsilon} \leq$$
$$\leq \|\boldsymbol{f}^\varepsilon\|_{0,\Omega^\varepsilon} \|(I - \Pi_{\mathcal{R}_\varepsilon})\, \boldsymbol{u}^\varepsilon\|_{1,\Omega^\varepsilon} = s_\varepsilon \|\boldsymbol{f}^\varepsilon\|_{0,\Omega^\varepsilon} ,$$

where $c > 0$ is a constant independent of ε. For $\boldsymbol{w}^\varepsilon = s_\varepsilon^{-1} \boldsymbol{u}^\varepsilon$, dividing (42) by s_ε^2, using (41) and the fact that $\mathfrak{W}_\varepsilon$ is a cone, we get

$$w^\varepsilon \in \mathfrak{W}_\varepsilon, \qquad \|(I - \Pi^\varepsilon_{\mathcal{R}})\, w^\varepsilon\|_{1,\,\Omega^\varepsilon} \to 0 \quad \text{as} \quad \varepsilon \to 0\,. \tag{43}$$

It is easy to see that

$$\begin{aligned} &(I - \Pi_{\mathcal{R}_\varepsilon})\, w^\varepsilon = (I - \Pi^\varepsilon_{\mathcal{R}})\, w^\varepsilon + (\Pi^\varepsilon_{\mathcal{R}} - \Pi_{\mathcal{R}_\varepsilon})\, w^\varepsilon, \\ &\|(I - \Pi_{\mathcal{R}_\varepsilon})\, w^\varepsilon\|^2_{1,\,\Omega^\varepsilon} = 1 = \|(I - \Pi^\varepsilon_{\mathcal{R}})\, w^\varepsilon\|^2_{1,\,\Omega^\varepsilon} + \|(\Pi^\varepsilon_{\mathcal{R}} - \Pi_{\mathcal{R}_\varepsilon})\, w^\varepsilon\|^2_{1,\,\Omega^\varepsilon}\,. \end{aligned} \tag{44}$$

Thus, $(\Pi^\varepsilon_{\mathcal{R}} - \Pi_{\mathcal{R}_\varepsilon})\, w^\varepsilon$ is a sequence of rigid displacements which is bounded in $(L^2(\Omega))^n$. Since $\mathfrak{R}$ is a finite dimensional space, there exists $\varrho^0 \in \mathfrak{R}$ such that $\|\varrho^0 - (\Pi^\varepsilon_{\mathcal{R}} - \Pi_{\mathcal{R}_\varepsilon})\, w^\varepsilon\|_{1,\,\Omega} \to 0$ as $\varepsilon \to 0$ over a subsequence (again denoted by ε). Now, from (43), (44) and Lemma 3, we see that

$$\|\varrho^0 - (I - \Pi_{\mathcal{R}_\varepsilon})\, w^\varepsilon\|_{1,\,\Omega^\varepsilon} \to 0 \quad \text{as} \quad \varepsilon \to 0\,,$$

$$1 = \|\varrho^0\|^2_{1,\,\Omega\setminus\Omega_0} + |\omega_\square|\, \|\varrho^0\|^2_{1,\,\Omega_0}\,,$$

and thus, for the extensions from Lemma 7, we have the convergence

$$\|\varrho^0 - P_\varepsilon(I - \Pi_{\mathcal{R}_\varepsilon})\, w^\varepsilon\|_{1,\,\Omega} \to 0 \quad \text{as} \quad \varepsilon \to 0\,. \tag{45}$$

Let us show that $\varrho^0 \in \mathfrak{W}_0$. Since $\mathfrak{W}_\varepsilon$ is a cone, we have $(I - \Pi_{\mathcal{R}_\varepsilon})\, w^\varepsilon \in \mathfrak{W}_\varepsilon$, and therefore, $\tilde{w}^\varepsilon = P_\varepsilon(I - \Pi_{\mathcal{R}_\varepsilon})\, w^\varepsilon$ satisfies the conditions

$$\Psi(\tilde{w}^\varepsilon, \varepsilon^{-1} x)\,|_{S_0^\varepsilon} = 0\,, \quad g(\tilde{w}^\varepsilon, x)\,|_{\partial\Omega} = 0\,. \tag{46}$$

Hence, by Lemma 2

$$0 = \varepsilon \int_{S_0^\varepsilon} \Psi(\tilde{w}^\varepsilon, \varepsilon^{-1} x)\, \mathrm{d}S \overset{\varepsilon\to 0}{\longrightarrow} |\partial_\square \omega| \int_{\Omega_0} \overline{\Psi}(\varrho^0)\, \mathrm{d}x = 0\,,$$

and by the trace theorem, we have $g(\varrho^0, x)\,|_{\partial\Omega} = 0$. Thus, $\varrho^0 \in \mathfrak{W}_0 \cap \mathfrak{R}$.

Since $\dim \mathfrak{R}_\varepsilon = \dim \mathfrak{R}_0$ by assumption, we may also assume that in the finite-dimensional spaces $\mathfrak{R}_\varepsilon$ and $\mathfrak{R}_0$ their respective bases e_j^ε and e_j^0, $j = 1, \ldots, k_0$, have been chosen in such a way that $e_j^\varepsilon \to e_j^0$ in $(H^1(\Omega))^n$ as $\varepsilon \to 0$ over a subsequence. Therefore, passing to the limit in the relations $((I - \Pi_{\mathcal{R}_\varepsilon})\, w^\varepsilon, e_j^\varepsilon)_{1,\,\Omega^\varepsilon} = 0$ and taking into account (45) and Lemma 3, we get $(\varrho^0, e_j^0)_{1,\,\Omega\setminus\Omega_0} + |\omega_\square|(\varrho^0, e_j^0)_{1,\,\Omega_0} = 0$ for $j = 1, \ldots, k_0$. Hence, $\varrho^0 \in (\mathfrak{W}_0 \cap \mathfrak{R}) \setminus \mathfrak{R}_0$. Dividing (42) by s_ε, using Lemma 3 and the condition of solvability (37) with $\boldsymbol{\varrho} = \boldsymbol{\varrho}^0$, we find that

$$0 \leqslant cs_\varepsilon \|(I - \Pi^\varepsilon_{\mathcal{R}})\, \boldsymbol{w}^\varepsilon\|^2_{1,\, \Omega^\varepsilon} \leqslant (\boldsymbol{f}^\varepsilon,\, \boldsymbol{w}^\varepsilon - \Pi_{\mathcal{R}_\varepsilon} \boldsymbol{w}^\varepsilon)_{0,\, \Omega^\varepsilon} \overset{\varepsilon \to 0}{\longrightarrow}$$

$$\overset{\varepsilon \to 0}{\longrightarrow} (\boldsymbol{f}^0,\, \boldsymbol{\varrho}^0)_{0,\, \Omega \setminus \Omega_0} + |\omega_\square|\, (\boldsymbol{f}^0,\, \boldsymbol{\varrho}^0)_{0,\, \Omega_0} < 0\,.$$

This contradiction proves the validity of (40).

In view of the weak compactness of a ball in Hilbert space and the compactness of the imbedding $H^1(\Omega) \subset L^2(\Omega)$, there exist $\boldsymbol{u}^0 \in (H^1(\Omega))^n$ and $\boldsymbol{\Gamma}^0(x) \in (L^2(\Omega))^{n^2}$ such that for a subsequence of $\varepsilon \to 0$ we have

$$P_\varepsilon(\boldsymbol{u}^\varepsilon - \Pi_{\mathcal{R}_\varepsilon} \boldsymbol{u}^\varepsilon) \rightharpoonup \boldsymbol{u}^0 \quad \text{in} \quad (H^1(\Omega))^n,$$

$$P_\varepsilon(\boldsymbol{u}^\varepsilon - \Pi_{\mathcal{R}_\varepsilon} \boldsymbol{u}^\varepsilon) \to \boldsymbol{u}^0 \quad \text{in} \quad (L^2(\Omega))^n,$$

$$\boldsymbol{\Gamma}^\varepsilon(x) \rightharpoonup \boldsymbol{\Gamma}^0(x) \quad \text{in} \quad (L^2(\Omega))^{n^2}.$$

Clearly, relations (46) hold for $\tilde{\boldsymbol{w}}^\varepsilon = \boldsymbol{u}^\varepsilon - \Pi_{\mathcal{R}_\varepsilon} \boldsymbol{u}^\varepsilon$. Therefore, Lemma 2 yields

$$0 = \varepsilon \int_{S_0^\varepsilon} \Psi(\tilde{\boldsymbol{w}}^\varepsilon,\, \varepsilon^{-1} x)\, \mathrm{d}S \overset{\varepsilon \to 0}{\longrightarrow} |\partial_\square \omega| \int_{\Omega_0} \overline{\Psi}(\boldsymbol{u}^0)\, \mathrm{d}x = 0\,,$$

and by the trace theorem for $\partial\Omega$, we have $g(\boldsymbol{u}^0, x)\,|_{\partial\Omega} = 0$. It follows that $\boldsymbol{u}^0 \in \mathcal{W}_0$.

Let us show that

$$\boldsymbol{\Gamma}^0\,|_{\Omega_0} = |\omega_\square|\, \widehat{\mathcal{A}}_0 \boldsymbol{e}(\boldsymbol{u}^0)\,.$$

To that end, as in the proof of Theorem 1, we apply Lemma 4 to the tensor $\mathcal{A}(\xi) \equiv \mathcal{A}_0(\xi)$ and verify the convergence (14) for

$$\Omega' = \Omega_0, \qquad \boldsymbol{w}^\varepsilon = \boldsymbol{u}^\varepsilon - \Pi_{\mathcal{R}_\varepsilon} \boldsymbol{u}^\varepsilon, \qquad V(\xi) \in (C^\infty_{\text{per}}(\overline{\omega}))^n.$$

Since the set $\mathcal{M} = \{\boldsymbol{\eta} \in \mathbb{R}^n \colon \overline{\Psi}(\boldsymbol{\eta}) = 0\}$ has interior points in $\mathbb{R}^n$ by assumption, it can be easily shown with the help of Lemma 6 that there exists $\boldsymbol{\eta}^0 \in \mathcal{M}$ such that $\varepsilon\psi(x)(\boldsymbol{\eta}^0 \pm V(\varepsilon^{-1}x)) \in \mathcal{W}_\varepsilon$ for any $\psi \in C_0^\infty(\Omega_0)$, $\psi \geqslant 0$. Taking $\boldsymbol{v}(x) = \boldsymbol{w}^\varepsilon + \varepsilon\psi(x)(\boldsymbol{\eta}^0 \pm V(\varepsilon^{-1}x))$ in (29) with $\boldsymbol{w}^\varepsilon = \boldsymbol{u}^\varepsilon -$

$-\Pi_{\mathcal{R}_\varepsilon} \boldsymbol{u}^\varepsilon$, we find that

$$\pm\varepsilon \int_{\Omega^\varepsilon} \psi(x)\, \nabla_x \boldsymbol{V} : \mathfrak{a}^\varepsilon \nabla_x \boldsymbol{w}^\varepsilon \, dx + \varepsilon \int_{\Omega^\varepsilon} ((\boldsymbol{\eta}^0 \pm \boldsymbol{V}) \otimes \nabla_x \psi) : \mathfrak{a}^\varepsilon \nabla_x \boldsymbol{w}^\varepsilon \, dx \geqslant$$

$$\geqslant \varepsilon \int_{\Omega^\varepsilon} \psi(x)(\boldsymbol{\eta}^0 \pm \boldsymbol{V}) \cdot \boldsymbol{f}^\varepsilon \, dx ,$$

where the product $\otimes$ is defined in the proof of Theorem 1. Hence we immediately obtain the convergence

$$\varepsilon \int_{\Omega^\varepsilon} \psi(x)\, \nabla_x \boldsymbol{V} : \mathfrak{a}^\varepsilon \nabla_x \boldsymbol{w}^\varepsilon \, dx \to 0 \quad \text{as} \quad \varepsilon \to 0 ,$$

which is equivalent to (14). Now, Lemma 4 and the symmetry properties of $\widehat{\mathfrak{a}}_0$ yield $\boldsymbol{\Gamma}^0 |_{\Omega_0} = |\omega_\square| \widehat{\mathfrak{a}}_0 \boldsymbol{e}(\boldsymbol{u}^0) = |\omega_\square| \widehat{\mathfrak{a}}_0 \nabla_x \boldsymbol{u}^0$. The relation $\boldsymbol{\Gamma}^0(x) |_{\Omega \setminus \Omega_0} = \mathfrak{a}_1(x)\, \boldsymbol{e}(\boldsymbol{u}^0)$ follows immediately from the weak convergence of $(\boldsymbol{u}^\varepsilon - \Pi_{\mathcal{R}_\varepsilon} \boldsymbol{u}^\varepsilon) |_{\Omega \setminus \Omega_0}$ to $\boldsymbol{u}^0 |_{\Omega \setminus \Omega_0}$ in $(H^1(\Omega \setminus \Omega_0))^n$.

In view of Lemma 5, it may be assumed that for a subsequence of ε and $\boldsymbol{w}^\varepsilon = \boldsymbol{u}^\varepsilon - \Pi_{\mathcal{R}_\varepsilon} \boldsymbol{u}^\varepsilon$ we have

$$\lim_{\varepsilon \to 0} \int_{\Omega^\varepsilon} \boldsymbol{e}(\boldsymbol{u}^\varepsilon) : \mathfrak{a}^\varepsilon \boldsymbol{e}(\boldsymbol{u}^\varepsilon) \, dx = \lim_{\varepsilon \to 0} \int_{\Omega^\varepsilon} \boldsymbol{e}(\boldsymbol{w}^\varepsilon) : \mathfrak{a}^\varepsilon \boldsymbol{e}(\boldsymbol{w}^\varepsilon) \, dx \geqslant$$

$$\geqslant \int_{\Omega \setminus \Omega_0} \boldsymbol{e}(\boldsymbol{u}^0) : \mathfrak{a}_1(x)\, \boldsymbol{e}(\boldsymbol{u}^0) \, dx + |\omega_\square| \int_{\Omega_0} \boldsymbol{e}(\boldsymbol{u}^0) : \widehat{\mathfrak{a}}_0 \boldsymbol{e}(\boldsymbol{u}^0) \, dx .$$

Now we can pass to the limit in (29) for a fixed $\boldsymbol{v} \in \mathcal{W}_0$, since $\boldsymbol{v} |_{\Omega^\varepsilon} \in \mathcal{W}_\varepsilon$ by Lemma 6. We get

$$\int_{\Omega} (\boldsymbol{v} - \boldsymbol{u}^0) \cdot (\chi_{\Omega \setminus \Omega_0}(x)\, \boldsymbol{f}^0 + |\omega_\square| \chi_{\Omega_0}(x)\, \boldsymbol{f}^0) \, dx =$$

$$= \lim_{\varepsilon \to 0} \int_{\Omega^\varepsilon} (\boldsymbol{v} - \boldsymbol{u}^\varepsilon) \cdot \boldsymbol{f}^\varepsilon \, dx \leqslant \lim_{\varepsilon \to 0} \int_{\Omega^\varepsilon} \boldsymbol{e}(\boldsymbol{v} - \boldsymbol{u}^\varepsilon) : \mathfrak{a}^\varepsilon \boldsymbol{e}(\boldsymbol{u}^\varepsilon) \, dx \leqslant$$

$$\leqslant \lim_{\varepsilon \to 0} \int_{\Omega^\varepsilon} \boldsymbol{e}(\boldsymbol{v}) : \mathfrak{a}^\varepsilon \boldsymbol{e}(\boldsymbol{u}^\varepsilon) \, dx - \lim_{\varepsilon \to 0} \int_{\Omega^\varepsilon} \boldsymbol{e}(\boldsymbol{u}^\varepsilon) : \mathfrak{a}^\varepsilon \boldsymbol{e}(\boldsymbol{u}^\varepsilon) \, dx \leqslant$$

$$\leqslant \int_{\Omega} \boldsymbol{e}(\boldsymbol{v} - \boldsymbol{u}^0) : (\chi_{\Omega \setminus \Omega_0}(x)\, \mathfrak{a}_1(x) + |\omega_\square| \chi_{\Omega_0}(x)\, \widehat{\mathfrak{a}}_0)\, \boldsymbol{e}(\boldsymbol{u}^0) \, dx ,$$

which shows that $\boldsymbol{u}^0$ is a solution of problem (33). Moreover, taking $\boldsymbol{v} = \boldsymbol{u}^0$ in the above relations, we obtain the convergence (39).

Since the solution of problem (33) is unique to within rigid displacements and the Korn inequality holds in Ω^ε uniformly with respect to ε, we see that (38), (39) hold for any subsequence of $\boldsymbol{u}^\varepsilon$, while $\boldsymbol{u}^0$ may be assumed the same.

The statement (ii) in the coercive case is established by similar arguments as (i), with due simplifications. ■

REMARK 3. Again it should be noted (cf. Remark 1) that the cone $\{\boldsymbol{\eta} : \overline{\Psi}(\boldsymbol{\eta}) = 0\}$ having interior points in $\mathbb{R}^n$ is an essential assumption in our proof of Theorem 2. Otherwise, the homogenized tensor $\widehat{\mathfrak{a}}_0$ may depend on the boundary conditions on the perforated boundary, as suggested by the example in [Y2], Sect. 4.

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