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p -adic Completions and Automorphisms of Nilpotent Groups.

RÜDIGER GÖBEL (*) - AGNES T. PARAS (**)

ABSTRACT - Given a group G , a new construction of a torsion-free, nilpotent group H of class two is given such that $\text{Aut } H/\text{Stab } H \cong G$. When $G = \{e\}$, it is shown that $\text{Aut } H = \text{Inn } H$.

1. Introduction.

It was shown in [4] that any group G is the outer automorphism group $\text{Aut } H/\text{Inn } H$ of some torsion-free metabelian group H . If the given group G has infinite cardinality $< 2^{\aleph_0}$, then we may also assume that $|H| = |G|$ (see [5]). It is natural to ask whether this result can be strengthened to nilpotent groups of class two.

In [2] and [3], two different constructions of a group H are presented, wherein H is a torsion-free nilpotent group of class two and $\text{Aut } H/\text{Stab } H$ is a prescribed group. (If K is any group, the group $\text{Stab } K$ is defined in this paper to be

$$\text{Stab } K = \{ \varphi \in \text{Aut } K : \varphi \upharpoonright_{Z(K)} = \text{id}_{Z(K)} \text{ and } \varphi \upharpoonright_{K/Z(K)} = \text{id}_{K/Z(K)} \}$$

where $Z(K)$ denotes the center of K . Clearly, $\text{Inn } H \subseteq \text{Stab } H$, if H is nilpotent of class two.) The first one made use of Zalesskii's construction of

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a torsion-free nilpotent group of class 2, with rank 3 but having no outer automorphisms. The second involved the creation of a 2-divisible, torsion-free abelian group X admitting an alternating bilinear map.

In this paper we consider the p -adic completion $\widehat{N}$ of a free nilpotent group N and construct a group H such that $N < H < \widehat{N}$ and $\text{Aut } H/\text{Stab } H$ is a prescribed group. It is hoped that the given proof, based entirely on group theory, will give more insight into this replacement of $\text{Inn } H$ by $\text{Stab } H$. The construction is very canonical and we can also show that in this setting *we can replace $\text{Stab } H$ by $\text{Inn } H$ if and only if the given group $G = \{e\}$* . Section 2 gives a description of the p -adic completion $\widehat{N}$ of a free nilpotent group N of class two and its elements. Section 3 contains a characterization of the elements of $\text{Inn } \widehat{N}$ and $\text{Stab } \widehat{N}$ and the main theorem.

2. p -adic completion of a free nilpotent group of class two.

If N is a nilpotent group and p is a prime, a topology on N , called the p -adic topology on N , can be defined by taking the set $\{N^{p^n} : n \in \omega\}$ as a base of open sets about $\{1\}$, with p -adic completion $\widehat{N}$ defined to be

$$\varprojlim_n N/N^{p^n} = \left\{ (a_i)_{i < \omega} \in \prod_{i < \omega} N/N^{p^i} : \Pi_i^j(a_j) = a_i \text{ for all } i < j \right\},$$

where $\Pi_i^j : N/N^{p^j} \rightarrow N/N^{p^i}$ is defined to be $\Pi_i^j(aN^{p^j}) = aN^{p^i}$ ($i < j$). If we take N to be free nilpotent of class two with free generating set $\{x_i : i \in I\}$, then the given base of open sets determines a Hausdorff topology on N and N embeds in $\widehat{N}$, where $x \in N$ is identified with $(xN^{p^i})_{i < \omega} \in \widehat{N}$. Moreover $\widehat{N}$ is also nilpotent of class two, $\widehat{N}' = \widehat{N}'$ and $\widehat{N}/\widehat{N}' \cong \widehat{N/N'}$ (see [7], p. 55). If N is free nilpotent, every element $g \in N$ can be represented as

$$\prod_{i \in I} x_i^{k_i} \prod_{i \neq j} [x_i, x_j]^{k_{ij}},$$

where only finitely many $k_i, k_{ij} \in \mathbb{Z}$ are non-zero. The set $\{x_i N' : i \in I\}$ is a free set of generators for the free abelian group N/N' . If, in addition, I is a linearly ordered set, then $\{[x_i, x_j] : i < j\}$ is a free set of generators for the free abelian group N' and every element g can be uniquely represented as

$$g = x_{i_1}^{k_1} \dots x_{i_n}^{k_n} \prod_{i < j} [x_i, x_j]^{k_{ij}},$$

where $i_1 < \dots < i_n$ and only finitely many k_{ij} are non-zero (see [6], p. 165).

If $x \in N$ and $\xi = (g_i + p^i \mathbb{Z})_{i < \omega} \in J_p$ is a p -adic integer, we let x^ξ denote the element $(a_i)_{i < \omega} \in \widehat{N}$, where $a_i = x^{g_i} N^{p^i}$. If $x, y \in N$, then

$$(xy)^\xi = x^\xi y^\xi [y, x]^{\frac{(\xi-1)\xi}{2}},$$

since $(ab)^n = a^n b^n [b, a]^{\frac{(n-1)n}{2}}$ for all $n \in \mathbb{Z}$. Let $\prod_{k < \omega} x_{ik}^{\xi_k}$ denote the infinite product $x_{i_1}^{\xi_1} x_{i_2}^{\xi_2} x_{i_3}^{\xi_3} \dots$. It is easy to see that every element of $\widehat{N}$ can be written (not necessarily uniquely) in the form

$$\prod_{k < \omega} x_{ik}^{\xi_k} \prod_{j < k < \omega} [x_{ij}, x_{ik}]^{\xi_{jk}},$$

where $\xi_i, \xi_{jk} \in J_p$ with only countably many ξ_k and ξ_{jk} non-zero and for all n , p^n divides all but finitely many ξ_k and ξ_{jk} .

3. Prescribing automorphism groups.

In this section we show how a given group can be realized as the automorphism group of a torsion-free nilpotent group of class two modulo its stabilizer. We begin by defining some preliminary notions which have appeared in [1] and [4] within the framework of modules, but which are now formulated in the context of nilpotent groups.

Let λ be a regular cardinal such that $\lambda \geq 2^{\aleph_0}$, and define the tree $T = {}^\omega \lambda$ to be the set of all functions $\tau : n \rightarrow \lambda$ ($n < \omega$). If σ and τ are two functions in T , define $\sigma \leq \tau$ if $\sigma \subseteq \tau$. Let G be any group such that $|G| \leq \lambda$ and e be the identity of G . Define N_G to be the free nilpotent group of class 2 with free generating set $\{g_\tau : g \in G, \tau \in T\}$. Note that G acts on N_G via

$$(3.1) \quad (g_\tau)^h = (gh)_\tau$$

and G embeds in $\text{Aut } N_G$. Moreover the same action makes N_G/N'_G a $\mathbb{Z}[G]$ -module. Let p be an odd prime and $\widehat{N}_G$ be the p -adic completion of N_G . If $y \in \widehat{N}_G$, then

$$y = \prod_{(g, \tau) \in G \times T} g_\tau^{\xi_{g_\tau}} \prod_{g_\tau \neq h_\mu} [g_\tau, h_\mu]^{\xi_{g_\tau h_\mu}},$$

where countably many $\xi_{g_\tau}, \xi_{g_\tau h_\mu}$ are non-zero p -adic integers, and, for all n , p^n divides all but finitely many ξ_{g_τ} and $\xi_{g_\tau h_\mu} - \xi_{h_\mu g_\tau}$ ($g_\tau \neq h_\mu$). Define

the T -support of y to be

$$[y] = \{\tau, \mu \in T : \xi_{g_\tau} \neq 0 \text{ or } \xi_{g_\tau h_\mu} - \xi_{h_\mu g_\tau} \neq 0\}.$$

Hence $[y]$ is the smallest subset S of T such that $y \in \langle g_\tau : g \in G, \tau \in S \rangle$. Define the *norm* of y to be

$$\|y\| = \min \{\nu \subseteq \lambda : [y] \subseteq^{\omega} \nu\}.$$

A *branch* v of T is defined to be a linearly ordered sequence $v = (v_n)_{n < \omega}$, where $v_n : n \rightarrow \lambda$ and $v_n \leq v_{n+1}$. Note that $v \in {}^\omega \lambda \setminus T$ and $v \upharpoonright_n = v_n$. If $X \subseteq T$, the set of all branches of T contained in X will be denoted by $Br(X)$. If v is a branch of T and ν is an ordinal such that $\nu < \lambda$, define *the part of v to the right of ν* to be

$$\downarrow[v] = \{v_n : \|v_n\| > \nu\}.$$

If $v = (v_n)_{n < \omega}$ is a branch of T , define

$$v^1 = \prod_{i < \omega} e_{v_i}^{p^i} \in \widehat{N}_G.$$

If H is a group such that $N < H < \widehat{N}$, define the *purification* H_* of H in $\widehat{N}$ to be

$$H_* = \{x \in \widehat{N} : x^{p^k} \in H \text{ for some } k \in \mathbb{Z}\}.$$

If p is an odd prime and $x, y \in \widehat{N}$ such that $x^{p^k}, y^{p^k} \in H$, then we obtain the equation $(xy)^{p^{2k}} = x^{p^{2k}} y^{p^{2k}} [y^{p^k}, x^{p^k}]^{\frac{p^{2k}-1}{2}}$. Thus it is clear that H_* is a subgroup of $\widehat{N}$, if p is an odd prime. Moreover, if p is an odd prime and N is any nilpotent group of class two, then if x and y are p -th powers in N , then so is xy . Define a *canonical subgroup* P of $\widehat{N}_G$ to be a subgroup of the form

$$N_{T_0} = \langle g_\tau : g \in G, \tau \in T_0 \rangle,$$

for some countable subset T_0 of T . We identify each $\tau \in T$ with the element e_τ of N .

DEFINITION 3.1. A *trap* (f, P, φ) is a triple, where $f : {}^\omega \omega \rightarrow T$ is a tree embedding, P is a canonical subgroup of $\widehat{N}_G$, and $\varphi \in \text{Aut } \widehat{P}$ such that

- (i) $\text{Im} f \subseteq P$,
- (ii) $[P] \subseteq P$, where $[P]$ is a subtree of T ,

- (iii) $\text{cf}(\|P\|) = \omega$, and
- (iv) $\|v\| = \|P\|$ for all $v \in \text{Br}(\text{Im } f)$.

We state without proof the following theorem (see [1]), which holds in ordinary set theory ZFC.

THEOREM 3.2 (The Black Box). *For some ordinal λ^* of cardinality λ , there exists a transfinite sequence of traps $(f_\alpha, P_\alpha, \varphi_\alpha)$ ($\alpha < \lambda^*$) such that for $\alpha, \beta < \lambda^*$,*

- (i) if $\beta < \alpha$, then $\|P_\beta\| \leq \|P_\alpha\|$;
- (ii) if $\beta \neq \alpha$, then $\text{Br}(\text{Im } f_\alpha) \cap \text{Br}(\text{Im } f_\beta) = \emptyset$;
- (iii) if $\beta + 2^{\aleph_0} \leq \alpha$, then $\text{Br}(\text{Im } f_\alpha) \cap \text{Br}([P_\beta]) = \emptyset$;
- (iv) for all $X \subset \widehat{N}$ with $|X| \leq \aleph_0$ and for all $\varphi \in \text{Aut } \widehat{N}$, there exists $\alpha < \lambda^*$ such that

$$X \subset \widehat{P}_\alpha, \|X\| < \|P_\alpha\|, \quad \varphi \upharpoonright_{P_\alpha} = \varphi_\alpha.$$

We now describe the construction of the torsion-free nilpotent group, which will possess the desired automorphism group. Let $\langle x^G \rangle$ denote the subgroup generated by the set $\{x^g: g \in G\}$.

Choose a transfinite sequence $(f_\alpha, P_\alpha, \varphi_\alpha)_{\alpha < \lambda^*}$ satisfying the conclusion of the Black Box. Let $H_0 = N$. Let $\mu \leq \lambda^*$ and assume we have found an ascending continuous chain of G -invariant subgroups

$$H_\alpha = \langle N, g_\beta^G: \beta < \alpha \rangle_* \quad (\alpha < \mu)$$

of $\widehat{N}_G$ such that the following hold inductively:

If $\mu = \alpha + 1$, choose a branch $v_\alpha \in \text{Br}(\text{Im } f_\alpha)$. Let $g_\alpha = v_\alpha^1$, if $(v_\alpha^1)^{\varphi_\alpha} \notin \langle H_\alpha, (v_\alpha^1)^G \rangle_*$. Otherwise, let $g_\alpha = x_\alpha \cdot v_\alpha^1$, where $x_\alpha^{\varphi_\alpha} \notin \langle H_\alpha, x_\alpha^G \rangle_*$ and $\|x_\alpha\|, \|x_\alpha\|^\varphi < \|v_\alpha\|$. If $H_{\alpha+1} = \langle H_\alpha, g_\alpha^G \rangle_*$, we also require that

$$(\dagger) \quad g_\alpha^{\varphi_\alpha} \notin H_{\alpha+1}$$

$$(*) \quad g_\beta^{\varphi_\beta} \notin H_{\alpha+1}, \quad \text{if } g_\beta^{\varphi_\beta} \notin H_\alpha \quad (\beta < \alpha)$$

If $(\dagger)$ does not occur, take $H_{\alpha+1} = H_\alpha$. If μ is a limit ordinal, take $H_\mu = \bigcup_{\alpha < \mu} H_\alpha$. Finally, let $H_{\lambda^*} = \langle N_G, g_\alpha^G: \alpha < \lambda^* \rangle_*$.

The next, by now standard, argument shows that $(*)$ can be arranged, while $(\dagger)$ depends on the choice of φ_α . Hence we will always choose

g_α as above with $(*)$ and, whenever possible, with $(\dagger)$. The latter case is called the *strong case* in [1]. The following theorem shows that condition $(*)$ holds for every ordinal α .

THEOREM 3.3. *Suppose H_α is defined as above. Then there exists a branch $v \in \text{Br}(\text{Im } f_\alpha)$ such that $H_{\alpha+1} = \langle H_\alpha, g_\alpha^G \rangle_*$ satisfies $(*)$, where $g_\alpha = v^1$.*

PROOF. Suppose that the above conclusion does not hold, i.e., if $v \in \text{Br}(\text{Im } f_\alpha)$ and $H_v = \langle H_\alpha, (v^1)^G \rangle_*$, there exists $\beta = \beta(v) < \alpha$ such that $g_\beta^{\mathcal{P}\beta} \in H_v \setminus H_\alpha$. Then for some integer s , the element $g_\beta^{\mathcal{P}\beta p^s}$ is a product of elements $g_v^{h_1}, g_{\beta_i}^{h_i}$ and elements from N_G . Using commutators, there exist integers s_v, n_i (not all zero), $n_{\beta_{ij}}; h_i \in G; n, u_{ik} \in N_G$; and $h_{vi} \in H_\alpha$ such that $g_\beta^{\mathcal{P}\beta p^s}$ is equal to

$$g_v^{n_1 h_1} \dots g_v^{n_k h_k} n \prod_i g_{\beta_i}^{n_{\beta_i 1} h_1} \dots g_{\beta_i}^{n_{\beta_i k} h_k} \prod_i [u_{i1}, g_{\beta_i}^{h_1}] \dots [u_{ik}, g_{\beta_i}^{h_k}] \cdot \prod_v [h_{v1}, g_v^{h_1}] \dots [h_{vk}, g_v^{h_k}].$$

Let $\nu < \|v\|$. Since ${}_\nu[v]$ is an infinite subset of v and not all n_i 's are zero, an infinite subset of v is contained in $[g_\beta^{\mathcal{P}\beta}] \subseteq [P_\beta]$. This means that $[v] \subseteq [P_\beta]$ and so $v \in \text{Br}(\text{Im } f_\alpha) \cap \text{Br}([P_\beta])$. By condition (iii) of the Black Box, $\alpha < \beta + 2^{\aleph_0}$. Hence if $v \in \text{Br}(\text{Im } f_\alpha)$, there exist $\beta(v) < \alpha; n, u_{ik} \in N_G; h_{vi} \in H_\alpha; h_i \in G$ and integers s_v, n_i (not all zero) such that $\beta(v) < \alpha < \beta(v) + 2^{\aleph_0}$ and

$$g_v^{-n_k h_k} \dots g_v^{-n_1 h_1} g_{\beta(v)}^{\mathcal{P}\beta(v) p^{s_v}} \prod_v [g_v^{h_1}, h_{v1}] \dots [g_v^{h_k}, h_{vk}] \in H_\alpha.$$

Hence there exist distinct branches $v, w \in \text{Br}(\text{Im } f_\alpha)$ such that $\beta(v) = \beta(w) = \beta$ (see [1], p. 457). Then

$$(3.2) \quad g_v^{-n_k h_k} \dots g_v^{-n_1 h_1} g_\beta^{\mathcal{P}\beta p^{s_v}} \prod_v [g_v^{h_1}, h_{v1}] \dots [g_v^{h_k}, h_{vk}]$$

$$(3.3) \quad g_w^{-m_k h_k} \dots g_w^{-m_1 h_1} g_\beta^{\mathcal{P}\beta p^{s_w}} \prod_w [g_w^{h_1}, h_{w1}] \dots [g_w^{h_k}, h_{wk}]$$

are both in H_α . Taking the p^{s_w} -th power of (3.2) and the p^{s_v} -th power of (3.3), we obtain

$$(3.4) \quad (g_v^{-n_k h_k} \dots g_v^{-n_1 h_1})^{p^{s_w}} g_\beta^{\mathcal{P}\beta p^{s_v + s_w}} \left(\prod_v [g_v^{h_1}, h_{v1}] \dots [g_v^{h_k}, h_{vk}] \right)^{p^{s_w}} \cdot [g_\beta^{\mathcal{P}\beta p^{s_v}}, g_v^{-m_k h_k} \dots g_v^{-m_1 h_1}]^{\frac{(p^{s_w} - 1) p^{s_w}}{2}}$$

$$(3.5) \quad (g_w^{-m_k h_k} \dots g_w^{-m_1 h_1})^{p^{s_v}} g_{\beta}^{\varphi \beta p^{s_w + s_v}} \left(\prod_w [g_w^{h_1}, h_{w1}] \dots [g_w^{h_k}, h_{wk}] \right)^{p^{s_v}} \\ [g_{\beta}^{\varphi \beta p^{s_w}}, g_w^{-m_k h_k} \dots g_w^{-m_1 h_1}]^{\frac{(p^{s_v} - 1) p^{s_v}}{2}}.$$

Multiplying (3.5) on the right by the inverse of (3.4), we have

$$(g_w^{-m_k h_k} \dots g_w^{-m_1 h_1})^{p^{s_v}} (g_v^{n_1 h_1} \dots g_v^{n_k h_k})^{p^{s_w}} \\ \cdot \left(\prod_w [g_w^{h_1}, h_{w1}] \dots [g_w^{h_k}, h_{wk}] \right)^{p^{s_v}} \left(\prod_v [h_{v1}, g_v^{h_1}] \dots [h_{vk}, g_v^{h_k}] \right)^{p^{s_w}} \\ \cdot [g_{\beta}^{\varphi \beta p^{s_w}}, g_w^{-m_k h_k} \dots g_w^{-m_1 h_1}]^{\frac{(p^{s_v} - 1) p^{s_v}}{2}} [g_{\beta}^{\varphi \beta p^{s_v}}, g_v^{n_k h_k} \dots g_v^{n_1 h_1}]^{\frac{(p^{s_w} - 1) p^{s_w}}{2}},$$

which is an element of H_{α} . By the definition of the supports of the elements of H_{α} , an infinite subset of v is contained in w or an infinite subset of w is contained in v . This gives a contradiction, since v and w have finite intersection. ■

Recall from equation (3.1) that G embeds in $\text{Aut } N_G$. By continuity, G also embeds in $\text{Aut } \widehat{N}_G$. Since the intersection $G \cap \text{Stab } \widehat{N}_G$ contains only the identity map and $\text{Stab } \widehat{N}_G$ is a normal subgroup of $\text{Aut } \widehat{N}_G$, then the semi-direct product $\text{Stab } \widehat{N}_G \rtimes G$ also embeds in $\text{Aut } \widehat{N}_G$. The following theorem describes the automorphisms which do not extend to the purification of every G -invariant extension of H_{α} .

THEOREM 3.4. *If $\varphi \in \text{Aut } \widehat{N}_G \setminus (\text{Stab } (\widehat{N}_G) \rtimes G)$, then there exists $x \in \widehat{N}_G$ such that $x^{\varphi} \notin \langle N_G, g_{\beta}^G, x^G : \beta < \alpha \rangle_*$, where the g_{β} 's are defined as in Theorem 3.3.*

PROOF. Let $H_{\alpha} = \langle N_G, g_{\beta}^G : \beta < \alpha \rangle_*$ and suppose that $x^{\varphi} \in \langle H_{\alpha}, x^G \rangle_*$ for all $x \in \widehat{N}_G$. Let τ and δ be distinct elements of T and $1, \xi, \varrho \in J_p$ be algebraically independent over $\mathbb{Z}$. Then there exists $k \in \mathbb{Z}$ such that

$$x_{\tau}^{\varphi p^k} \equiv \prod_i g_{\beta_i}^{a_i} \prod_{\mu} e_{\mu}^{b_{\mu}} \text{mod } \widehat{N}'_G \\ y_{\delta}^{\varphi p^k} \equiv \prod_i g_{\beta_i}^{c_i} \prod_{\mu} e_{\mu}^{d_{\mu}} \text{mod } \widehat{N}'_G \\ (x_{\tau}^{\xi} y_{\delta}^{\varrho})^{\varphi p^k} \equiv \prod_i g_{\beta_i}^{e_i} \prod_{\mu} e_{\mu}^{f_{\mu}} (x_{\tau}^{\xi} y_{\delta}^{\varrho})^n \text{mod } \widehat{N}'_G$$

for some $a_i, b_{\mu}, c_i, d_{\mu}, n$ in $\mathbb{Z}[G]$. Since $(x_{\tau}^{\xi} y_{\delta}^{\varrho})^{\varphi} = x_{\tau}^{\varphi \xi} y_{\delta}^{\varphi \varrho}$, we have

$$\prod g_{\beta_i}^{a_i \xi} \prod e_{\mu}^{b_{\mu} \xi} \prod g_{\beta_i}^{c_i \varrho} \prod e_{\mu}^{d_{\mu} \varrho} \equiv \prod g_{\beta_i}^{e_i} \prod e_{\mu}^{f_{\mu}} (x_{\tau}^{\xi} y_{\delta}^{\varrho})^n \text{mod } \widehat{N}'_G$$

Thus $\prod g_{\beta_i}^{a_i \xi - e_i + c_i \varrho} \prod e_{\mu}^{b_{\mu} \xi - f_{\mu} + d_{\mu} \varrho} x_{\tau}^{\xi n} y_{\delta}^{\varrho n} \equiv 1 \pmod{\widehat{N}'_G}$. Since the elements $g_{\beta} \widehat{N}'_G$ and $h_{\tau} \widehat{N}'_G$ ($\beta < \alpha$, $h \in G$, $\tau \in T$) are independent in $\widehat{N}_G / \widehat{N}'_G$ and $1, \xi, \varrho$ are algebraically independent over $\mathbb{Z}$, we have that

$$a_i = e_i = c_i = 0 \quad \text{for all } i \quad \text{and} \quad b_{\mu} = f_{\mu} = d_{\mu} = 0 \quad \text{for all } \mu \neq \tau, \delta.$$

Hence

$$x_{\tau}^{\varphi p^k} \equiv e_{\tau}^{b_{\tau}} e_{\delta}^{b_{\delta}}, \quad y_{\delta}^{\varphi p^k} \equiv e_{\tau}^{d_{\tau}} e_{\delta}^{d_{\delta}}, \quad e_{\tau}^{b_{\tau} \xi} e_{\delta}^{b_{\delta} \xi} e_{\tau}^{d_{\tau} \varrho} e_{\delta}^{d_{\delta} \varrho} \equiv e_{\tau}^{f_{\tau}} e_{\delta}^{f_{\delta}} (x_{\tau}^{\xi} y_{\delta}^{\varrho})^n.$$

Applying once more the algebraic independence of $1, \xi, \varrho$ over $\mathbb{Z}$ and the independence of g_{τ} modulo $\widehat{N}_G / \widehat{N}'_G$ ($g \in G$, $\tau \in T$) to the preceding congruence, we obtain $b_{\delta} = d_{\tau} = 0$, $e_{\tau}^{b_{\tau}} = x_{\tau}^n$, $e_{\delta}^{d_{\delta}} = y_{\delta}^n$,

$$x_{\tau}^{\varphi p^k} \equiv x_{\tau}^n \quad \text{and} \quad y_{\delta}^{\varphi p^k} \equiv y_{\delta}^n.$$

Suppose now that $n = \sum_{i=1}^k w_i h_i \in \mathbb{Z}[G]$. Then $x_{\tau}^{\sum w_i h_i} \equiv \prod (x h_i)_{\tau}^{w_i}$, by equation (3.1). Similarly, we obtain

$$(3.6) \quad x_{\tau}^{\varphi p^k} = (x h_1)_{\tau}^{w_1} \dots (x h_k)_{\tau}^{w_k} \cdot m_1$$

$$(3.7) \quad y_{\delta}^{\varphi p^k} = (y h_1)_{\delta}^{w_1} \dots (y h_k)_{\delta}^{w_k} \cdot m_2$$

$$(3.8) \quad (x_{\tau}^{\xi} y_{\delta}^{\varrho})^{\varphi p^k} = (x_{\tau}^{\xi} y_{\delta}^{\varrho})^{w_1 h_1} \dots (x_{\tau}^{\xi} y_{\delta}^{\varrho})^{w_k h_k} \cdot q$$

for some $m_i \in H_{\alpha}'$ and for some $q \in \langle H_{\alpha}, (x_{\tau}^{\xi} y_{\delta}^{\varrho})^G \rangle'$. Let

$$q = m_3 \prod_{h_i} [a_i, x_{\tau}^{\xi} y_{\delta}^{\varrho}]^{h_i} \prod_{i < j} [(x_{\tau}^{\xi} y_{\delta}^{\varrho})^{h_i}, (x_{\tau}^{\xi} y_{\delta}^{\varrho})^{h_j}]^{b_{ij}},$$

where $m_3 \in H_{\alpha}'$, $a_i \in H_{\alpha}$ and $b_{ij} \in \mathbb{Z}$. But

$$\begin{aligned} (3.9) \quad (x_{\tau}^{\xi} y_{\delta}^{\varrho})^{\varphi p^k} &= x_{\tau}^{\xi \varphi p^k} y_{\delta}^{\varrho \varphi p^k} [y_{\delta}^{\varrho \varphi}, x_{\tau}^{\xi \varphi}]^{\frac{(p^k - 1)p^k}{2}} = \\ &= ((x h_1)_{\tau}^{w_1} \dots (x h_n)_{\tau}^{w_n})^{\xi} ((y h_1)_{\delta}^{w_1} \dots (y h_n)_{\delta}^{w_n})^{\varrho} m_1^{\xi} m_2^{\varrho} [y_{\delta}^{\varrho \varphi}, x_{\tau}^{\xi \varphi}]^{\frac{(p^k - 1)p^k}{2}} = \\ &= (x h_1)_{\tau}^{w_1 \xi} \dots (x h_n)_{\tau}^{w_n \xi} (y h_1)_{\delta}^{w_1 \varrho} \dots (y h_n)_{\delta}^{w_n \varrho} m_1^{\xi} m_2^{\varrho} [y_{\delta}^{\varrho \varphi}, x_{\tau}^{\xi \varphi}]^{\frac{(p^k - 1)p^k}{2}} \\ &\quad \cdot \prod_{i < j} [(x h_j)_{\tau}^{w_j}, (x h_i)_{\tau}^{w_i}]^{\frac{(\xi - 1)\xi}{2}} \prod_{i < j} [(y h_j)_{\delta}^{w_j}, (y h_i)_{\delta}^{w_i}]^{\frac{(\varrho - 1)\varrho}{2}}. \end{aligned}$$

Now equation (3.8) can be rewritten as

$$(3.10) \quad (xh_1)_\tau^{w_1\xi} \dots (xh_n)_\tau^{w_n\xi} (yh_1)_\delta^{w_1\varrho} \dots (yh_n)_\delta^{w_n\varrho} m_3 \prod_i [a_i, x_\tau^\xi y_\delta^\varrho]^{h_i} \\ \cdot \prod_{i < j} [(x_\tau^\xi y_\delta^\varrho)^{h_i}, (x_\tau^\xi y_\delta^\varrho)^{h_j}]^{b_{ij}} \prod_i [(yh_i)_\delta^\varrho, (xh_i)_\tau^\xi]^{-\frac{(w_i-1)w_i}{2}} \prod_{i < j} [(yh_i)_\delta^\varrho, (xh_j)_\tau^\xi]^{w_i w_j}.$$

Equations (3.9) and (3.10) yield

$$(3.11) \quad m_1^\xi m_2^\varrho [y_\delta^\varrho, x_\tau^{\xi\varrho}]^{-\frac{(p^k-1)p^k}{2}} \prod_{i < j} [(xh_j)_\tau^{w_j}, (xh_i)_\tau^{w_i}]^{-\frac{(\xi-1)\xi}{2}} \prod_{i < j} \\ \cdot [(yh_j)_\delta^{w_j}, (yh_i)_\delta^{w_i}]^{-\frac{(\varrho-1)\varrho}{2}} = m_3 \prod_i [a_i, x_\tau^\xi y_\delta^\varrho]^{h_i} \prod_{i < j} [(x_\tau^\xi y_\delta^\varrho)^{h_i}, (x_\tau^\xi y_\delta^\varrho)^{h_j}]^{b_{ij}} \\ \cdot \prod_i [(yh_i)_\delta^\varrho, (xh_i)_\tau^\xi]^{-\frac{(w_i-1)w_i}{2}} \prod_{i < j} [(yh_i)_\delta^\varrho, (xh_j)_\tau^\xi]^{w_i w_j}.$$

Taking the p^k -th power of equation (3.11) and collecting the commutators with $\xi\varrho$, we use equations (3.6) and (3.7) to get

$$(3.12) \quad [(yh_1)_\delta^{w_1} \dots (yh_n)_\delta^{w_n}, (xh_1)_\tau^{w_1} \dots (xh_n)_\tau^{w_n}]^{\frac{p^k-1}{2}} \\ = \prod_{i < j} [(xh_i)_\tau, (yh_j)_\delta]^{b_{ij}} [(yh_i)_\delta, (xh_j)_\tau]^{b_{ij}} \\ \cdot \prod_i [(yh_i)_\delta, (xh_i)_\tau]^{-\frac{(w_i-1)w_i}{2}} \prod_{i < j} [(yh_i)_\delta, (xh_j)_\tau]^{w_i w_j}.$$

Since the commutators $\{[(yh_i)_\delta, (xh_j)_\tau]: i, j = 1, \dots, n\}$ form a linearly independent set in $\widehat{N}'_G$, combine like commutators to get the equations

$$(3.13) \quad w_i^2 \cdot \frac{p^k-1}{2} = \frac{(w_i-1)w_i}{2}$$

$$(3.14) \quad w_i w_j p^k + b_{ij} p^k = w_i w_j \cdot \frac{p^k-1}{2}$$

$$(3.15) \quad -b_{ij} p^k = w_i w_j \cdot \frac{p^k-1}{2}.$$

Equation (3.13) implies that either $w_i = 0$ or $w_i = p^k$. Equations (3.14)

and (3.15) imply that $w_i w_j = 0$ for all i, j . Since φ is assumed to be an automorphism, there is an i such that $w_i = p^k$. Thus it follows that $n = p^k h$ for some $h \in G$ and

$$x_\tau^\varphi \equiv (xh)_\tau \quad \text{and} \quad y_\delta^\varphi \equiv (yh)_\delta \bmod \widehat{N}'_G.$$

Hence there exists $h \in G$ such that for all $\tau \in T$ and for all $x \in G$, $(x_\tau)^\varphi \equiv (xh)_\tau \bmod \widehat{N}'_G$, i.e., φ induces h on $\widehat{N}_G/\widehat{N}'_G$. Since $\widehat{N}_G$ is nilpotent of class two, $[x_\tau, y_\delta]^\varphi = [(xh)_\tau, (yh)_\delta] = [x_\tau, y_\delta]^h$. Thus φ also induces h on $\widehat{N}'_G$, and so $\varphi \in \text{Stab}(\widehat{N}_G) \rtimes G$. ■

Let $\varphi = \varphi_{gh}$ be the automorphism defined for all $x \in \widehat{N}_G$ to be $x^\varphi = x[h, x^g]$, for some fixed $g \in G$ and $h \in \widehat{N}_G$. Clearly $\varphi_{eh^{-1}}$ is conjugation by h .

COROLLARY 3.5. *If $h \in H_{\lambda^*}$ and g is any element of G , then $\varphi_{gh} \upharpoonright_{H_{\lambda^*}} \in \text{Stab } H_{\lambda^*}$ and extends to any extension of H_{λ^*} which is G -invariant.*

The corollary shows that if G is non-trivial, then there exist elements of $\text{Stab } H_{\lambda^*}$ which are not in $\text{Inn } H_{\lambda^*}$.

COROLLARY 3.6. *Suppose $G = \{e\}$ and $H = \langle N_G, g_\beta: \beta < \alpha \rangle$ is defined as in Theorem 3.3. If $\varphi \in \text{Aut } \widehat{N}_G \setminus \text{Inn } \widehat{N}_G$, then there exists $x \in \widehat{N}_G$ such that $x^\varphi \notin \langle H, x \rangle_*$.*

PROOF. If $G = \{e\}$, then $e_\tau^\varphi \equiv e_\tau \bmod \widehat{N}'_G$ for all $\tau \in T$, by Theorem 3.4. Let τ and δ be distinct elements of T . Then there exists an integer k such that $e_\tau^{\varphi p^k} = e_\tau^{p^k} h_1$ and $e_\tau^{\xi \varphi p^k} = e_\tau^{\xi p^k} h_2 [e_\tau^\xi, h]$, for some $h_i \in H'$ and $h \in H$. By continuity of φ , we also have $e_\tau^{\xi \varphi p^k} = e_\tau^{\xi p^k} h_1^\xi$. Thus $h_2 = 1$ and $h_1 = [e_\tau, h]$. As in Theorem 3.4, we take and compare the images $e_\tau^\varphi, e_\delta^\varphi$ and $(e_\tau^\xi e_\delta^\varphi)^\varphi$ to show that for any pair τ, δ of distinct elements of T , there exist an integer k and $h \in H$ such that

$$e_\tau^{\varphi p^k} = e_\tau^{p^k} [e_\tau, h] \quad \text{and} \quad e_\delta^{\varphi p^k} = e_\delta^{p^k} [e_\delta, h].$$

By taking three distinct elements $\tau, \delta, \mu \in T$ and applying the preceding observation to the three distinct pairs of elements of T , it is easy to see that there exist $g \in H$ and an integer k such that

$$e_\tau^{\varphi p^k} = e_\tau^{p^k} [e_\tau, g] \quad \text{for all } \tau \in T.$$

Since $e_\tau^{\varphi p^k}, e_\tau^{p^k} \in \widehat{N}_G^{p^k}$ for all τ , it follows that $[e_\tau, g] \in \widehat{N}_G^{p^k}$ for all τ . Thus

there exists $g_* \in \widehat{N}_G$ such that $g_*^{p^k} \equiv g \pmod{\widehat{N}'_G}$ and $e_\tau^\varphi = e_\tau[e_\tau, g_*] = e_\tau^g$ for all τ , i.e., $\varphi \in \text{Inn } \widehat{N}_G$. ■

THEOREM 3.7. *Let $H_{\lambda^*} = \langle N_G, g_\alpha^G: \alpha < \lambda^* \rangle_*$. Then $\text{Aut } H_{\lambda^*} \cong \text{Stab}(H_{\lambda^*}) \rtimes G$.*

PROOF. Let $\varphi \in \text{Aut } H_{\lambda^*}$. Then φ extends to $\varphi \in \text{Aut } \widehat{N}_G$. Note that

$$H_{\lambda^*} \cap \widehat{N}'_G = Z(H_{\lambda^*}), (H_{\lambda^*}')_* = Z(H_{\lambda^*}), \quad \text{and} \quad H_{\lambda^*}/Z(H_{\lambda^*}) \cong H_{\lambda^*}\widehat{N}'_G/\widehat{N}'_G.$$

If $\varphi \in \text{Stab}(\widehat{N}_G) \rtimes G$, then $\varphi \upharpoonright_{\widehat{N}_G/\widehat{N}'_G} = g \upharpoonright_{\widehat{N}_G/\widehat{N}'_G}$ and $\varphi \upharpoonright_{\widehat{N}'_G} = g$, for some $g \in G$. Hence $\varphi \upharpoonright_{H_{\lambda^*}/Z(H_{\lambda^*})} = g$ and $\varphi \upharpoonright_{Z(H_{\lambda^*})} = g$, i.e., $\varphi \in \text{Stab}(H_{\lambda^*}) \rtimes G$. Suppose $\varphi \in \text{Aut } \widehat{N}_G \setminus (\text{Stab}(\widehat{N}_G) \rtimes G)$. By Theorem 3.4, there exists $x \in \widehat{N}_G$ such that $x^\varphi \notin \langle H_{\lambda^*}, x^G \rangle_*$. Theorem 3.2 implies that there exists $\alpha < \lambda^*$ such that $x, x^\varphi \in \widehat{P}_\alpha$; $\|x\|, \|x^\varphi\| < \|P_\alpha\|$ and $\varphi_\alpha = \varphi \upharpoonright_{\widehat{P}_\alpha}$. We show that there exists $g_\alpha \in \widehat{P}_\alpha$ and $v_\alpha \in \text{Br}(\text{Im } f_\alpha)$ such that $\|x\| < \|g_\alpha\|$ and if $H_{\alpha+1} = \langle H_\alpha, g_\beta^G: \beta < \alpha \rangle_*$ with $g_\beta^{\varphi^\beta} \notin H_\alpha$ for all $\beta < \alpha$, then $g_\alpha^{\varphi^\alpha}, g_\beta^{\varphi^\beta} \notin H_{\alpha+1}$ for all $\beta < \alpha$. Let $v \in \text{Br}(\text{Im } f_\alpha)$, then $v \neq v_\beta$ for all $\beta < \alpha$. We show that either $(v^1)^\varphi \notin \langle H_\alpha, (v^1)^G \rangle_*$ or $(v^1 x)^\varphi \notin \langle H_\alpha, (v^1 x)^G \rangle_*$, i.e., the sought after g_α is either v^1 or $v^1 x$. Suppose that the preceding is false. Then there exists an integer k such that

$$(3.16) \quad (v^1 x)^{\varphi p^k} = (v^1 x)^{n_1 g_1} \dots (v^1 x)^{n_l g_l} h_1 \prod_g [h_{2g}, (v^1 x)^g]$$

$$(3.17) \quad (v^1)^{\varphi p^k} = (v^1)^{m_1 g_1} \dots (v^1)^{m_l g_l} h_3 \prod_g [h_{4g}, (v^1)^g]$$

for some integers $m_i, n_i, g_i \in G$ and $h_i, h_{ig} \in H_\alpha$. Then we have the following congruences mod $\widehat{N}'_G$:

$$\begin{aligned} (v^1 x)^{\varphi p^k} &\equiv (v^1)^{\varphi p^k} x^{\varphi p^k} \equiv (v^1)^{m_1 g_1} \dots (v^1)^{m_l g_l} h_3 x^{\varphi p^k} \\ (v^1 x)^{\varphi p^k} &\equiv (v^1 x)^{n_1 g_1} \dots (v^1 x)^{n_l g_l} h_1 \equiv (v^1)^{n_1 g_1} \dots (v^1)^{n_l g_l} x^{n_1 g_1} \dots x^{n_l g_l} h_1. \end{aligned}$$

By choice of support and, hence, norm of v , it follows that $n_i = m_i$ ($i = 1, \dots, l$) and

$$(3.18) \quad x^{\varphi p^k} \equiv h_3^{-1} x^{n_1 g_1} \dots x^{n_l g_l} h_1 \pmod{\widehat{N}'_G}.$$

So

$$\begin{aligned}
 (3.19) \quad (v^1 x)^{\varphi p^{2k}} &= (v^1)^{\varphi p^{2k}} x^{\varphi p^{2k}} [x^\varphi, (v^1)^\varphi]^{\frac{(p^{2k}-1)p^{2k}}{2}} = \\
 &= ((v^1)^{m_1 g_1} \dots (v^1)^{m_l g_l})^{p^k} h_3^{p^k} x^{\varphi p^{2k}} \prod_g [h_{4g}, (v^1)^g]^{p^k} \cdot \\
 &\quad \cdot [h_3, (v^1)^{m_1 g_1} \dots (v^1)^{m_l g_l}]^{\frac{(p^k-1)p^k}{2}} [x^{\varphi p^k}, (v^1)^{\varphi p^k}]^{\frac{p^{2k}-1}{2}}.
 \end{aligned}$$

But $[x^{\varphi p^k}, (v^1)^{\varphi p^k}] = [h_3^{-1} x^{m_1 g_1} \dots x^{m_l g_l} h_1, (v^1)^{m_1 g_1} \dots (v^1)^{m_l g_l} h_3]$. We also have $(v^1 x)^{\varphi p^{2k}}$ equal to

$$\begin{aligned}
 (3.20) \quad ((v^1 x)^{m_1 g_1} \dots (v^1 x)^{m_l g_l})^{p^k} h_1^{p^k} [h_1, (v^1 x)^{m_1 g_1} \dots (v^1 x)^{m_l g_l}]^{\frac{(p^k-1)p^k}{2}} \cdot \\
 \cdot \prod_g [h_{2g}, (v^1 x)^g]^{p^k}.
 \end{aligned}$$

Since $\|x\|, \|x^\varphi\| < \|v\|$ and $v \neq v_\beta$ for all $\beta < \alpha$, it follows from equations (3.17), (3.18), (3.19) and (3.20) that

$$\begin{aligned}
 x^{\varphi p^{2k}} &= h_3^{-p^k} (x^{m_1 g_1} \dots x^{m_l g_l})^{p^k} h_1^{p^k} [h_1, x^{m_1 g_1} \dots x^{m_l g_l}]^{\frac{(p^k-1)p^k}{2}} \cdot \\
 &\quad \cdot \prod_g [h_{2g}, x^g]^{p^k} [h_3, x^{m_1 g_1} \dots x^{m_l g_l} h_1]^{\frac{p^{2k}-1}{2}},
 \end{aligned}$$

which is an element of $\langle H_\alpha, x^G \rangle \subset \langle H_{\lambda^*}, x^G \rangle$. This contradicts the assumption that $x^\varphi \notin \langle H_{\lambda^*}, x^G \rangle_*$. Therefore $\varphi \in \text{Stab}(\widehat{N}_G) \rtimes G$ and $\text{Aut } H_{\lambda^*} = \text{Stab}(H_{\lambda^*}) \rtimes G$. ■

COROLLARY 3.8. *If $G = \{e\}$, there exists a torsion-free nilpotent group H_* of class two such that $\text{Aut } H_* = \text{Inn } H_*$.*

PROOF. Suppose $G = \{e\}$. Let $H = \langle N_G, g_\beta: \beta < \lambda^* \rangle$ be as in Theorem 3.7 and H_* its purification in $\widehat{N}_G$. Suppose $\varphi \in \text{Aut } H_*$. Then φ extends to an automorphism of $\widehat{N}_G$. Using Corollary 3.6 and the same argument as in Theorem 3.7, it follows that $\varphi \in \text{Inn } \widehat{N}_G$. Thus φ is conjugation by some element x in the normalizer $N_{\widehat{N}_G}(H_*)$ of H_* in $\widehat{N}_G$. So $[g, x] \in H_*$ for all $g \in H_*$. If τ and μ are distinct, there exists an integer k such that $[e_\tau, x]^{p^k}$ and $[e_\mu, x]^{p^k}$ are both in H , i.e.,

$$[e_\tau, x]^{p^k} = [e_\tau, h_\tau] \quad \text{and} \quad [e_\mu, x]^{p^k} = [e_\mu, h_\mu],$$

for some $h_\tau, h_\mu \in H$. It follows from these equations that $x^{p^k} \equiv e_{\tau_i}^{\xi_\tau} h_\tau$ and $x^{p^k} \equiv e_{\mu_i}^{\xi_\mu} h_\mu \pmod{\widehat{N}'_G}$, for some $\xi_\mu, \xi_\tau \in J_p$. Thus $e_{\mu_i}^{-\xi_\mu} e_{\tau_i}^{\xi_\tau} \in H \widehat{N}'_G$. By the choice of the supports of the elements of H , elements in H with finite support must be in N_G . Hence it must be that ξ_μ and ξ_τ are integers, $x_{\tau_i}^{\xi_\tau} \in H$ and $x_{\mu_i}^{p^k} \equiv x_{\tau_i}^{\xi_\tau} h_\tau \in H$. It follows that $x \in (H \widehat{N}'_G)_*$ and so $N_{\widehat{N}'_G}(H_*) = (H \widehat{N}'_G)_*$.

Finally we show that $(H \widehat{N}'_G)_* = H_* \widehat{N}'_G$, for then $\text{Aut } H_* = N_{\widehat{N}'_G}(H_*) / \widehat{N}'_G = \text{Inn } H_*$. Clearly $H_* \widehat{N}'_G \subseteq (H \widehat{N}'_G)_*$. Since $\widehat{N}'_G \subset H_* \widehat{N}'_G$, it suffices to show that if $\prod_{i < \omega} e_{\tau_i}^{\xi_i} \in (H \widehat{N}'_G)_*$, then there exists $g \in H_*$ and $\eta \in \widehat{N}'_G$ such that $\prod_{i < \omega} e_{\tau_i}^{\xi_i} = g\eta$. By the definition of $\widehat{N}'_G$, for all n , p^n divides ξ_i for all but finitely many ξ_i in J_p . Now there exists an integer k such that

$$\prod_{i < \omega} e_{\tau_i}^{\xi_i p^k} \equiv \prod_{i=1}^m e_{\tau_i}^{a_i} \prod_{i=1}^m g_{\beta_i}^{b_i} \in H,$$

for some integers a_i, b_i . However

$$\prod_{i=1}^m e_{\tau_i}^{a_i} \prod_{i=1}^m g_{\beta_i}^{b_i} \equiv \prod_{i < \omega} e_{\tau_i}^{\xi_i p^k} \prod_{i < j < \omega} [e_{\tau_i}, e_{\tau_j}]^{\xi_{ij}}$$

and p^k divides ξ_{ij} for almost all ξ_{ij} . If p^k does not divide some ξ_{ij} , then there exists an integer n_{ij} such that $\xi_{ij} + n_{ij} \in p^k J_p$. Let $\mu_{ij} = \xi_{ij}$, if p^k divides ξ_{ij} , and $\mu_{ij} = \xi_{ij} + n_{ij}$, otherwise. Then

$$\prod e_{\tau_i}^{\xi_i p^k} \prod_{i < j} [e_{\tau_i}, e_{\tau_j}]^{\mu_{ij}} \in H \cap \widehat{N}_G^{p^k}.$$

Hence it must equal $(\prod e_{\tau_i}^{\xi_i} \prod_{i < j} [e_{\tau_i}, e_{\tau_j}]^{\delta_{ij}})^{p^k}$, for some $\delta_{ij} \in J_p$. Thus $\prod e_{\tau_i}^{\xi_i} \prod_{i < j} [e_{\tau_i}, e_{\tau_j}]^{\delta_{ij}}$ is an element of H_* , and so $\prod_{i < \omega} e_{\tau_i}^{\xi_i} \in H_* \widehat{N}'_G$. ■

COROLLARY 3.9. *Let $N = \langle X \rangle$ be a free nilpotent group with basis X and G a non-trivial group such that G acts faithfully on X . If $N \subset H \subset \widehat{N}$, $H_* = H$ and H is G -invariant, then $\text{Inn } H \neq \text{Stab } H$ and $G \cap \text{Stab } H = 1$.*

PROOF. Let $g \in G \setminus \{e\}$ and $h \in H \setminus Z(H)$. Define the map φ by $x^\varphi = x[h, x^g]$, for all $x \in H$. Clearly $\varphi \in \text{Stab } H$ and $\varphi \notin \text{Inn } H$. Since G acts faithfully on X , the intersection $G \cap \text{Stab } H$ must contain only the identity map. ■

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