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A Confinement Result for Axisymmetric Fluids.

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ABSTRACT - In this paper we consider an incompressible, inviscid, axisymmetric fluid moving in $\boldsymbol{R}^3$ without swirl (a vortex ring). We study its time evolution via the Euler Equation, assuming that initially the vorticity is concentrated at a finite distance, say r_0, from the symmetry axis. We prove a bound on the growth of the support of the vorticity. Namely we show that the fluid is confined in a cylinder with radius $r \leqslant r_0 + constant\, t^{1/4} \log(e+t)$.

1. Introduction.

The interest in studying vortex rings can be found in many different physical contexts. Just to remind some examples, one can note that questions related to the behavior of vortex rings are crucial in the theory of transport and mixing (expansion of rings due to the buoyancy [1], or growth of jets [2]) or in the problem of generation of sound (the sound field can be expressed in terms of vorticity unsteadiness [3], the sound source can be modelized in terms of interacting rings [4]), or else, in the study of turbulence (accelerated ions in superfluid helium create quantized vortex rings [5]).

For an extensive review of results and problems on the subject one can see [6].

In the present paper we are concerned with a particular problem of inviscid swirl-free dynamics of a fluid, namely we are interested

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in saying «how fast, and therefore how far, can go a ring of vorticity».

It is known, in fact, that the main part of the vorticity remains close to the symmetry axis (see eq. (2.12) below), but this does not exclude that some filaments of vorticity go far away. We give here a bound on this effect. We consider, in particular, the time evolution of an incompressible, inviscid, axisymmetric fluid moving in $\boldsymbol{R}^3$ without swirl and, assuming that the vorticity is initially concentrated around the symmetry axis at a distance r_0, we prove that the distance must be less than $r_0 +$ + constant $t^{1/4}\log(e+t)$.

The paper is organized as follows: in the next section 2 we state the problem and give our main theorem. The sections 3 and 4 are respectively devoted to the proof of the theorem and to some preliminary results.

2. Preliminaries and main theorem.

Let us consider an incompressible inviscid fluid of unitary density in $\boldsymbol{R}^3$. If $u = u(\xi, t)$, $\xi \in \boldsymbol{R}^3$, denotes the velocity field of the fluid and $p = p(\xi, t)$ is the pressure, the Euler equations can be written

$$\begin{aligned} \partial_t u + (u \cdot \nabla)\, u &= -\nabla p \\ \nabla \cdot u &= 0\,. \end{aligned} \tag{2.1}$$

We assume the initial condition

$$u(\xi, 0) = u_0(\xi)\,,$$

and we suppose, moreover, that the initial velocity field is axisymmetric without swirl, i.e. if $\xi = (z, r, \theta)$ are cylindrical coordinates, then

$$u_0(\xi) = (u_z(z, r, 0), u_r(z, r, 0), 0)\,. \tag{2.2}$$

We assume, furthermore, that the boundary condition

$$u(\xi, t) \to 0 \quad \text{as} \quad |\xi| \to \infty$$

holds.

If $\omega(\xi, t) = \nabla \wedge u(\xi, t)$ denotes the vorticity, it is well known that the velocity can be expressed in terms of ω as

$$u(\xi, t) = -\frac{1}{4\pi} \int_{\boldsymbol{R}^3} \frac{(\xi - \eta) \wedge \omega(\eta, t)}{|\xi - \eta|^3}\, d\eta\,. \tag{2.3}$$

Taking into account the fact that the cylindrical symmetry is preserved by the time evolution, the problem (2.1), can be considered as bidimensional. More precisely, if we set $\omega = (0, 0, \omega)$, the equation (2.1) can be rewritten in terms of ω as

$$\begin{aligned} \partial_t \omega + (u \cdot \nabla)\, \omega - \frac{u_r \omega}{r} &= 0 \\ \partial_z (r u_z) + \partial_r (r u_r) &= 0\,. \end{aligned} \tag{2.4}$$

If $x(t) = (z(t, x_0), r(t, x_0))$, is the time evolution of $x(0) = x_0 = (z_0, r_0)$ along the velocity field

$$\begin{aligned} \dot z &= u_z(z, r, t) \\ \dot r &= u_r(z, r, t)\,, \end{aligned} \tag{2.5}$$

it is well known that $(2.4)_1$ says that

$$\frac{\omega(x(t), t)}{r(t)} = \frac{\omega(x_0, 0)}{r_0}\,, \tag{2.6}$$

that is, ω/r is constant along the evolution (2.5). (We recall that in the case of planar symmetry the conserved quantity is ω).

The equations (2.5), (2.3), and (2.6) give a weak formulation of the Euler Equation.

It is also well known that the momentum of inertia

$$M = \int_{\boldsymbol{R}} dz \int_0^\infty r^2 \omega(z, r, t)\, dr\,, \tag{2.7}$$

and the total vorticity

$$\Omega = \int_{\boldsymbol{R}} dz \int_0^\infty \omega(z, r, t)\, dr \tag{2.8}$$

are conserved along the evolution (2.5). (The fact that M is constant can be checked directly by performing the time derivative of M along (2.5) and is related to the invariance of the problem under rotations; Ω is constant as a direct consequence of (2.6)).

The main result we prove in this paper is the following.

THEOREM 2.1. Assume that $\omega(z, r, t)$ is a solution of the problem (2.3), (2.5), (2.6). Call

$$D_0 = \{x = (z, r) \in \boldsymbol{R}^2 \colon |z| \le a,\ 0 < r \le b\}\,,$$

with a, b positive constants, the support of the vorticity at time $t=0$. Suppose, finally that

$$\omega(x,0)\geqslant 0 \quad x\in D_0, \qquad \omega(x,0)=0 \quad x\notin D_0$$

and that

$$\frac{\omega(x,0)}{r}\leqslant d<\infty\,. \tag{2.9}$$

Then there exists a positive constant C, depending only on the initial conditions, such that for all $t>0$ the support D_t of ω is contained in the strip $S_t=\{x=(z,r),\ z\in\boldsymbol{R},\ r<r_0+Ct^{1/4}\log(e+t)\}$, i.e.

$$\omega(z,r,t)=0 \qquad \text{if } \mathrm{r}\geqslant \mathrm{r}_0+\mathrm{Ct}^{1/4}\log(\mathrm{e}+\mathrm{t})\,. \tag{2.10}$$

(The proof is given in the next section).

REMARK. It is trivial to give a rough bound on the growth of D_t by using the fact that the velocity field is bounded. The previous result states this bound in a quite sharp way.

We remind, moreover, that a bound similar to (2.10) has been proved in the planar case, see [7], [8], [9].

COROLLARY 2.2. A trivial consequence of the theorem is the confinement result

$$|r(t)-r_0|<Ct^{1/4}\log(e+t)\,.$$

From now on, C always denotes a positive constant depending on the initial conditions.

Before concluding the section, we give a final definition.

DEFINITION 2.1. If $S^{r^*}=\{x=(z,r): z\in\boldsymbol{R},\ r<r^*\}$ denotes a strip in the (z,r)-plane, then

$$m_{r^*}(t)=\int_{\boldsymbol{R}} dz\int_{r^*}^{\infty}\omega(z,r,t)\,dr\,, \tag{2.11}$$

is the vorticity external to the strip S^{r^*} at time t.

Let us note that, as a consequence of (2.7), a trivial bound on $m_{r^*}(t)$, is given by

$$m_{r^*}(t) \leqslant M/(r^*)^2. \tag{2.12}$$

3. Three lemmas and the proof of theorem 2.1.

We remark, first of all, that the velocity $u(x, t)$, with respect to the coordinates (z, r), can explicitly be written as

$$u(x, t) = \frac{1}{2\pi} \int_{\mathbf{R}} dz' \int_0^\infty r'\,dr' \int_0^\pi \omega(x', t)\, H(x, x', \theta)\, d\theta\,, \tag{3.1}$$

$$H(x, x', \theta) =$$

$$= \left(-\frac{(r\cos\theta - r')}{[\,|x - x'|^2 + 2rr'(1-\cos\theta)]^{3/2}}, \frac{(z - z')\cos\theta}{[\,|x - x'|^2 + 2rr'(1-\cos\theta)]^{3/2}} \right).$$

We anticipate that the main step in the proof of the theorem consists in showing that, for all $r \geqslant r_0 + Ct^{1/4}\log(e+t)$, the radial component of the velocity (3.1) satisfies an estimate of the form

$$|u_r(x, t)| \leqslant \frac{C}{r^3}. \tag{3.2}$$

To prove, the previous (3.2) we need some preliminary results, that we state without proof. All the proofs can be found in the next section.

LEMMA 3.1. *Assume the hypotheses of the Theorem 2.1. Then the following estimate holds*

$$|u_r(x, t)| \leqslant C\left(\frac{1}{r^3} + r(m_{r/2}(t))^{1/2} \right), \tag{3.3}$$

where $m_{r/2}(t)$ is defined in the (2.11).

The next two results allow us to say that, for r sufficiently large, $m_{r/2}(t)$ is as small as we want. More precisely the lemmas hold.

LEMMA 3.2. *Under the same hypotheses of the theorem 2.1., given two positive numbers r_1, r_2, $r_0 \leqslant r_1 < r_2$, the vorticity external to the strip*

S^{r_2} satisfies the estimate

$$m_{r_2}(t) \leq F(r_1, r_2) \int_0^t m_{r_1}(s)\, ds\,, \tag{3.4}$$

where

$$F(r_1, r_2) = C\left[\frac{r_2 + (r_2 - r_1)(1 + (r_2/r_1)^2)}{r_1^3 (r_2 - r_1)^2}\right]. \tag{3.5}$$

LEMMA 3.3. Assume that $r \geq r_0 + Ct^{1/4}\log(e+t)$. Then for all $n \in N$, it is

$$m_r(t) \leq \frac{C}{r^{n+2}}\,. \tag{3.6}$$

We are ready now for our main result.

PROOF OF THEOREM 2.1. Fix n of Lemma 3.3 equal to 6. The estimate (3.2) easily follows from the (3.3) and (3.6).

If the (3.2) holds, then the theorem is proved by contradiction.

4. Proofs of the lemmas.

We start by giving two formulas that we need in what follows.

Call

$$B(x, x') = \int_0^\pi \frac{\cos\theta}{[\,|x - x'|^2 + 2rr'(1-\cos\theta)]^{3/2}}\, d\theta\,.$$

If $x \neq x'$, there exists a positive constant c such that

$$|B(x, x')| \leq c\frac{rr'}{|x - x'|^5}\,. \tag{4.1}$$

Moreover

$$B(x, x') = \frac{1}{(rr')^{3/2}}\left\{\left(\frac{rr'}{|x-x'|^2 + 4rr'}\right)^{1/2}\cdot\left(\frac{2rr'}{|x-x'|^2} + \frac{3}{8}\right) + \right.$$

$$\left. -\frac{3}{8}\left[\ln\frac{|x-x'|}{(rr')^{1/2}} - \ln\left(2 + \left(\frac{|x-x'|^2 + 4rr'}{rr'}\right)^{1/2}\right)\right] + R(|x-x'|^2)\right\},$$

where $R(|x - x'|^2)$ is bounded.

In what follows, in particular, we use that

$$\lim_{x \to x'} |x - x'|^2 B(x, x') = 1/r . \tag{4.2}$$

(The proof of (4.1), (4.2) can be found in the Appendix).

PROOF OF LEMMA 3.1. To show that the (3.3) holds assume that, for all r, the interval $[0, \infty)$ is divided as follows: $[0, \infty) = \mathfrak{I} \cup \mathcal{N}$, where $\mathcal{N}$ is a neighborhood of r as small as we want, say for definiteness $\mathcal{N} = [r - r/c, r + r/c]$, with c a positive constant, and $\mathfrak{I} = [0, \infty)/\mathcal{N}$.

From (3.1) one can write

$$u_r(x, t) = \frac{1}{2\pi}\left(\int_{\mathbf{R} \times \mathfrak{I}} (z - z')\, \omega(x', t)\, r' B(x, x')\, dx' + \right. \tag{4.3}$$

$$\left. + \int_{\mathbf{R} \times \mathcal{N}} (z - z')\, \omega(x', t)\, r' B(x, x')\, dx' \right),$$

where $B(x, x')$ is defined above.

We estimate the first term in the previous sum. Taking into account the estimates (4.1) and the (2.7) we get

$$T_1 \equiv \frac{1}{2\pi} \left| \int_{\mathbf{R} \times \mathfrak{I}} (z - z')\, \omega(x', t)\, r' B(x, x')\, dx' \right| \leqslant \tag{4.4}$$

$$\leqslant \int_{\mathbf{R} \times \mathfrak{I}} \omega(x', t) \frac{r r'^2}{|x - x'|^4} dx' \leqslant \frac{C}{r^3} .$$

For the second integral we note that, from the (4.2) it follows that

$$T_2 \equiv \frac{1}{2\pi} \left| \int_{\mathbf{R} \times \mathcal{N}} (z - z')\, \omega(x', t)\, r' B(x, x')\, dx' \right| \leqslant$$

$$\leqslant c \int_{\mathbf{R} \times \mathcal{N}} \frac{r'^{1/2} |z - z'|}{(r)^{1/2} |x - x'|^2} \omega(x', t)\, dx' \leqslant c \int_{\mathbf{R} \times \mathcal{N}} \frac{\omega(x', t)}{|x - x'|} dx' .$$

Given $k = (m_{r - r/c}(t))^{1/2} > 0$, let us consider now the ball $B_k = \{x' \in \mathbf{R} \times \times [0, \infty) : |x - x'| < k\}$, and set

$$\mathbf{R} \times \mathcal{N} = (B_k \cap (\mathbf{R} \times \mathcal{N})) \cup ((\mathbf{R} \times \mathcal{N})/B_k) \equiv i_1 \cup i_2 .$$

Then, using hypothesis (2.9) it is

$$T_2 \leqslant \int_{i_1} \frac{\omega(x',t)}{r'} \frac{r'}{|x-x'|} dx' + \int_{i_2} \frac{\omega(x',t)}{|x-x'|} dx' \leqslant cdr 2\pi k + \frac{1}{k} m_{r-r/c}(t),$$

where c is a suitable constant, and, as usual, the first integral is performed in polar coordinates.

Summing T_1 and T_2, the lemma is proved.

PROOF OF LEMMA 3.2. Given two positive numbers $r_1, r_2, r_1 < r_2$, let us define the function

$$\mu_{r_1 r_2}(t) = \int_R dz \int_0^\infty G_{r_1 r_2}(r)\, \omega(x,t)\, dr,$$

where

$$G_{r_1 r_2}(r) = \begin{cases} 0 & \text{if } r < r_1 \\ \text{a smooth function} & \text{if } r_1 \leqslant r \leqslant r_2 \\ 1 & \text{if } r > r_2 \end{cases}$$

and

$$\text{(4.5)} \qquad \begin{aligned} &|G'_{r_1 r_2}(r)| \leqslant \frac{c}{r_2 - r_1}, \\ &|G'_{r_1 r_2}(r) - G'_{r_1 r_2}(r')| \leqslant c \frac{|r - r'|}{(r_2 - r_1)^2}, \end{aligned}$$

where c is a positive constant.

From the definition of $\mu_{r_1 r_2}(t)$ it follows that

$$m_{r_2}(t) \leqslant \mu_{r_1 r_2}(t) \leqslant \mu_{r_1 r_2}(0) + \int_0^t |\dot{\mu}_{r_1 r_2}(s)|\, ds.$$

If b of D_0 is such that $b \leqslant r_1$, it follows that

$$m_{r_2}(t) \leqslant \int_0^t |\dot{\mu}_{r_1 r_2}(s)|\, ds.$$

Let us compute the time derivative of $\mu_{r_1 r_2}(t)$ along the evolution (2.5).

One has, taking into account the (3.1),

$$\dot{\mu}_{r_1 r_2}(t) = \int_{\boldsymbol{R}\times\boldsymbol{R}^+} G'_{r_1 r_2}(r)\, u_r(x, t)\, \omega(x, t)\, dx =$$

$$= \int_{\boldsymbol{R}\times\boldsymbol{R}^+}\int_{\boldsymbol{R}\times\boldsymbol{R}^+} (z - z')\, G'_{r_1 r_2}(r)\, r'\, B(x, x')\, \omega(x', t)\, \omega(x, t)\, dx'\, dx\,.$$

As already remarked, for $|x - x'| \to 0$, $B(x, x')$ becomes singular. We overcome this problem by using the hypothesis $(4.5)_2$.

More precisely, taking into account that $B(x, x') = B(x', x)$, one can write

$$\dot{\mu}_{r_1 r_2}(t) = -\int_{\boldsymbol{R}\times\boldsymbol{R}^+}\int_{\boldsymbol{R}\times\boldsymbol{R}^+} (z - z')\, G'_{r_1 r_2}(r')\, rB(x, x')\, \omega(x, t)\, \omega(x', t)\, dx\, dx'\,.$$

It follows then

$$\dot{\mu}_{r_1 r_2}(t) = \frac{1}{2}\int_{\boldsymbol{R}\times\boldsymbol{R}^+}\int_{\boldsymbol{R}\times\boldsymbol{R}^+} [r'(G'_{r_1 r_2}(r) - rG'_{r_1 r_2}(r'))]\cdot$$

$$\cdot (z - z')\, B(x, x')\, \omega(x', t)\, \omega(x, t)\, dx\, dx' =$$

$$= \frac{1}{2}\left\{\int_{\boldsymbol{R}\times\boldsymbol{R}^+}\int_{\boldsymbol{R}\times\mathfrak{I}} [r'(G'_{r_1 r_2}(r) - G'_{r_1 r_2}(r')) +\right.$$

$$+ (r' - r)\, G'_{r_1 r_2}(r')](z - z')\, B(x, x')\, \omega(x', t)\, \omega(x, t)\, dx\, dx' +$$

$$+ \int_{\boldsymbol{R}\times\boldsymbol{R}^+}\int_{\boldsymbol{R}\times\mathcal{N}} [r'(G'_{r_1 r_2}(r) - G'_{r_1 r_2}(r')) +$$

$$\left. + (r' - r)\, G'_{r_1 r_2}(r')](z - z')\, B(x, x')\, \omega(x', t)\, \omega(x, t)\, dx\, dx'\right\},$$

where $\mathfrak{I}$ and $\mathcal{N}$ are the same as in the previous lemma.

Let us evaluate the first integral in the sum. Taking into account the (4.5), (4.1), (2.7) and (2.12), we have

$$\Bigg| \int_{\mathbf{R}\times\mathbf{R}^+} \int_{\mathbf{R}\times\mathfrak{Z}} [r'(G'_{r_1r_2}(r) - G'_{r_1r_2}(r')) +$$

$$+(r'-r)\,G'_{r_1r_2}(r')](z-z')\,B(x,x')\,\omega(x',t)\,\omega(x,t)\,dx'\,dx \Bigg| \leqslant$$

$$\leqslant \frac{c}{(r_2-r_1)^2} \int_{\mathbf{R}\times\mathbf{R}^+} \int_{\mathbf{R}\times\mathfrak{Z}} \frac{r\omega(x',t)(r')^2}{|x-x'|^3}\,\omega(x,t)\,dx'\,dx +$$

$$+\frac{c}{r_1^3(r_2-r_1)}\Bigg(\int_{\mathbf{R}\times\mathbf{R}^+} \int_{r_1}^{r(1-1/c)} rr'\,\omega(x,t)\,\omega(x',t)\,dx'\,dx +$$

$$+ \int_{\mathbf{R}\times\mathbf{R}^+} \int_{r(1+1/c)}^{r_2} (r')^2\,\omega(x,t)\,\omega(x',t)\,dx'\,dx \Bigg) \leqslant$$

$$\leqslant m_{r_1}(t)\,C\left(\frac{1}{(r_2-r_1)^2 r_1^2} + \frac{1}{(r_2-r_1)\,r_1^3}(r_2/r_1)^2 \right).$$

From the (4.2), following the same path as in the previous case, one gets furthermore,

$$\Bigg| \int_{\mathbf{R}\times\mathbf{R}^+} \int_{\mathbf{R}\times\mathcal{N}} [r'(G'_{r_1r_2}(r) - G'_{r_1r_2}(r')) +$$

$$+(r'-r)\,G'_{r_1r_2}(r')](z-z')\,B(x,x')\,\omega(x',t)\,\omega(x,t)\,dx'\,dx \Bigg| \leqslant$$

$$\leqslant \frac{c}{(r_2-r_1)^2} \int_{\mathbf{R}\times\mathbf{R}^+} \int_{\mathbf{R}\times\mathcal{N}} \frac{(r')^{1/2}\,\omega(x',t)}{r^{1/2}}\,\omega(x,t)\,dx'\,dx +$$

$$+\frac{c}{(r_2-r_1)} \int_{\mathbf{R}\times\mathbf{R}^+} \int_{\mathbf{R}\times\mathcal{N}} \frac{\omega(x',t)}{(r')^{1/2}r^{1/2}}\,\omega(x,t)\,dx'\,dx \leqslant$$

$$\leqslant Cm_{r_1}(t)\left(\frac{1}{r_1^2(r_2-r_1)^2} + \frac{1}{r_1^3(r_2-r_1)} \right).$$

Then the lemma is proved.

PROOF OF LEMMA 3.3. In this lemma we finally prove that for r sufficiently large, $m_{r/2}(t)$ is as small as we want. To this aim we divide the interval $[r/2, r]$ in a suitable numer of parts and using the (3.4) and (3.5) we prove our thesis.

Let us set, more precisely, $[r/2, r] = \bigcup_{i=1}^{k} I_i$ where $I_i = \left[r\left(1-\frac{i}{2k}\right), r\left(1-\frac{i-1}{2k}\right)\right]$, and k is a positive constant to be fixed in (4.9).

Let us remark, first of all, that if one sets $r_1 = r\left(1-\frac{i}{2k}\right)$, $r_2 = r\left(1-\frac{i-1}{2k}\right)$, for all $i = 1, \ldots, k$, it follows from (3.5) that

$$F(r_1, r_2) \leqslant C\frac{k^2}{r^4}.$$

If one iterates the (3.4) in the intervals I_i one gets

$$m_r(t) \leqslant \left(\frac{Ck^2}{r^4}\right)^k \int_0^t ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{k-1}} m_{r/2}(s_k)\, ds_k. \tag{4.6}$$

From the (2.12), taking into account the Stirling formula $\ln k! > k(\ln k - 1)$, it follows that the (4.6) can be rewritten

$$m_r(t) \leqslant \frac{(Ckt)^k}{r^{4k+2}}. \tag{4.7}$$

Let us define

$$R(t) = r_0 + Ct^{1/4}\log(e+t). \tag{4.8}$$

We show that if $r \geqslant R(t)$, then also

$$r \geqslant r_0 + C' t^{1/4}\log(e+r), \tag{4.9}$$

with C' a suitable constant. Call in fact t^* the solution of $R(t) = t$, (t^* is finite), and set $r^* = t^*$. If $r \geqslant R(t)$, then it is i) $r_0 \leqslant r \leqslant r^*$, or ii) $r > r^*$. In the case i) one has

$$r \geqslant r_0 + Ct^{1/4}\log(e+t) \geqslant r_0 + Ct^{1/4}.$$

Setting $C^* = \dfrac{C}{\ln(e+r^*)}$, then $C^*\ln(e+r) < C$, and therefore

$$r \geqslant r_0 + C^* t^{1/4}\log(e+r).$$

In the case ii) $t > r$, and

$$r \geqslant r_0 + Ct^{1/4} \log(e + r).$$

Then (4.9) holds with $C' = \max(C, C^*)$.

The lemma is proved from (4.7) if, for any $n \in N$ the number k is chosen such that

$$r^{n/k} k = \log^4(e + r). \tag{4.10}$$

APPENDIX. We prove that the (4.1), (4.2), hold. If $x \neq x'$, it is, in fact,

$$\left| \int_0^\pi \frac{\cos\theta}{[\,|x - x'\,|^2 + 2rr'(1 - \cos\theta)]^{3/2}} d\theta \right| =$$

$$= \frac{1}{|x - x'\,|^3} \left| \int_0^\pi \frac{\cos\theta}{[1 + a(1 - \cos\theta)]^{3/2}} d\theta \right|,$$

where $a = \dfrac{2rr'}{|x - x'\,|^2}$. If one notes that

$$\frac{1}{[1 + a(1 - \cos\theta)]^{3/2}} = 1 + \frac{3}{2} \alpha \frac{\cos\theta - 1}{[1 + \alpha(1 - \cos\theta)]^{5/2}},$$

$\alpha \in (0, a)$, the (4.1) easily follows by integration.

(ii) The (4.2) can be proved by noting that, if $b^2 = \dfrac{|x - x'\,|^2}{rr'}$,

$$\int_0^\pi \frac{\cos\theta}{[\,|x - x'\,|^2 + 2rr'(1 - \cos\theta)]^{3/2}} d\theta =$$

$$= (rr')^{-3/2} \left(\int_0^\pi \frac{\cos\theta/2}{[b^2 + 4\sin^2\theta/2]^{3/2}} d\theta - \frac{3}{2} \int_0^\pi \frac{\sin^2\theta/2 \cos\theta/2}{[b^2 + 4\sin^2\theta/2]^{3/2}} d\theta + \right.$$

$$\left. + \int_0^\pi \frac{\cos\theta - \cos\theta/2 + (3/2)\sin^2\theta/2 \cos\theta/2}{[b^2 + 4\sin^2(\theta/2)]^{3/2}} d\theta \right) \equiv (rr')^{-3/2}(i_1 + i_2 + i_3).$$

The first and second integral can be computed exactly and are equal to

$$i_1 = 2 \int_0^1 \frac{dx}{(b^2 + 4x^2)^{3/2}} = \frac{2}{b^2 (b^2 + 4)^{1/2}},$$

$$i_2 = \frac{3}{4} \frac{1}{(b^2 + 4)^{1/2}} - \frac{3}{8} (\log (2 + (b^2 + 4)^{1/2}) - \log b).$$

The remaing part is bounded as $b^2 \to 0$. This allows us to say that (4.2) holds.

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