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Rendiconti del Seminario Matematico della Università di Padova, tome 104 (2000), p. 63-70

<http://www.numdam.org/item?id=RSMUP_2000__104__63_0>
On the Automorphism Group of a Second Order Structure.

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Introduction.

This paper will deal with some property of the automorphism group of special second order structures which are associated to semisimple flat homogeneous spaces. We will consider a second order structure $Q$, i.e. a subbundle of the 2-frame bundle $P^2(M)$ over a smooth manifold $M$, which is associated to a semisimple flat homogeneous space $L/L_0$ with $\dim M = \dim L/L_0$. Such structures have been intensively studied by Ochiai ([3]) and generalize some well-known structures, such as projective and conformal ones. We will confine ourself to the case when $M$ is a reductive homogeneous space and $Q$ is a second order structure, that it has a Cartan connection which is preserved by any automorphism; such class of semisimple flat homogeneous space $L/L_0$, has been classified by Ochiai and the classification will be given in Table 1.

In section 1 we will briefly recall basic facts about semisimple flat homogeneous space according to the paper by Ochiai [3] to which we shall refer throughout the following.

In section 2 we will prove our main theorem, which can be formulated as follows:

THEOREM 1. Let $M = G/K$ be a reductive homogeneous space and let $Q$ be a second order structure over $M$ associated to a semisimple flat homogeneous space $L/L_0$ where $(L, L_0)$ belongs to Table 1. If $G$ acts as an automorphism group of $Q$, then there exists a $G$-invariant torsion-free affine connection $\Gamma$ belonging to $Q$.

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Our result generalizes the one in [4], where only the projective structures are taken into consideration. As far as we know, it is still unknown, which of the special semisimple flat homogeneous space considered by Ochiai have some meaningful geometric interpretation, besides the projective and conformal ones. In any case, our result greatly simplifies the study of transitive group $G$ of automorphism, since it realize $G$ as a group of affine transformation of some torsionfree connection.

1. Preliminaries.

Let $L/L_0$ be a connected homogeneous space on which a semisimple Lie group $L$ acts effectively and transitively. The homogeneous space $L/L_0$ is called semisimple flat homogeneous space if the Lie algebra $l$ of $L$ has a graded structure $l = g_{-1} \oplus g_0 \oplus g_1$, $[g_i, g_j] \subset g_{i+j}$, such that $g_0 \oplus g_1$ is the Lie algebra of $L_0$; we may therefore write $L_0 = G_0 \cdot G_1$, where $G_0$ is a closed subgroup with Lie algebra $g_0$ and $G_1$ is a connected Lie group corresponding to $g_1$; more precisely $G_0$ is a normalizer of $g_i$, i.e. $G_0 = \{ a \in L_0 : \text{Ad}(a)(g_i) = g_i \}$, $i = -1, 0, 1$ (see [3]).

Let $M$ be a manifold of dimension $n = \dim L/L_0$. We denote by $P^r(M)$ the $r$-frame bundle over $M$ with structure group $G^r(n)$; $P^1(M)$ is usually called the frame bundle and $G^1(n)$ is isomorphic to $GL(n, R)$. It turns out that $L_0$ is realizable as a subgroup of $G^2(n)$ and $G_0 = L_0 \cap G^1(n)$.

Now, let $P$ be a $G_0$-structure over $M$, i.e a $G_0$-reduction of $P^1(M)$, and let $\Gamma$ be a $G_0$-connection without torsion, i.e. a torsionfree connection $\Gamma$.
on $P^1(M)$ such that $\Gamma$ has values into the Lie algebra of $G_0$, when restricted to $P$. By a result due to Kobayashi (see [2]), there exists a one-to-one correspondence between affine connections without torsion and admissible cross-section, i.e. an application $s : P^1(M) \to P^2(M)$, satisfying $s(u \alpha) = s(u) \alpha$, $\forall \alpha \in G^1(n)$. Hence each $G_0$-connection without torsion on $P$ gives a $G_0$-subbundle $s(P)$ of $P^2(M)$. Since $G_0 \subset L_0 \subset G^2(n)$, we have a $L_0$-subbundle $Q(\Gamma)$ of $P^2(M)$ obtained by extending the structure group of $s(P)$ from $G_0$ to $L_0$.

The above consideration allows to define an equivalence relation, called $L_0$-equivalence, in the set $A(P)$, the set of $G_0$-connection without torsion of $P$, as follows: if $\Gamma$, $\Theta$ are element of $A(P)$, then

$$\Gamma \text{ is equivalent to } \Theta \text{ if and only if } Q(\Gamma) = Q(\Theta).$$

The $L_0$-subbundle $Q(\alpha)$ of $P^2(M)$, where $\alpha$ is any class of $A(P)$, will be called a $L_0$-structure of second order. The $L_0$-equivalence is a generalization of some well-know structures such as projective and conformal; we will briefly recall some facts about such structures:

**Projective Geometry.**

In this case $L/L_0$ is a projective space, $G_0$ is $GL(n, \mathbb{R})$ and the $L_0$-equivalence is the projective equivalence: two torsionfree connection $\omega$, $\omega' \in \Lambda^1(P)$ are equivalent if and only if there exists $p : P^1(M) \to \mathbb{R}^n$ such that:

$$\omega - \omega' = \theta p + (p \theta) \text{id},$$

where $\theta$ is a canonical form of $P^1(M)$. Geometrically, the condition above, means that $\omega$ and $\omega'$ have the same geodesic up to reparametrization. The projective geometry studies the invariant properties of a projective equivalence class (see [2] for detailed exposition).

**Conformal Geometry.**

In this case $L/L_0$ is the n-dimensional Möbius space, $G_0$ is $CO(n)$ and the $L_0$-equivalence relation is the conformal equivalence: $\omega$ is equivalent to $\omega'$ if and only if there exists $p : Q \to \mathbb{R}^n$, where $Q$ is a $CO(n)$-structure, such that

$$\omega - \omega' = p \theta - p^t \theta^t - (\theta p) \text{id}.$$

It can be proved that, given a $CO(n)$-structure on $M$, then its first pro-
longation is a conformal structure as a subbundle of $P^2(M)$ and vice versa. On the other hand, $CO(n)$-structure on $M$ are in a natural one to one correspondence with conformal equivalence class of Riemannian metrics on $M$. We refer to [2] for a detailed exposition.

We now fix a $L_0$-structure of second order $Q$, relative to a $G_0$-structure $P$ and a $G_0$ torsionfree connection $\Gamma$ and we consider the group $Aut(M, Q)$ of the automorphism of such structure; a diffeomorphism $\phi$ belongs to $Aut(M, Q)$ if and only if its natural lift $\phi^{(2)}$ to $P^2(M)$ preserves $Q$. It is clear that any $\phi$ in $Aut(M, Q)$ acts as a permutation on $A(P)$. For some special class of semisimple flat homogeneous space, Ochiai proved the existence and uniqueness of a special normal Cartan connection $\omega_Q$ on $Q$, which is automatically preserved by any automorphism in $Aut(M, Q)$. This theorem is a generalization of existence and uniqueness of normal Cartan connection of projective and conformal structures; he showed that the existence of a normal Cartan connection is related to the vanishing of certain Spencer cohomology group; we will give all such Lie algebras $l$, in Table 1.

We recall that a Cartan connection in a bundle $Q$ is a 1-form $\omega$ on $Q$ with values in the Lie algebra $l$ of $L$ such that:

1. $\omega(A^*) = A$, for all $A \in g_0 \oplus g_1$;
2. $R_\alpha^* \omega = Ad(\alpha^{-1}) \omega$, for all $\alpha \in L_0$;
3. $\omega(X) = 0$ if and only if $X = 0$.

Here we denote by $A^*$ the fundamental vector field corresponding on element $A$ of the Lie algebra $g_0 \oplus g_1$. By definition of Cartan connection it follows in particular that $\omega$ gives an absolute parallelism on $Q$; by a classical Theorem of Kobayashi the mapping

$$\Psi_u : G \rightarrow Q$$

$$g \rightarrow g^{(2)}(u),$$

where $u$ is any fixed element of $Q$, gives a closed embedding of $G$ into $Q$.

The aim of this note is to give a simple criterium on the Lie group structure of $G = Aut(M, Q)$ in order that there exists an affine connection $\Gamma$ in $A(P)$ left fixed by $G$ or, is equivalent, a fixed point by the action of $G$ on $A(P)$. The existence of such fixed connection greatly simplifies the study of the $G$-action on $M$.

We consider the case when $M = G/K$, where $G$ is a Lie group and $K$ is a Lie closed subgroup of $G$. We will assume that $G \subset Aut(M, Q)$, where $Q$...
is a second order structure on $M = G/K$ and $M = G/K$ is a reductive, that is, if $g$ and $k$ are the Lie algebras of $G$ and $K$ respectively, there is $Ad(K)$-invariant subspace $m$ of $g$ so that

$$g = k \oplus m.$$  

In the next section we will give a proof of the main theorem.

2. Proof of the Theorem 1.

Let $\xi$ be a projection of $G$ over $G/K = M$, $\pi$ the projection of $Q$ over $G/K = M$, $\omega$ a normal Cartan connection on $Q$ and, since $\omega$ is $1$-valued form, we put $\omega = \omega_{-1} \oplus \omega_0 \oplus \omega_1$. For every $u \in Q$, we get an embedding of $G$ into $Q$ given by

$$\Psi_u : G \rightarrow Q,$$  

$$g \rightarrow g^{(2)}(u),$$

where $g^{(2)}$ is a lift of the trasformation $g$ of $G/K = M$ to an automorphism of the bundle $Q$. We use that embedding to define, for every $u \in Q$, a subspace $H_u$ of $T_u Q$ as follows:

$$H_u = (\Psi_u)_* [(e)(Ad(g)(m))],$$

where $g$ is any element of $G$ with $\xi(g) = \pi(u)$; the definition is well-posed because the subspace $Ad(g)(m)$ does not depend on the choice of element $g$ thanks to the reductivity. We can easily prove, see [4], that $H_u$ is isomorphic, by the differential of $\pi$, to $T_{\pi(u)} M$ and the distribution $H$ is $G$-invariant and $L_0$-invariant. Our aim is to show that

$$\gamma_0 = \{ u \in Q : \text{Tr} (\omega_0 |_{u}(X)) = 0, \forall X \in H_u \},$$

is a principal $G_0$-subbundle isomorphic to $P$; if we prove this, then the restriction of $\Theta_0$ to $\gamma_0$, where $\Theta_0$ is the $\mathfrak{gl}(n; \mathbb{R})$-component of the canonical form $\Theta$ of $P^2(M)$, gives a torsionfree $G_0$-connection, which is $G$-invariant.

We fix a point $u \in Q$ then there exist unique vector $X_1, \ldots, X_n$ of $H_u$ that is $\omega_{-1}(X_i) = e_i$, where $e_1, \ldots, e_n$ is a base of $g_{-1}$; that is because $\pi_* |H_u$ is an isomorphism onto $T_{\pi(u)} M$. We can identify the Lie algebra $g_0$ as a Lie subalgebra of $\mathfrak{gl}(g_{-1})$, see [3], as follows:

$$X \rightarrow \text{ad}(X) |_{g_{-1}}.$$
Now we consider the bilinear application $B : (X, Y) \to \text{Tr}([X, Y])$; this is a duality between $g_{-1}$ and $g_1$ because $-2B(X, Y) = B(X, Y)$, where $B$ is the Killing-Cartan form of $l$ (see [3]). We can construct, thanks to the duality, an element $y \in g_1$ in the following way:

$$\phi(a^i e_i) = a^i \text{Tr}(\omega_0 |_{u(X_i)}) = B(y, a^i e_i),$$

and we claim that if we put $w = \exp(y) \in L_0$, then

$$\text{Tr}(\omega_0 |_{uw}(X)) = 0, \quad \forall X \in H_{uw}.$$

Indeed we have $H_{uw} = (R_w) \ast H_u$ and

$$\omega_0 |_{uw}((R_w) \ast X)) = \omega_0 |_{u(X)} - [y, \omega_{-1} |_{u(X)}],$$

because $\omega$ is a Cartan connection and $l$ is a graded Lie algebra. On the other hand $\text{Tr}([y, \omega_{-1} |_{u(X)}]) = B(y, \omega_{-1} |_{u(X)})$ from the definition of $y$.

Now let $w_1$ and $w_2$ in $L_0$ such that $\text{Tr}(\omega_0 |_{uw_i} = 0)$, $\forall X \in H_{uw_i}$, $i = 1, 2$, and note that $uw_2 = uw_1 w_1^{-1} w_2$; we can write $w_1^{-1} w_2 = j_0 \exp(x)$, for some $x \in g_1$ and $j_0 \in G_0$ so for every $X \in H_{uw_1}$ we have

$$\omega_0 |_{uw_2}((R_{w_1^{-1} w_2}) \ast X) = (\text{Ad}(\exp(-x)) j_0^{-1}(\omega(X)))_{g_0}, \quad \forall X \in H_{uw_1},$$

where $(\_)$ means the $g_0$-component. Hence

$$0 = \text{Tr}(g_0^{-1} \omega_0 |_{uw_1}(X) g_0) - \text{Tr}([x, \text{Ad}(j_0^{-1}(\omega_{-1} |_{uw_1}(X))]),$$

by the action of $\text{Ad}$ into the Lie algebra $g_0$. Now we recall that the $g_1$ values 1-form $\omega_1 |_{uw_1} : H_{uw_1} \to g_{-1}$ is surjective, $G_0$ normalizes $g_i$, $i = -1, 0, 1$ in $L_0$ and $\text{Tr}(\omega_0 |_{uw_1}(X)) = 0$. Hence $x = e$ and $w_1^{-1} w_2 \in G_0$; more precisely we have shown that $\forall u \in Q$ there exist a unique element $\eta \in G_1$ such that $\text{Tr}(\omega_{\eta}(X)) = 0$, $\forall X \in H_{\eta}$. So we can define a differential map

$$\lambda : Q \to L_0/G_0,$$

where for every $u \in Q$ we define $\lambda(u)$ to be the class $[h]$ in $L_0/G_0$ of any element $h \in L_0$ with

$$\text{Tr}(\omega_0 |_{uh}(X)) = 0, \quad \forall X \in H_{uh}.$$

It is easy to check that $\lambda$ is a $L_0$-equivariant map, i.e. $\lambda(ug) = g^{-1} \lambda(u)$, $\forall u \in Q$ and $\forall g \in L_0$. We note that $\gamma_0 = \lambda^{-1}([e])$; we obtain that $\gamma_0$ is a $G_0$-subbundle by the following general result.
LEMMA 2.1. Let \((Q(M, L_0), M, \pi)\) be a principal fibre bundle over \(M\) with structure group \(L_0\). Suppose now \(G_0\) to be a closed subgroup of \(L_0\) and suppose there is a differential map \(\lambda\)

\[ \lambda : Q(M, L_0) \to L_0/G_0 \]

that is \(L_0\)-equivariant, i.e. \(\lambda(ug) = g^{-1}\lambda(u), \forall u \in Q(M, L_0)\) and \(\forall g \in L_0\). Then \(\gamma_0 = \lambda^{-1}(id)\) with projection \(\pi|_{\gamma_0}\) is a \(G_0\)-principal bundle.

The proof can be visited in [4].

Now let \(\Gamma\) be a \(G_0\) torsionfree connection on \(P\) belonging to an equivalence class \(\alpha\) that generates \(Q\), and let \(s\) be an admissible cross-section corresponding to \(\Gamma\). We can define the map

\[ F : P \to \gamma_0, \]

\[ u \mapsto s(u) \phi(s(u)), \]

where \(\phi(s(u))\) is the unique element of \(G_1\) for which

\[ \text{Tr}(\omega_0 |_{s(u)\phi(s(u))}(X)) = 0, \quad \forall X \in H_{s(u)\phi(s(u))}. \]

Now, it is easy to check that \(F\) is a fibre bundle isomorphism; indeed we only need to prove that \(F\) is injective. If \(u_1, u_2 \in P\) are such that \(F(u_1) = F(u_2)\) then we can find \(g_0 \in G_0\) with \(u_2 = u_1 g_0\) and

\[ s(u_1) \phi(s(u_1)) = s(u_2) \phi(s(u_2)) = s(u_1) g_0 \phi(s(u_2)). \]

Hence \(\phi(s(u_1)) = g_0 \phi(s(u_2))\); since \(\phi(s(u_i)) \in G_1\) for \(i = 1, 2\) and \(G_0 \cap G_1 = \{ e \}\), then \(g_0 = e\) and \(u_1 = u_2\). Q.E.D.

We will briefly recall now the notations used in Table 1.

Let \(K\) denote the field of real number \(R\), or the complex field \(C\) or quaternions field \(Q\); in a natural way, \(R \subset C \subset Q\). For each element \(x \in K\) we define the element \(\bar{x}\) and \(\tilde{x}\) as follows:

If \(x = x_0 + x_1 i + x_2 j + x_3 k\), with \(x_i \in R\) then

\[ \bar{x} = x_0 - x_1 i - x_2 j - x_3 k, \quad \tilde{x} = x_0 + x_1 i - x_2 j - x_3 k. \]

We use the following notation:

1. \(gl(n; K) = \{\text{all matrices of order } n \text{ over the field } K\}\);
2. \(sl(n; K)\) = the semisimple part of \(gl(n; K)\);
3. \(so(p, q; K) = \{A \in gl(n; K) : \bar{A}'I_{p, q} + I_{p, q}A = 0\}\), where the
matrix $I_{p, q}$ is:

$$\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

$so(n; K) = so(n, 0; K)$;

(4) $u(p, q; K) = \{A \in gl(p + q; K) : \tilde{A}'I_{p, q} + I_{p, q}A = 0\}$,

$u(n; K) = u(n, 0, K)$;

(5) $su(p, q; K) = u(p, q; K) \cap sl(p, q; K)$;

(6) $sp(n; K) = \{A \in gl(n; K) : \tilde{A}'J + JA = 0\}$.

We remember also that the Lie algebra $\mathfrak{l} = g_{-1} \oplus g_0 \oplus g_1$.

REFERENCES


Manoscritto pervenuto in redazione il 15 gennaio 1999.