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Fitting Height of A-Nilpotent Groups.

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ABSTRACT - Let A be a finite supersolvable group acting coprimely on a finite solvable group G in such a manner that the fixed point subgroup $C_G(A)$ normalizes every A-invariant Sylow subgroup of G. Let $h(G)$ and $k(A)$ denote the Fitting height of G and the composition length of A respectively. It is shown that under certain assumtions on A the inequality $h(G) \leq k(A) + 1$ holds.

Let A be a finite group acting coprimely on a finite group G. Following [1] we say that G is A-nilpotent if the fixed point subgroup $C_G(A)$ normalizes every A-invariant Sylow subgroup of G. This generalizes the notion of the fixed-point-free action, i. e. the action with $C_G(A) = 1$. The results obtained in [1], [2], [3] show that the relation with the «fixed-point-free» case extends far beyond formal definitions.

Assume that G and A are solvable and let $h(G)$ and $k(A)$ denote the Fitting height of G and the composition length of A respectively. (Thus, $k(A)$ is the number of primes dividing $|A|$ counting multiplicities.) There is a conjecture that if $C_G(A) = 1$ then $h(G) \leq k(A)$. This has been confirmed in many cases (see [5]). In particular, A. Turull showed that the conjecture is true if A is supersolvable and no section of A is isomorphic to $\mathbb{Z}_r \wr \mathbb{Z}_s$ or to $GN(p^{q^e})$, where p is a prime dividing $|G|$ (see [5] for the necessary definitions).

The question on the Fitting height of an A-nilpotent group was considered by E. Jabara [2]. He proved that if A is cyclic of prime-power or-

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der then, under some additional assumptions on G, the inequality $h(G) \leqslant k(A) + 1$ holds. The goal of the present paper is to show that the technique developed by A. Turull can be quite effective in the treatment of length problems for A-nilpotent groups. In fact most of the argument contained in this paper originates from Turull's work [4].

THEOREM. Let G be a solvable A-nilpotent group, where A is supersolvable and no section of A is isomorphic to $\mathbb{Z}_r \wr \mathbb{Z}_s$ or to $GN(p^{q^e})$ for any prime p dividing $|G|$. Then $h(G) \leqslant k(A) + 1$.

In fact the hypothesis that G is solvable is superfluous in the above theorem as it is shown in [1] that any A-nilpotent group is solvable. Our first lemma is quite obvious and so we omit the proof.

LEMMA 1. Let A act coprimely on a finite group G and assume that G is A-nilpotent. Let N be an A-invariant normal subgroup of G and H an A-invariant subgroup of G. Then A induces an action on G/N and H under which both groups G/N and H are A-nilpotent.

LEMMA 2. Let G be a finite solvable group with $h = h(G) \geqslant 2$. Suppose that A acts coprimely on G and G is A-nilpotent. Let M be a minimal A-invariant normal subgroup of G and assume that $h(G/C_G(M)) = h - 1$. Then $C_M(A) = 1$.

PROOF. Since G is solvable, M is an elementary abelian p-group for some prime p. Set $C = C_G(M)$ and $F/C = F(G/C)$. Then F/C is a nilpotent p'-group. Let S be an A-invariant Sylow q-subgroup of F for some prime $q \neq p$. By the hypothesis $C_M(A)$ normalizes S and therefore $[S, C_M(A)] \leqslant M \cap S = 1$. Since M is minimal and S does not centralize M, it follows that $C_M(A) = 1$.

We now require the notion of A-support of G as introduced by A. Turull.

DEFINITION. Let a group A act on a finite solvable group G. A subgroup $P \leqslant G$ is called a generating A-support subgroup of G if:

1) P is normal in AG and P is p-group for some prime p.

2) There are AG-invariant subgroups P_1 and H such that

A) $P_1 \leqslant Z(P)$, P/P_1 is elementary abelian and AG-completely reducible,

B) $H \leqslant C_G(P_1)$,
C) $H/H \cap C_G(P/P_1)$ is elementary abelian for some prime r,
D) H acts non-trivially on each H-chief factor of P/P_1.

Then the A-support of G (denoted by $\mathrm{supp}_A(G)$) is the subgroup generated by all normal in AG subgroups $S \leqslant G$ such that S is either abelian or a generating A-support.

LEMMA 3 ([4, 4.3]). Let G be a finite solvable group and A act on G. Then

1. $\cap C_G(X) \leqslant F(G)$, where X runs through the AG-chief factors of $\mathrm{supp}_A(G)$. In particular $C_G(\mathrm{supp}_A(G)) \leqslant F(G)$.

2. If $N \leqslant G$ and N is normal in AG then $\mathrm{supp}_A(G)N/N \leqslant \mathrm{supp}_A(G/N)$.

3. If $B \leqslant A$ and $(|A|, |G|) = 1$ then $\mathrm{supp}_B(G) \geqslant \mathrm{supp}_A(G)$.

4. $C_A(\mathrm{supp}_A(G)) \leqslant C_A(G/F(G))$.

The next lemma is immediate from Theorem 4.6 of [4].

LEMMA 4. Let AG be a solvable finite group with G normal in AG and A supersolvable without sections isomorphic to $\mathbb{Z}_r \wr \mathbb{Z}_s$ or to $GN(p^{q^e})$ for any p dividing $|G|$. Let k be a field of characteristic not dividing $|A|$ and M an irreducible kAG-module. Assume

1. M is faithful for G;

2. $B_1 > B_2$ are normal subgroups of A with $|B_1/B_2|$ a prime;

3. $C_M(B_1) = 0$ and $C_M(B_2) \neq 0$.

If $S = \mathrm{supp}_A(G)$, we have $C_S(B_1) = C_S(B_2)$ and $C_{G/F(G)}(B_1) = C_{G/F(G)}(B_2)$.

THEOREM. Let G be a solvable A-nilpotent group, where A is supersolvable and no section of A is isomorphic to $\mathbb{Z}_r \wr \mathbb{Z}_s$ or to $GN(p^{q^e})$ for any prime p dividing $|G|$. Then $h(G) \leqslant k(A) + 1$.

PROOF. Let $1 = A_0 < A_1 < \ldots < A_n = A$ be a chief series of A. Set $h = h(G)$. By Lemma 3, we can choose an AG-chief factor F_1 of $\mathrm{supp}_A(G)$ such that $h(G/C_G(F_1)) = h - 1$. If $h - 1 > 0$ we can choose an AG-chief factor F_2 of $\mathrm{supp}_A(G/C_G(F_1))$ such that $h(G/C_G(F_2)) = h - 2$. Continuing this process we obtain AG-chief factors $F_1, F_2, \ldots, F_h$ of G such that for any $1 \leqslant i < j \leqslant h$ either F_j is a factor of $\mathrm{supp}_A(G/C_G(F_i))$ or F_j is a factor of $(G/C_G(F_i))/F(G/C_G(F_i))$. Lemma 2 shows that if $i \leq h-1$ then $C_{F_i}(A) = 1$. We therefore can define the map

$$f : \{1, \ldots, h-1\} \to \{1, \ldots, n\},$$

where $f(i)$ is the smallest k such that $C_{F_i}(A_k) = 1$.

If $k = f(i) = f(j)$ for some $j > i$ then, by Lemma 4 applied with A_{k-1} and A_k in place of B_2 and B_1 respectively, we have $C_{F_j}(A_k) = C_{F_j}(A_{k-1})$. But then $f(j) \leq k-1$, a contradiction. Therefore f is one-to-one and $h \leq n+1$.

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