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Stabilizing influence of a Skew-symmetric operator in semilinear parabolic equation

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Stabilizing Influence of a Skew-Symmetric Operator in Semilinear Parabolic Equation (*).

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ABSTRACT - Sufficient conditions for asymptotic stability of the zero solution of a nonlinear parabolic differential equation in a Hilbert space are formulated by means of spectral properties of a certain linear operator L. The operator L need not be dissipative and its spectrum may have a continuous part touching the imaginary axis. Stability is a consequence of an appropriate influence of a skew-symmetric part of the operator L.

1. Introduction.

This paper deals with stability of the zero solution of the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Lu + N(t, u) \tag{1}$$

where $L = A + B$, A is a nonpositive selfadjoint operator in a real Hilbert space H which does not have zero as its eigenvalue, B is a linear operator «of a lower order» than A and $N(t, .)$ is a nonlinear operator in H. Many works have studied the same problem on a more or less abstract level under various conditions on the operators A, B and N.

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The situation when the operator L is essentially dissipative is studied in detail in the work of G. P. Galdi - M. Padula (1990). Conditions leading to a similar situation are also used in the papers of K. Masuda (1975), P. Maremonti (1984), G. P. Galdi - S. Rionero (1985), W. Borchers - T. Miyakawa (1992) and H. Kozono - M. Yamazaki (1995). All these conditions involve the requirement that operator B is in some sense «sufficiently small» in comparison with A.

If the symmetric part L_s of operator L has some eigenvalues on the positive side of the real axis then operator L is non-dissipative. The zero solution of equation (1) can be stable even in this case if the skew-symmetric ($\equiv$ antisymmetric) part of L has an appropriate influence on the behaviour of solutions of equation (1). This influence is involved in the widely used assumption that $\operatorname{Re} \lambda \leqslant -\delta$ for some $\delta > 0$ and all $\lambda \in \sigma(L)$, where $\sigma(L)$ denotes the spectrum of L (see e.g. G. Prodi (1962), D. H. Sattinger (1970) and H. Kielhöfer (1976)). However, this assumption cannot be satisfied if the spectrum of L has an essential part which has a nonempty intersection with the imaginary axis. Such a case is typical for problems in exterior domains. Sufficient conditions for the stability of the zero solution of equation (1) which can be fulfilled if L is not dissipative and the spectrum of L touches the imaginary axis are derived in J. Neustupa (1994). However, these conditions are not formulated as conditions on the spectrum of L only. Operator L is supposed to have the form $-A + B_1 + B_2$ where A is a nonnegative selfadjoint operator which does not have zero as its eigenvalue, B_1, B_2 are certain operators «of the lower order» and the conditions used in J. Neustupa (1994) also involve certain boundedness of B_1 and B_2 with respect to A.

The present paper deals with the case when operator L is not dissipative and its spectrum has an essential part which has a non-empty intersection with the imaginary axis, but sufficient conditions for stability are expressed mainly by an assumption abount $R_\lambda(L)$ (the resolvent operator of L). This assumption (see condition (iv) in Section 2) does not require «sufficient smallness» of operator B relative to A and it can be regarded as a generalization of the condition «$\operatorname{Re} \lambda \leqslant -\delta$ for all $\lambda \in \sigma(L)$».

This paper has the following structure: Section 2 contains basic assumptions and auxiliary lemmas. The main result on stability at an abstract level is proved in Section 3. A simple example in one space dimension is given in Section 4. The results from Sections 2 and 3 can also be applied to a general parabolic system in three space dimensions and to the Navier-Stokes equations in an exterior domain. Follow-up papers are being prepared on these themes.

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2. Basic assumptions and auxiliary lemmas.

Let H be a real Hilbert space with a scalar product $(.,.)_0$ and an associated norm $\|.\|_0$. Suppose that

$$L = A + B_s + B_a$$

where A is a selfadjoint operator in H which is nonpositive (i.e. its spectrum is a subset of the interval $(-\infty, 0]$) and it does not have 0 as its eigenvalue. B_s and B_a are linear operators in H such that their domains $D(B_s)$ and $D(B_a)$ contain $D(A)$, B_s is symmetric and B_a is skew-symmetric. $N(t,.)$ is for each $t \in [0, +\infty)$ a nonlinear operator in H with the domain $D(N)$ which does not depend on t and $D(A) \subset D(N)$.

Throughout this paper, $\sigma(L)$ will denote the spectrum of L, $\varrho(L)$ will denote the resolvent set of L and $R_\lambda(L)$ will be the resolvent of L (i.e. $R_\lambda(L) = (L - \lambda I)^{-1}$). We put

$$\|\phi\|_1 = \|(-A)^{1/2}\phi\|_0 \quad \text{for } \phi \in D((-A)^{1/2}), \qquad \|\phi\|_2 = \|A\phi\|_0 \quad \text{for } \phi \in D(A).$$

H_1 will be the completion of $D((-A)^{1/2})$ in the norm $\|.\|_1$ and H_2 will be the completion of $D(A)$ in the norm $\|.\|_2$. We shall use the following assumptions for operators B_s and B_a:

(i) $\exists c_1 > 0 : \|B_s\phi\|_0 \leq c_1 \|\phi\|_1$ for $\phi \in H_1$,

(ii) $\exists \alpha \in [1/2, 1)$ $\exists c_2, c_3 \geq 0 : \|B_a\phi\|_0 \leq c_2 \|(-A)^\alpha \phi\|_0 + c_3 \|\phi\|_1$ for $\phi \in D((-A)^\alpha)$.

It can be verified (for example by means of the resolution of identity for the operator $(-A)$ and the Hölder inequality) that $\|(-A)^\alpha\phi\|_0 \leq \|\phi\|_2^{2\alpha - 1}\|\phi\|_1^{2-2\alpha}$. Hence it follows from condition (ii) that if $\mu > 0$ is given and $k(\mu) = (2c_2^2)^{1/(2-2\alpha)}\mu^{(1-2\alpha)/(2-2\alpha)} + 2c_3^2$ then

$$\|B_a\phi\|_0^2 \leq \mu\|\phi\|_2^2 + k(\mu)\|\phi\|_1^2 \tag{2}$$

for all $\phi \in D(A)$.

It can be derived from condition (i) that operator B_s is A-bounded with an A-bound arbitrarily small. Hence operator $A+B_s$ is selfadjoint (see T. Kato (1996), p. 287). Let us denote its resolution of identity by $E(\lambda)$. Put

$$P' = \int_0^{+\infty} \mathrm{d}E(\lambda), \qquad P'' = I - P', \qquad H' = P'H, \qquad H'' = P''H.$$

P', P'' are orthogonal projections in H and H', H'' are closed orthogonal subspaces of H such that $H = H' \oplus H''$. Both projections P' and P'' commute with $A+B_s$ on $D(A+B_s) \equiv D(A)$ and so $P'D(A) \subset D(A)$ and $P''D(A) \subset D(A)$.

LEMMA 1. *If condition* (i) *is satisfied then there exist positive constants* c_4, c_5, c_6 *and* c_7 *so that*

$$\|P'\phi\|_1^2 + \|P''\phi\|_1^2 \leqslant c_4\|\phi\|_1^2 + c_5\|\phi\|_0^2 \tag{3}$$

$$\|P'\phi\|_2^2 + \|P''\phi\|_2^2 \leqslant c_6\|\phi\|_2^2 + c_7\|\phi\|_0^2 \tag{4}$$

for $\phi \in D(A)$.

PROOF.

$$\begin{aligned}
&\|P'\phi\|_1^2 + \|P''\phi\|_1^2 = \\
&\quad = (-AP'\phi, P'\phi)_0 + (-AP''\phi, P''\phi)_0 = ((-A-B_s)P'\phi, P'\phi)_0 + \\
&\quad + ((-A-B_s)P''\phi, P''\phi)_0 + (B_sP'\phi, P'\phi)_0 + (B_sP''\phi, P''\phi)_0 \leqslant \\
&\quad \leqslant ((-A-B_s)\phi, \phi)_0 + \|B_sP'\phi\|_0\|P'\phi\|_0 + \|B_sP''\phi\|_0\|P''\phi\|_0 \leqslant \\
&\quad \leqslant (-A\phi, \phi)_0 + [\|B_s\phi\|_0 + \|B_sP'\phi\|_0 + \|B_sP''\phi\|_0]\cdot\|\phi\|_0 \leqslant \\
&\quad \leqslant \|\phi\|_1^2 + c_1[\|\phi\|_1 + \|P'\phi\|_1 + \|P''\phi\|_1]\cdot\|\phi\|_0 \leqslant \\
&\quad \leqslant \|\phi\|_1^2 + \varepsilon c_1[\|\phi\|_1 + \|P'\phi\|_1 + \|P''\phi\|_1]^2 + \frac{c_1}{4\varepsilon}\|\phi\|_0^2 \leqslant \\
&\qquad\qquad \leqslant (1+3\varepsilon c_1)\|\phi\|_1^2 + 3\varepsilon c_1\|P'\phi\|_1^2 + 3\varepsilon c_1\|P''\phi\|_1^2 + \frac{c_1}{4\varepsilon}\|\phi\|_0^2,
\end{aligned}$$

$$(1-3\varepsilon c_1)[\|P'\phi\|_1^2 + \|P''\phi\|_1^2] \leqslant (1+3\varepsilon c_1)\|\phi\|_1^2 + \frac{c_1}{4\varepsilon}\|\phi\|_0^2.$$

If ε is chosen for example so that $1-3\varepsilon c_1 = 1/2$ then we obtain the estimate (3). The inequality (4) can be derived in a similar way.

Then next condition we shall need is:

(iii) $\exists c_8 \in (0, 1): ((A + B_s)\,\phi, \phi)_0 \leqslant -c_8 \|\phi\|_1^2$ for $\phi \in H'' \cap D(A)$.

The following lemma shows the case when (iii) is fulfilled.

LEMMA 2. *Let there exist $\varepsilon > 0$ such that $\sigma(A + B_s + \varepsilon P'' B_s)\,|_{H''})$ (i.e. the spectrum of the operator $A + B_s + \varepsilon P'' B_s$ reduced to H'') is a subset of the interval $(-\infty, 0]$. Then condition* (iii) *is satisfied.*

PROOF. It follows from the assumption of the lemma that $((A + B_s + \varepsilon P'' B_s)\,\phi, \phi)_0 \leqslant 0$ for all $\phi \in H''$. This can be rewritten as

$$((A + B_s)\,\phi, \phi)_0 \leqslant \frac{\varepsilon}{1+\varepsilon}(A\phi, \phi)_0.$$

Since $(A\phi, \phi)_0 = -\|\phi\|_1^2$, the above inequality confirms the validity of (iii).

The projection P'' is identical with $I - P'$. Since space H' can be finite-dimensional in many practical cases, P' can easily be expressed and there exists a good possibility to verify the assumptions of Lemma 2. We shall show this verification in a concrete example in Section 4 and we formulate other conditions implying the validity of the assumptions of Lemma 2 in another concrete situation in Section 5.

If $z \in \mathbb{C}$, $z \neq 0$ then $\arg z$ will denote the number $\varphi \in (-\pi, \pi]$ such that $z = |z| e^{\varphi i}$.

We shall denote by H_C a so called complexification of H. It is the space of all elements of the type $\phi_r + i\phi_i$, where $\phi_r, \phi_i \in H$. The scalar product of two elements $\phi \equiv \phi_r + i\phi_i$, $\psi \equiv \psi_r + i\psi_i$ in H_C is defined by

$$(\phi, \psi)_0 = [(\phi_r, \psi_r)_0 + (\phi_i, \psi_i)_0] + i[(\phi_i, \psi_r)_0 - (\phi_r, \psi_i)_0].$$

Operators A, B_s and B_a can be extended in the usual way to the operators in H_C so that, for example, if $\phi \equiv \phi_r + i\phi_i \in H_C$ and $\phi_r, \phi_i \in D(A)$ then $A\phi = A\phi_r + iA\phi_i$.

Since operator $(-A)$ is selfadjoint and nonnegative in H, it is sectorial in H. Using conditions (i), (ii) and applying Theorem 1.3.2 from D. Henry (1981), p. 19, we can derive that operator $(-L)$ is sectorial, too. Thus, there exists $\varphi \in (\pi/2, \pi)$, $a \in \mathbb{R}$ (we can assume that $a \geqslant 0$ without loss of generality) and $c_9 > 0$ so that

$$S \equiv \{\lambda \in \mathbb{C};\ \lambda \neq a \text{ and } |\arg(\lambda - a)| < \varphi\} \subset \varrho(L)$$

and

$$\|R_\lambda(L)\,\phi\|_0 \leqslant \frac{c_9}{|\lambda - a|}\,\|\phi\|_0 \tag{5}$$

for all $\lambda \in S$ and $\phi \in H$. Moreover, $R_\lambda(L)\,\phi$ is an analytic H_C-valued function of λ in $\varrho(L)$ for every $\phi \in H$. It follows from D. Henry (1981), p. 20, that operator L is a generator of an analytic semigroup e^{Lt} in H and there exists $c_{10} > 0$ so that

$$\|\mathrm{e}^{Lt}\,\phi\|_0 \leqslant c_{10}\,\mathrm{e}^{at}\,\|\phi\|_0 \tag{6}$$

for all $t \geqslant 0$ and $\phi \in H$.

Denote $\mathbb{C}_+(-\delta) = \{z \in \mathbb{C};\ \mathrm{Re}\, z > -\delta\}$. The next condition we shall use is:

(iv) $\exists \delta > 0$ *so that if* $\phi \in H'$ *then* $P'R_\lambda(L)\,\phi$ *can be extended* (*in dependence on* λ) *from* $\varrho(L) \cap \mathbb{C}_+(-\delta)$ *to an* H_C*-valued analytic function in* $\mathbb{C}_+(-\delta)$.

The validity of this condition will be verified in an example in Section 4.

REMARK 1. It is obvious that condition (iv) is satisfied if there exists $\delta > 0$ so that for each $\phi \in H'$ the equation

$$(L - \lambda I)\,y_\lambda = \phi \tag{7}$$

has a solution $y_\lambda(\phi)$ such that $P'y_\lambda(\phi)$ can be extended from $\varrho(L) \cap \cap\, \mathbb{C}_+(-\delta)$ to an H_C-valued analytic function of λ in $\mathbb{C}_+(-\delta)$.

We shall denote by δ_1 the number $\delta/2$ in the rest of this paper.

LEMMA 3. *If conditions* (i), (ii) *and* (iv) *are satisfied then there exists* $c_{11} > 0$ *so that* $\|P'\,\mathrm{e}^{Lt}\phi\|_0 \leqslant c_{11}\,\mathrm{e}^{-\delta_1 t}\|\phi\|_0$ *for all* $\phi \in H'$ *and* $t \geqslant 0$.

PROOF. Assume first that $t \geqslant 1$. Put $\eta = -\delta_1 + i\xi$, where ξ is a positive real number which is so large that $\eta \in S$. Denote $\psi = \arg \eta$. It is clear that $\psi \in (\pi/2, \pi)$. Let us define the curves Γ_1, Γ_2, Γ_3 and Γ_4 in $\mathbb{C}$ by means of their parametrizations — in order not to complicate the notation we denote the parametrizations by the same letters as the curves:

$$\Gamma_1(s) = -s\,\mathrm{e}^{-\psi i};\ s \in (-\infty, -|\eta|], \qquad \Gamma_2(s) = -\delta_1 + is;\ s \in [-\xi, \xi],$$
$$\Gamma_3(s) = s\,\mathrm{e}^{\psi i};\ s \in [|\eta|, +\infty), \qquad \Gamma_4(s) = a + |\eta - a|\,\mathrm{e}^{si};\ s \in [-\psi, \psi].$$

Curves Γ_1, Γ_4 and Γ_3 are subsets of S, *the end point of* Γ_1 = *the initial point of* $\Gamma_4 = \bar{\eta}$ and *the end point of* Γ_4 = *the initial point of* $\Gamma_3 = \eta$.

The semigroup e^{Lt} can be defined by the formula

$$e^{Lt} = \frac{1}{2\pi i} \int_{\Gamma_1 \cup \Gamma_4 \cup \Gamma_2} e^{\lambda t} R_\lambda(L)\, d\lambda \,.$$

Since projector P' is closed in H, we have

$$P' e^{Lt} = \frac{1}{2\pi i} \int_{\Gamma_1 \cup \Gamma_4 \cup \Gamma_2} e^{\lambda t} P' R_\lambda(L)\, d\lambda \,.$$

Suppose that $\phi \in H'$. Denote by $y'_\lambda(\phi)$ the analytic extension of $P' R_\lambda(L)\, \phi$ to the domain $\mathbb{C}_+(-\delta)$. It follows from Cauchy's theorem that the integral of $e^{\lambda t} y'_\lambda(\phi)$ on Γ_4 is equal to the integral of the same function on Γ_2. This means that

$$(8) \qquad P' e^{Lt} \phi = \frac{1}{2\pi i} \int_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3} e^{\lambda t} y'_\lambda(\phi)\, d\lambda = \frac{1}{2\pi i} \int_{\Gamma_1} e^{\lambda t} P' R_\lambda(L)\, \phi\, d\lambda + \\ + \frac{1}{2\pi i} \int_{\Gamma_2} e^{Lt} y'_\lambda(\phi)\, d\lambda + \frac{1}{2\pi i} \int_{\Gamma_3} e^{\lambda t} P' R_\lambda(L)\, \phi\, d\lambda \,.$$

Using estimate (5) and the expression of the integral on Γ_1 by means of parametrization, we get

$$\left\| \frac{1}{2\pi i} \int_{\Gamma_1} e^{\lambda t} P' R_\lambda(L)\, \phi\, d\lambda \right\|_0 =$$

$$= \frac{1}{2\pi} \left\| \int_{-\infty}^{-|\eta|} e^{-s \exp(-\psi i)\, t} \cdot P' R_{-s\exp(-\psi i)}(L)\, \phi \cdot e^{-\psi i}\, ds \right\|_0 \leqslant$$

$$\leqslant \frac{1}{2\pi} \int_{-\infty}^{-|\eta|} e^{-s(\cos\psi)\, t} \|R_{-s\exp(-\psi i)}(L)\, \phi\|_0\, ds \leqslant \frac{1}{2\pi} \int_{-\infty}^{-|\eta|} e^{-s(\cos\psi)\, t} c_{12} \|\phi\|_0\, ds =$$

$$= \frac{c_{12}}{2\pi} \frac{(-1)}{\cos\psi \cdot t} e^{|\eta|(\cos\psi)\, t} \|\phi\|_0 \leqslant \frac{c_{12}}{2\pi} \frac{(-1)}{\cos\psi} e^{|\eta|(\cos\psi)\, t} \|\phi\|_0 .$$

Since $\operatorname{Re}\eta = |\eta|(\cos\psi) = -\delta_1$, we have

$$\left\| \frac{1}{2\pi i}\int_{\Gamma_1} e^{\lambda t} P' R_\lambda(L)\,\phi\,d\lambda \right\|_0 \leqslant c_{13}\, e^{-\delta_1 t}\|\phi\|_0 \tag{9}$$

where $c_{13} = -c_{12}/(2\pi\cdot\cos\psi)$. We can also derive the same estimate for the integral over Γ_3. Further, we have

$$\left\| \frac{1}{2\pi i}\int_{\Gamma_2} e^{\lambda t} y_\lambda'(\phi)\,d\lambda \right\|_0 = \frac{1}{2\pi}\left\| \int_{-\xi}^{\xi} e^{-\delta_1 t + is} y'_{-\delta_1+is}(\phi)\, i\,ds \right\|_0 \leqslant$$

$$\leqslant \frac{1}{2\pi} e^{-\delta_1 t}\int_{-\xi}^{\xi} \|y'_{-\delta_1+is}(\phi)\|_0\,ds = \frac{1}{2\pi} e^{-\delta_1 t}\left| \int_{\Gamma_2} \|y_\lambda'(\phi)\|_0\,d\lambda \right| .$$

Let $L^2(\Gamma_2; H_C)$ be the Banach space of mappings f_λ which are defined a.e. in Γ_2 and their values belong to H_C. The norm of f_λ in $L^2(\Gamma_2; H_C)$ is $\left| \int_{\Gamma_2} \|f_\lambda\|_0^2\,d\lambda \right|^{1/2}$. The mapping $\mathcal{C} : \phi \to y_\lambda(\phi)$ can be regarded as a linear mapping of H' to $L^2(\Gamma_2; H_C)$. Let us show that this mapping is closed:

Assume that $\{\phi_n\}$ is a sequence in H' such that $\phi_n \to \phi$ in H' and $\mathcal{C}\phi_n \to \Phi$ in $L^2(\Gamma_2; H_C)$. The behaviour of the functions $\sin[k\pi(\lambda - \bar{\eta})/(2\xi i)]$ for $\lambda \in \Gamma_2$ is the same as the behaviour of the functions $\sin k\lambda$ for $\lambda \in [0, \pi]$. Hence the set M of all functions of the type

$$\Psi(\lambda) = \sum_{k=1}^{n} \psi_k \cdot \sin[k\pi(\lambda - \bar{\eta})/(2\xi i)] \tag{10}$$

(where $n \in \mathbb{N}$ and $\psi_1, \ldots, \psi_n \in H_C$) is dense in $L^2(\Gamma_2; H_C)$. This can be proved by a contradiction: If M is not dense in $L^2(\Gamma_2; H_C)$ then there exists $\varphi \in L^2(\Gamma_2; H_C)$, $\varphi \neq 0$, which is orthogonal to $\overline{M}$. In particular, this means that if $\psi \in H_C$ and Ψ has the form (10) with $\psi_k = \alpha_k \psi$ ($\alpha_k \in \mathbb{C}$, $k = 1, \ldots, n$) then

$$(\varphi, \Psi)_{L^2(\Gamma_2; H_C)} = \int_{\Gamma_2} (\varphi(\lambda), \psi)_0 \cdot \sum_{k=1}^{n} \alpha_k \sin[k\pi(\lambda - \bar{\eta})/(2\xi i)]\,d\lambda = 0 .$$

Hence $(\varphi(\lambda), \psi)_0 = 0$ for a.a. $\lambda \in \Gamma_2$. Since ψ was chosen arbitrarily in H_C, φ is equal to the zero element of $L^2(\Gamma_2; H_C)$. But this is a contradiction. Suppose now that Ψ is an arbitrary function of type (10). Then Ψ is analytic (in dependence on λ) in $\mathbb{C}$. Using Cauchy's theorem and the convergence

$$\left| \int_{\Gamma_4} \|y_\lambda'(\phi_n) - y_\lambda'(\phi)\|_0^2 \, \mathrm{d}\lambda \right| = \left| \int_{\Gamma_4} \|P' R_\lambda(L)\, \phi_n - P' R_\lambda(L)\, \phi\|_0^2 \, \mathrm{d}\lambda \right| \leq$$

$$\leq \text{const.} \left| \int_{\Gamma_4} \|\phi_n - \phi\|_0^2 \, \mathrm{d}\lambda \right| \leq \text{const.}\, \|\phi_n - \phi\|_0^2 \to 0\,,$$

we can write,

$$\int_{\Gamma_2} (y_\lambda'(\phi_n),\, \Psi(\lambda))_0 \, \mathrm{d}\lambda = \int_{\Gamma_4} (y_\lambda'(\phi_n),\, \Psi(\lambda))_0 \, \mathrm{d}\lambda \to$$

$$\to \int_{\Gamma_4} (y_\lambda'(\phi),\, \Psi(\lambda))_0 \, \mathrm{d}\lambda = \int_{\Gamma_2} (y_\lambda'(\phi),\, \Psi(\lambda))_0 \, \mathrm{d}\lambda\,.$$

So $y_\lambda'(\phi_n) \to y_\lambda'(\phi)$ weakly in $L^2(\Gamma_2;\, H_C)$. Since $\mathscr{T}\phi_n \equiv y_\lambda'(\phi_n) \to \Phi$ in $L^2(\Gamma_2;\, H_C)$, we have $\Phi = y_\lambda'(\phi)$. Thus, the operator $\mathscr{T}$ is closed.

The domain of definition of $\mathscr{T}$ is the whole space H' and hence, due to the closed graph theorem, $\mathscr{T}$ is bounded. There exists $c_{14} > 0$ (which does not depend on ϕ) so that

$$\left| \int_{\Gamma_2} \|y_\lambda'(\phi)\|_0 \, \mathrm{d}\lambda \right| \leq \sqrt{2\xi} \left| \int_{\Gamma_2} \|y_\lambda'(\phi)\|_0^2 \, \mathrm{d}\lambda \right|^{1/2} = \sqrt{2\xi}\, \|\mathscr{T}\phi\|_{L^2(\Gamma_2;\, H_C)} \leq c_{14} \|\phi\|_0.$$

So we obtain the estimate

$$\left\| \frac{1}{2\pi i} \int_{\Gamma_2} \mathrm{e}^{\lambda t} y_\lambda'(\phi) \, \mathrm{d}\lambda \right\|_0 \leq \frac{c_{14}}{2\pi} \mathrm{e}^{-\delta_1 t} \|\phi\|_0. \tag{11}$$

It follows from (8)-(11) that

$$\|P' \mathrm{e}^{Lt} \phi\|_0 \leq \left(2c_{13} + \frac{c_{14}}{2\pi} \right) \mathrm{e}^{-\delta_1 t} \|\phi\|_0.$$

We have derived this estimate for $t \geq 1$ and $\phi \in H'$. However, if we also use inequality (6) for $t \in [0,\, 1)$, we can easily obtain the desired estimate $\|P' \mathrm{e}^{Lt} \phi\|_0 \leq c_{11} \mathrm{e}^{-\delta_1 t} \|\phi\|_0$ for all $t \geq 0$ and $\phi \in H'$.

We shall use the following assumption about nonlinear operator N:

(v) $\exists \beta \in [0,\, 1]$ $\exists \gamma \geq 2 - \beta$ $\exists c_{15} > 0 : \|N(t,\, \phi)\|_0 \leq c_{15} \|\phi\|_1^\gamma [\|\phi\|_2 + \|\phi\|_1]^\beta$ *for* $t \geq 0$ *and* $\phi \in D(A)$.

Under solutions of equation (1) or another analogous equation in a time interval $[0, T)$ (where $T \in (0, +\infty]$), we understand functions u such that:

a) if J is a compact interval in $[0, T)$ then $u \in L^2(J; H_2) \cap \cap L^2(J; H)$ and $\mathrm{d}u/\mathrm{d}t \in L^2(J; H)$,

b) u satisfies a given equation a.e. in $(0, T)$.

It follows from the theory of interpolation spaces (see J. L. Lions - E. Magenes (1972)) that if a solution u has the regularity which is required in condition a) then it is (after a possible change on a subset of $[0, T)$ whose measure is zero) a continuous mapping from $[0, T)$ to H_1.

LEMMA 4. *Let $\tau > 0$ and $f \in L^2(0, \tau; H)$. Then there exists a unique solution v of the equation*

$$\frac{\mathrm{d}v}{\mathrm{d}t} = Av + f(t) \tag{12}$$

with the initial condition $v(0) = 0$ in the time interval $[0, \tau)$ and

$$\|v\|_{L^2(0, \tau; H_2)} + \left\| \frac{\mathrm{d}v}{\mathrm{d}t} \right\|_{L^2(0, \tau; H)} \leq \sqrt{2}\, \|f\|_{L^2(0, \tau; H)} + \sqrt{2}\, \|v\|_{L^2(0, \tau; H)}. \tag{13}$$

PROOF. The idea of the proof is the same as that used in the proof of Theorem IV.1 in O. A. Ladyzhenskaya (1970). Let us define

$$D(\mathcal{L}) = \left\{ u;\ u(t) = \int_0^t \varphi(s)\, \mathrm{d}s,\ \varphi \in L^2(0, \tau; H_2) \right\}, \qquad \mathcal{L}u = \frac{\mathrm{d}u}{\mathrm{d}t} - Au\,.$$

$D(\mathcal{L})$ is dense in $L^2(0, \tau; H)$. The adjoint operator $\mathcal{L}^*$ to $\mathcal{L}$ is densely defined in $L^2(0, \tau; H)$ and so $\mathcal{L}$ is closable. Let $\overline{\mathcal{L}}$ be the closure of $\mathcal{L}$.

Let $u \in D(\mathcal{L})$ and $t \in [0, \tau]$ now. Then

$$(\mathcal{L}u, \mathcal{L}u)_{L^2(0, \tau; H)} = \int_0^t \left(\frac{\mathrm{d}u}{\mathrm{d}s} - Au,\ \frac{\mathrm{d}u}{\mathrm{d}s} - Au \right)_0 \mathrm{d}s =$$

$$= \int_0^t \left[\left\| \frac{\mathrm{d}u}{\mathrm{d}s} \right\|_0^2 + \|u\|_2^2 + 2\left(\frac{\mathrm{d}u}{\mathrm{d}s}, (-A)\, u \right)_0 \right] = \int_0^t \left[\left\| \frac{\mathrm{d}u}{\mathrm{d}s} \right\|_0^2 + \|u\|_2^2 \right] \mathrm{d}s + u(t)\|_1^2.$$

It is seen from these equalities that if $u_n \in D(\mathcal{L})$, $u_n \to u$ in $L^2(0, \tau; H)$,

$\mathcal{L}u_n \to \mathcal{L}u$ in $L^2(0, \tau; H)$ then $\{du_n/dt\}$ converges in $L^2(0, \tau; H)$, $\{u_n\}$ converges in $L^2(0, \tau; H_2)$ and $\{u_n(t)\}$ converges in H_1 uniformly for $t \in [0, \tau]$. Thus, we have $du/dt \in L^2(0, \tau; H)$ and $u \in L^2(0, \tau; H_2) \cap C([0, \tau]; H_1)$ for $u \in D(\overline{\mathcal{L}})$.

Let us now show that $R(\overline{\mathcal{L}}) = L^2(0, \tau; H)$. ($R(\mathcal{L})$ is the range of $\overline{\mathcal{L}}$.) Suppose that this is not true. Then there exists $g \in L^2(0, \tau; H)$, $g \neq 0$, which is orthogonal to $R(\overline{\mathcal{L}})$ and consequently, also to $R(\mathcal{L})$. Since the element $\int_0^t A^{-1} g(s)\, ds$ belongs to $D(\mathcal{L})$, it holds:

$$0 = \left(g, \mathcal{L}\int_0^t A^{-1} g\, ds\right)_{L^2(0, \tau; H)} = \int_0^\tau \left(g(t), A^{-1} g(t) - \int_0^t g(s)\, ds\right)_0 dt =$$

$$= -\int_0^\tau \left[\|(-A)^{-1/2} g(t)\|_0^2 + \frac{1}{2}\frac{d}{dt}\left\| \int_0^t g(s)\, ds \right\|_0^2 \right] dt .$$

So $(-A)^{-1/2} g(t) = 0$ for a.a. $t \in [0, \tau]$, which means that g is the zero element in $L^2(0, \tau; H)$. This is the desired contradiction. So $R(\overline{\mathcal{L}}) = L^2(0, \tau; H)$. This implies the existence of the solution v of equation (12) with initial condition $v(0) = 0$ in the interval $[0, \tau]$.

Suppose that v_1, v_2 are two such solutions. Put $w = v_1 - v_2$. Then $dw/dt = Aw$ and $w(0) = 0$. Hence $w(t) = e^{at} w(0) = 0$. This proves the uniqueness of the solution.

Inequality (13) can be obtained if we multiply equation (12) by v in $L^2(0, \tau; H)$.

LEMMA 5. *Let conditions* (i), (ii) *be fulfilled and let u be a solution of equation* (1) *in the interval* $[0, T)$. *Then the initial-value problem given by the equation*

$$\frac{dv}{dt} = Av + B_s v + P'' B_a v + P'' N(t, u) \tag{14}$$

and the initial condition $v(0) = P'' u(0)$ *has a unique solution v in the interval* $[0, T)$ *such that* $v(t) \in H''$ *for a.a.* $t \in [0, T)$.

PROOF. Let us denote $u' = P'u$ and $u'' = P''u$. Applying projection P'' to equation (1), we obtain

$$\frac{du''}{dt} = Au'' + B_s u'' + P''B_a u'' + P''B_a u' + P''N(t, u). \tag{15}$$

So if we prove that there exists a solution v_1 of the equation

$$\frac{dv_1}{dt} = Av_1 + B_s v_1 + P''B_a v_1 - P''B_a u' \tag{16}$$

with the intial condition $v_1(0) = 0$ in the interval $[0, T)$, we can put $v = u'' + v_1$ and if we add equations (15), (16) and the intial values of functions u'' and v_1, we can see that v is the desired solution of equation (14) which satisfies the initial condition $v(0) = P''u(0)$.

Let $\tau < T$. It follows from inequalities (3), (4), condition (ii) and the fact that $u \in L^2(0, \tau; H_2) \cap L^2(0, \tau; H)$ that $P''B_a u' \in L^2(0, \tau; H'')$. Using operator $\overline{\mathcal{L}}$ from the proof of Lemma 4, we can write equation (16) with the initial condition $v_1(0) = 0$ in the equivalent form

$$v_1 = (\overline{\mathcal{L}})^{-1}(B_s v_1 + P''B_a v_1) - (\overline{\mathcal{L}})^{-1}P''B_a u'. \tag{17}$$

Applying inequality (13) and standard but rather laborious estimates, we can obtain:

$$\begin{aligned}\|(\overline{\mathcal{L}})^{-1}(B_s v_1 + P''B_a v_1)\|_{L^2(0, \tau; H_2)} + \left\| \frac{d}{dt}(\overline{\mathcal{L}})^{-1}(B_s v_1 + P''B_a v_1) \right\|_{L^2(0, \tau; H)} &\leqslant \\ \leqslant \sqrt{2}\varepsilon \|v_1\|_{L^2(0, \tau; H_2)} + K(\varepsilon)\, \tau^{1/2} \left\| \frac{dv_1}{dt} \right\|_{L^2(0, \tau; H)} &+ \\ + \sqrt{2}\tau^{1/2} \left\| \frac{d}{dt}(\overline{\mathcal{L}})^{-1}(B_s v_1 + P''B_a v_1) \right\|_{L^2(0, \tau; H)}&\end{aligned}$$

for each $\varepsilon > 0$. Let $\varepsilon > 0$ and $\tau_0 > 0$ be chosen so small that

$$\sqrt{2}\varepsilon \leqslant \frac{1}{4}, \qquad K(\varepsilon)\, \tau_0^{1/2} \leqslant \frac{1}{4}, \qquad \sqrt{2}\tau_0^{1/2} \leqslant \frac{1}{2}.$$

Assume that $\tau \leqslant \tau_0$ at first. Then we have

$$\|(\bar{\mathcal{L}})^{-1}(B_s v_1 + P'' B_a v_1)\|_{L^2(0,\tau;H_2)} + \left\| \frac{\mathrm{d}}{\mathrm{d}t}(\bar{\mathcal{L}})^{-1}(B_s v_1 + P'' B_a v_1) \right\|_{L^2(0,\tau;H)} \leqslant$$

$$\leqslant \frac{1}{2}\left[\|v_1\|_{L^2(0,\tau;H_2)} + \left\| \frac{\mathrm{d}v_1}{\mathrm{d}t} \right\|_{L^2(0,\tau;H)} \right].$$

Thus, the operator $I - (\bar{\mathcal{L}})^{-1}(B_s + P'' B_a)$ is invertible and equation (17) is uniquely solvable in $[0, \tau]$.

Now let $\tau > \tau_0$. We can assume without loss of generality that τ_0 was chosen so that $\tau = 2^k \tau_0$ for some $k \in \mathbb{N}$. We have proved the existence of solution v_1 of equation (16) with the initial condition $v_1(0) = 0$ in the time interval $[0, \tau_0]$. Put

$$v_2(t) = \begin{cases} v_1(t) & \text{for } t \in [0, \tau_0], \\ v_2(2\tau_0 - t) & \text{for } t \in [\tau_0, 2\tau_0]. \end{cases}$$

Let v_3 be the solution of the equation

$$\frac{\mathrm{d}v_3}{\mathrm{d}t} = Av_3 + B_s v_3 + P'' B_a v_3 - P'' B_a u'(t + \tau_0) +$$

$$+ P'' B_a u'(\tau_0 - t) + 2\frac{\mathrm{d}v_1}{\mathrm{d}t}(\tau_0 - t)$$

with the initial condition $v_3(0) = 0$ in the interval $[0, \tau_0]$. (Its existence can be proved in the same way as the existence of solution v_1 in $[0, \tau_0]$.) Put $v_4(t) = 0$ for $t \in [0, \tau_0]$, $v_4(t) = v_3(t - \tau_0)$ for $t \in [\tau_0, 2\tau_0]$. It can easily be verified that $v_1 = v_2 + v_4$ is the solution of equation (16) with the initial condition $v_1(0) = 0$ in the interval $[0, 2\tau_0]$. This solution can be extended in the same way to the time interval $[0, \tau]$. Since τ can be chosen arbitrarily near to T (if $T < +\infty$) or arbitrarily large (if $T = +\infty$), the solution exists on the interval $[0, T)$.

Uniqueness of the solution can be proved by the standard procedure: we suppose that we have two solutions, we subtract them and we prove that their difference is equal to the zero element of H identically in $[0, T)$. To do that, we can use the fact that the operator $A + B_s + P'' B_a$ generates an analytic semigroup in H, which can be shown in the same way as in the case of operator L. Since equation (14) and the initial condition $v(0) = P''u(0)$ represent the problem in H'', the values of solution v remain in H''.

LEMMA 6. *Let conditions* (i), (ii) *be fulfilled. Then u is a solution of equation* (1) *in the interval* $[0, T)$ *if and only if $u = v + w$ where the functions v, w are solutions of the equations*

$$\frac{dv}{dt} = Av + B_s v + P'' B_a v + P'' N(t, v + w), \tag{18}$$

$$\frac{dw}{dt} = Aw + B_s w + B_a w + P' B_a v + P' N(t, v + w) \tag{19}$$

in the interval $[0, T)$, *satisfying the initial conditions* $v(0) = P''u(0)$, $w(0) = P'u(0)$.

PROOF. Let u be a solution of equation (1) in $[0, T)$. It follows from Lemma 5 that there exists a solution v of equation (14) in $[0, T)$, satisfying the condition $v(0) = P''u(0)$. If we put $w = u - v$, we can see that equation (14) is identical with equation (18) and subtracting equations (1), (18), we can see that w is a solution of equation (19) in $[0, T)$. Subtracting also the initial conditions which are satisfied by functions u and v, we get: $w(0) = P'u(0)$.

On the other hand, if v and w are solutions of equations (18) and (19) on the interval $[0, T)$, satisfying the initial conditions $v(0) = P''u(0)$ and $w(0) = P'u(0)$, then we can add equations (18), (19) and we can see that $u = v + w$ is a solution of equation (1) on $[0, T)$.

3. Main theorem about stability.

We shall not treat the question of the existence of solutions of equation (1) in this section. We are going to derive estimates of each solution u whose value at time $t = 0$ is «small enough» and these estimates will be valid as long as the solution exists, i.e. in a time interval where solution u is defined. Thus, u cannot finish with a «blow up» in the neighbourhood of the right end point of its domain of definition. It is natural to expect that u can be defined in the time interval $[0, +\infty)$. In fact, to prove this, it would be necessary to use some additional assumptions about the nonlinear operator N (see e.g. D. Henry (1981)) and in order not to complicate this paper, we do not want to do this here.

THEOREM 1. *Let conditions* (i), (ii), (iii) [*or* (iii)′], (iv) *and* (v) *be satisfied. Then to any given* $\varepsilon > 0$, *there exists* $\kappa > 0$ *so that if* u *is a solution of equation* (1) *in the interval* $[0, T)$, $\|u(0)\|_0 + \|u(0)\|_1 \leq \kappa$ *then* $\|u(t)\|_0 + \|u(t)\|_1 \leq \varepsilon$ *for all* $t \in [0, T)$. *Moreover, if* $T = +\infty$ *then* $\lim_{t \to +\infty} (\|u(t)\|_1 + \|P' u(t)\|_0) = 0$.

PROOF. Let u be a solution of equation (1) in a time interval $[0, T)$. It follows from Lemma 6 that $u = v + w$, where v and w are solutions of equations (18) and (19) in $[0, T)$, satisfying the initial conditions $v(0) = P'' u(0)$ and $w(0) = P' u(0)$. We are first going to derive estimates which will be valid a.e. in the interval $(0, T)$.

If we multiply equation (18) by v, use condition (iii) and the fact that $v(t) \in H''$ for a.a. $t \in [0, T)$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_0^2 = ((A + B_s)\, v,\, v)_0 + (P'' B_a v,\, v)_0 + (P'' N(t,\, v + w),\, v)_0 \leq \tag{20}$$

$$\leq -\, c_8 \|v\|_1^2 + \|N(t,\, v + w)\|_0 \|v\|_0 .$$

Multiplying the equation (18) by $(-Av)$ and using conditions (i), (ii), we get

$$\left(\frac{\mathrm{d}v}{\mathrm{d}t},\, -Av\right)_0 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(v,\, -Av)_0 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_1^2 =$$

$$= -\, \|v\|_2^2 + (B_s v,\, -Av)_0 + (P'' B_a v,\, -Av)_0 + (P'' N(t,\, v + w),\, -Av)_0 \leq$$

$$\leq -\frac{1}{2}\|v\|_2^2 + \frac{3}{2}\|B_s v\|_0^2 + \frac{3}{2}\|B_a v\| + \frac{3}{2}\|N(t,\, v + w)\|_0^2 \leq$$

$$\leq -\frac{1}{2}\|v\|_2^2 + \frac{3}{2}c_1^2\|v\|_1^2 + \frac{3}{2}\mu_1\|v\|_2^2 + \frac{3}{2}k(\mu_1)\|v\|_1^2 + \frac{3}{2}\|N(t,\, v + w)\|_0^2 .$$

If we choose $\mu_1 = 1/6$ and denote $c_{16} = 3(c_1^2 + k(1/6))$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_1^2 \leq -\frac{1}{2}\|v\|_2^2 + c_{16}\|v\|_1^2 + 3\|N(t,\, v + w)\|_0^2 . \tag{21}$$

The solution w of equation (19) can be expressed in the form

$$w(t) = \mathrm{e}^{Lt} w(0) + \int_0^t \mathrm{e}^{L(t-s)} F(s)\, \mathrm{d}s\, ,$$

where $F(s) = P' B_a v(s) + P' N(s, v(s) + w(s))$. Thus, we have

$$P' w(t) = P' e^{Lt} w(0) + \int_0^t P' e^{L(t-s)} F(s)\, ds .$$

Using Lemma 3, we obtain

$$\|P' w(t)\|_0 \leqslant c_{11} e^{-\delta_1 t} \|w(0)\|_0 + \int_0^t c_{11} e^{-\delta_1 (t-s)} \|F(s)\|_0 \, ds .$$

Denote by $h(t)$ the right hand side of this inequality. Then

$$\|P' w(t)\|_0 \leqslant h(t) \tag{22}$$

and moreover, it can be verified that h satisfies the equation

$$\frac{dh}{dt} + \delta_1 h = c_{11} \|F(t)\|_0 \tag{23}$$

and the initial condition $h(0) = c_{11} \|w(0)\|_0$. Multiplying equation (23) by h, one gets

$$\tag{24} \left\{ \begin{aligned} & \frac{1}{2} \frac{d}{dt} h^2 \leqslant - \delta_1 h^2 + c_{11} \|F\|_0 h , \\ & \frac{d}{dt} h^2 \leqslant - \delta_1 h^2 + \frac{c_{11}^2}{\delta_1} \|F\|_0^2 \leqslant - \delta_1 h^2 + \\ & \qquad + \frac{2c_{11}^2}{\delta_1} \|B_a v\|_0^2 + \frac{2c_{11}^2}{\delta_1} \|N(t, v + w)\|_0^2 \leqslant \\ & \qquad \leqslant - \delta_1 h^2 + c_{17} \mu_2 \|v\|_2^2 + c_{17} k(\mu_2) \|v\|_1^2 + c_{17} \|N(t, v + w)\|_0^2 \end{aligned} \right.$$

where $c_{17} = 2c_{11}^2 / \delta_1$. The number μ_2 in (24) can be chosen arbitrarily in the interval $(0, 1)$.

Let us use the notation $w' = P' w$, $w'' = P'' w$ for a while. If we multiply the equation (19) by w, we obtain:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|w\|_0^2 &= ((A + B_s) w, w)_0 + (B_a w, w)_0 + (P' B_a v, w)_0 + (P' N(t, v + w), w)_0 = \\ &= ((A + B_s) w', w')_0 + ((A + B_s) w'', w'')_0 + (P' B_a v, w')_0 + \\ &+ (P' N(t, v + w), w')_0 \leqslant (A w', w')_0 + (B_s w', w')_0 - c_8 \|w''\|_1^2 + \end{aligned}$$

$$+\|w'\|_0^2+\frac{1}{2}\|P'B_a v\|_0^2+\frac{1}{2}\|P'N(t,v+w)\|_0^2\leqslant$$

$$\leqslant -c_8(\|w'\|_1^2+\|w''\|_1^2)+(c_8-1)\|w'\|_1^2+\mu_3\|B_s w'\|_0^2+\frac{1}{4\mu_3}\|w'\|_0^2+\|w'\|_0^2+$$

$$+\frac{1}{2}\mu_4\|v\|_2^2+\frac{1}{2}k(\mu_4)\|v\|_1^2+\frac{1}{2}\|N(t,v+w)\|_0^2\leqslant$$

$$\leqslant -\frac{1}{2}c_8\|w\|_1^2+(c_8-1)\|w'\|_1^2+\mu_3 c_1^2\|w'\|_1^2+[1/(4\mu_3)+1]\|w'\|_0^2+\frac{1}{2}\mu_4\|v\|_2^2+$$

$$+\frac{1}{2}k(\mu_4)\|v\|_1^2+\frac{1}{2}\|N(t,v+w)\|_0^2.$$

If we choose $\mu_3=(1-c_8)/c_1^2$ and use (22), we obtain:

$$\frac{d}{dt}\|w\|_0^2\leqslant -c_8\|w\|_1^2+c_{18}h^2+\mu_4\|v\|_2^2+k(\mu_4)\|v\|_1^2+\|N(t,v+w)\|_0^2 \tag{25}$$

where $c_{18}=c_1^2/(2-2c_8)+2$. μ_4 can be an arbitrary number in the interval $(0,1)$.

If we multiply equation (19) by $(-Aw)$, we obtain:

$$\left(\frac{dw}{dt},-Aw\right)_0=\frac{1}{2}\frac{d}{dt}(w,-Aw)_0=\frac{1}{2}\frac{d}{dt}\|w\|_1^2=$$

$$=-\|w\|_2^2+(B_s w,-Aw)_0+(B_a w,-Aw)_0+(P'B_a v,-Aw)_0+$$

$$+(P'N(t,v+w),-Aw)_0\leqslant$$

$$\leqslant -\frac{1}{2}\|w\|_2^2+2\|B_s w\|_0^2+2\|B_a w\|_0^2+2\|B_a v\|_0^2+2\|N(t,v+w)\|_0^2\leqslant$$

$$\leqslant -\frac{1}{2}\|w\|_2^2+2\mu_5\|w\|_2^2+2(c_1^2+k(\mu_5))\|w\|_1^2+2\mu_6\|v\|_2^2+2k(\mu_6)\|v\|_1^2+$$

$$+2\|N(t,v+w)\|_0^2.$$

If we choose $\mu_5=\mu_6=1/8$ and denote $c_{19}=4(c_1^2+k(1/8))$, we get:

$$\frac{d}{dt}\|w\|_1^2\leqslant -\frac{1}{2}\|w\|_2^2+c_{19}\|w\|_1^2+\frac{1}{2}\|v\|_2^2+4k\left(\frac{1}{8}\right)\|v\|_1^2+4\|N(t,v+w)\|_0^2. \tag{26}$$

Let ξ, ζ, η and ω be positive numbers. (Their concrete values will be specified later.) Put

$$V(v, h, w) = \|v\|_0^2 + \xi\|v\|_1^2 + \zeta h^2 + \eta\|w\|_0^2 + w\|w\|_1^2.$$

It follows from (20), (21), (24), (25), (26) that

$$\frac{\mathrm{d}}{\mathrm{d}t} V(v, h, w) \leq \left[-2c_8 + c_{16}\xi + c_{17}k(\mu_2)\,\zeta + k(\mu_4)\,\eta + 4k\left(\frac{1}{8}\right)\omega \right] \|v\|_1^2 + \tag{27}$$

$$+ \left[-\frac{1}{2}\xi + c_{17}\mu_2\zeta + \mu_4\eta + \frac{1}{2}\omega \right] \|v\|_2^2 + [-\delta_1\zeta + c_{18}\eta]\, h^2 +$$

$$+ [-c_8\eta + c_{19}\omega] \|w\|_1^2 - \frac{1}{2}\omega\|w\|_2^2 +$$

$$+ [3\xi + c_{17}\zeta + \eta + 4\omega] \|N(t, v+w)\|_0^2 + 2\|N(t, v+w)\|_0 \|v\|_0 .$$

Let us choose ξ, ζ, η and ω so that

$$c_{16}\xi + c_{17}k(\mu_2)\,\zeta + k(\mu_4)\,\eta + 4k\left(\frac{1}{8}\right)\omega = c_8,$$

$$-\frac{1}{4}\xi + c_{17}\mu_2\zeta + \mu_4\eta + \frac{1}{2}\omega = 0\,,$$

$$-\frac{1}{2}\delta_1\zeta + c_{18}\eta = 0\,,$$

$$-\frac{1}{2}c_8\eta + c_{19}\omega = 0\,.$$

This system has a positive solution:

$$\xi = \frac{c_8}{\Delta}\left[c_{17}\mu_2 + \frac{\delta_1\mu_4}{2c_{18}} + \frac{c_8\delta_1}{8c_{18}c_{19}} \right], \quad \zeta = \frac{c_8}{4\Delta}, \quad \eta = \frac{c_8\delta_1}{8\Delta c_{18}}, \quad \omega = \frac{c_8^2\delta_1}{16\Delta c_{18}c_{19}}$$

where

$$\Delta = c_{16}\left[c_{17}\mu_2 + \frac{\delta_1\mu_4}{2c_{18}} + \frac{c_8\delta_1}{8c_{18}c_{19}} \right] + \frac{1}{4}\left[c_{17}k(\mu_2) + \frac{\delta_1 k(\mu_4)}{2c_{18}} + \frac{c_8\delta_1 k(1/8)}{c_{18}c_{19}} \right].$$

If we substitute these values of ξ, ζ, η and ω to (27), we obtain

$$\frac{d}{dt}V(v,h,w) \leqslant - c_8\|v\|_1^2 - \frac{1}{4}\xi\|v\|_2^2 - \frac{1}{2}\delta_1\zeta h^2 - \frac{1}{2}c_8\eta\|w\|_1^2 - \frac{1}{2}\omega\|w\|_2^2 + \tag{28}$$

$$+ c_{20}\|N(t, v+w)\|_0^2 + \|N(t, v+w)\|_0\|v\|_0,$$

where $c_{20} = 3\xi + c_{17}\zeta + \eta + 4\omega$. Let us denote

$$|||(v, w)||| = \left[c_8\|v\|_1^2 + \frac{1}{4}\xi\|v\|_2^2 + \frac{1}{2}\delta_1\zeta h^2 + \frac{1}{2}\eta c_8\|w\|_1^2 + \frac{1}{2}\omega\|w\|_2^2\right]^{1/2}.$$

Using condition (v), we can derive that there exist $c_{21} > 0$ and $c_{22} > 0$ so that

$$\|N(t, v+w)\|_0^2 \leqslant c_{21} V(v, h, w)^{2(\beta+\gamma-1)}\,|||(v, w)|||^2,$$

$$\|N(t, v+w)\|_0\|v\|_0 \leqslant c_{22} V(v, h, w)^{\beta+\gamma-1}\,|||(v, w)|||^2.$$

Substituting this into (28), we obtain

$$\frac{d}{dt}V(v, h, w) \leqslant$$

$$\leqslant |||(v, w)|||^2[-1 + c_{20}c_{21}V(v, h, w)^{2(\beta+\gamma-1)} + c_{20}c_{22}V(v, h, w)^{\beta+\gamma-1}].$$

Thus, if

$$c_{20}c_{21}V(v(0), h(0), w(0))^{2(\beta+\gamma-1)} + c_{20}c_{22}V(v(0), h(0), w(0))^{\beta+\gamma-1} \leqslant 1$$

then

$$\frac{d}{dt}V(v, h, w) \leqslant 0$$

for a.a. $t \in (0, T)$. This means that

$$V(v(t), h(t), w(t)) \leqslant V(v(0), h(0), w(0)) \tag{29}$$

for a.a. $t \in [0, T)$. Hence we have

$$\|u(t)\|_0^2 + \|u(t)\|_1^2 = \|v(t) + w(t)\|_0^2 + \|v(t) + w(t)\|_1^2 \leqslant \tag{30}$$

$$\leqslant c_{23}[\|v(t)\|_0^2 + \xi\|v(t)\|_1^2 + \eta\|w(t)\|_0^2 + \omega\|w(t)\|_1^2] \leqslant$$

$$\leqslant c_{23}V(v(t), h(t), w(t)) \leqslant c_{23}V(v(0), h(0), w(0)) =$$

$$= c_{23}[\|v(0)\|_0^2 + \xi\|v(0)\|_1^2 + (\zeta c_{11}^2 + \eta)\|w(0)\|_0^2 + w\|w(0)\|_1^2] =$$

$$= c_{23}[\|P''u(0)\|_0^2 + \xi\|P''u(0)\|_1^2 + (\zeta c_{11}^2 + \eta)\|P'u(0)\|_0^2 + \omega\|P'u(0)\|_1^2] \leqslant$$

$$\leqslant c_{24}[\|P''u(0)\|_0^2 + \|P'u(0)\|_0^2] + c_{25}[\|P''u(0)\|_1^2 + \|P'u(0)\|_1^2] \leqslant$$

$$\leqslant c_{24}\|u(0)\|_0^2 + c_{25}[c_4\|u(0)\|_1^2 + c_5\|u(0)\|_0^2] \leqslant c_{26}[\|u(0)\|_0^2 + \|u(0)\|_1^2].$$

These estimates complete the proof of the first part of the theorem.

Suppose that $T = +\infty$ and the initial values of v and w are so small that

$$c_{20}c_{21}V(v(0), h(0), w(0))^{2(\beta+\gamma-1)} + c_{20}c_{22}V(v(0), h(0), w(0))^{\beta+\gamma-1} \leqslant \frac{1}{2}$$

now. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}V(v, h, w) \leqslant -\frac{1}{2}\,|||(v, w)|||^2$$

for a.a. $t \in (0, +\infty)$ and hence

$$\lim_{t\to+\infty} V(v(t), h(t), w(t)) + \frac{1}{2}\int_0^{+\infty} |||(v(s), w(s))|||^2\,\mathrm{d}s \leqslant V(v(0), h(0), w(0)).$$

Put $W(v, h, w) = \|v\|_1^2 + ah^2 + b\|w\|_1^2$ where a, b are such positive constants that $-1/2 + ac_{17}\mu_2 + (1/2)\,b < -1/4$. Since $W(v, h, w) \leqslant$ $\leqslant \text{const.}\ |||(v, w)|||^2$, $W(v, h, w)$ is integrable on $(0, +\infty)$. It follows from (21), (26) and the boundedness of $V(v, h, w)$ (see (29)) that

$$\frac{\mathrm{d}}{\mathrm{d}t}W(v, h, w) \leqslant \left[-\frac{1}{2} + ac_{17}\mu_2 + \frac{1}{2}b\right]\|v\|_2^2 - a\delta_1 h^2 - \frac{1}{2}b\|w\|_2^2 +$$

$$+\left[c_{16} + ac_{17}k(\mu_2) + 4bk\left(\frac{1}{8}\right)\right]\|v\|_1^2 + bc_{19}\|w\|_1^2 + [3 + ac_{17} + 4b]\|N(t, v+w)\|_0^2 \leqslant$$

$$\leqslant -\frac{1}{4}\|v\|_2^2 - a\delta_1 h^2 - \frac{1}{2}\|w\|_2^2 + c_{27} + c_{28}V(v, h, w)^{2(\beta+\gamma-1)}\,|||(v, w)|||^2 \leqslant$$

$$\leqslant -\frac{1}{4}\|v\|_2^2 - a\delta_1 h^2 - \frac{1}{2}\|w\|_2^2 +$$

$$+c_{29}V(v(0),\,h(0),\,w(0))^{2(\beta+\gamma-1)}\left(\frac{1}{4}\|v\|_2^2+a\delta_1h^2+\frac{1}{2}\|w\|_2^2\right)+c_{30}=$$

$$=[c_{29}V(v(0),\,h(0),\,w(0))^{2(\beta+\gamma-1)}-1]\left(\frac{1}{4}\|v\|_2^2+a\delta_1h^2+\frac{1}{2}\|w\|_2^2\right)+c_{30}.$$

Thus, if the initial data are so small that $c_{29}V(v(0),\,h(0),\,w(0))^{2(\beta+\gamma-1)}\leq 1$ then the time derivative of $W(v,\,h,\,w)$ is bounded above a.e. in $(0,\,+\infty)$. This, together with the continuity and integrability of $W(v,\,h,\,w)$ on the interval $(0,\,+\infty)$, implies:

$$\lim_{t\to+\infty}W(v(t),\,h(t),\,w(t))=0\,.$$

Thus, we have: $\lim\limits_{t\to+\infty}(\|u(t)\|_1+\|P'u(t)\|_0)=0$.

4. An example in one space dimension.

This section contains an illustrative example of the operator $L\equiv A+ + B_s+B_a$ which is not dissipative, and conditions (i)-(iv) are satisfied. Hilbert space H is $L^2((0,\,+\infty))$ here. $D(A)$ is $W^{2,\,2}((0,\,+\infty))\cap \cap W_0^{1,\,2}((0,\,+\infty))$ and

$$Lu=u''+[U(x)\,u]',$$

where $U(x)=2\pi^2x$ for $x\in[0,\,1]$ and $U(x)=2\pi^2$ for $x\in[1,\,+\infty)$. Operators A, B_s and B_a are:

$$Au=u''\,,$$

$$B_su=\frac{1}{2}U'(x)\,u=\begin{cases}\pi^2u & \text{for } x\in[0,\,1),\\ 0 & \text{for } x\in(1,\,+\infty),\end{cases}$$

$$B_au=U(s)\,u'+\frac{1}{2}U'(x)\,u=\begin{cases}2\pi^2xu'+\pi^2u & \text{for } x\in[0,\,1),\\ 2\pi^2u' & \text{for } x\in(1,\,+\infty).\end{cases}$$

The validity of conditions (i), (ii) is obvious. We are going to show that operator L is not dissipative and conditions (iii) and (iv) are fulfilled, too.

The spectrum of the operator $L_s=A+B_s$ is a subset of the real axis. It is not difficult to verify that the interval $(-\infty\,,\,0]$ represents a continuous part of $\sigma(L_s)$ and it contains no eigenvalues of L_s. By standard calculation, we can find out that L_s has the unique eigenvalue $\lambda_0=4.516239$

(exactly, $\lambda_0 = \pi^2 - s^2$, where s is the positive solution of the equation $s \cdot \cot s = -\sqrt{\pi^2 - s^2}$). The algebraic multiplicity of λ_0 is equal to one and the corresponding eigenfunction is

$$u_0(x) = \begin{cases} c_{31} \sin(\sqrt{\pi^2 - \lambda_0} x) & \text{for } x \in [0, 1], \\ c_{32} \mathrm{e}^{-\sqrt{\lambda_0} x} & \text{for } x \in (1, +\infty) \end{cases}$$

where $c_{31} = 1.166203$ and $c_{32} = 7.192440$. $\left(\text{Exactly, the constants } c_{31} \text{ and } c_{32} \text{ are chosen so that } c_{31} \sin\sqrt{\pi^2 - \lambda_0} = c_{32} \mathrm{e}^{-\sqrt{\lambda_0}} \text{ and } \int_0^{+\infty} u_0^2(x)\, \mathrm{d}x = 1.\right)$ Hence subspace H' is one-dimensional and it is generated by function u_0. H'' is the orthogonal complement to H' in H.

Let us now turn our attention to condition (iii). Due to Lemma 2, we can investigate the spectrum of the operator $A + B_s + \varepsilon P'' B_s$ in space H''. Since $\sigma((A + B_s)|_{H''}) = (-\infty, 0]$ and $\varepsilon P'' B_s$ is $(A + B_s)$-compact, $\sigma((A + B_s)|_{H''})$ and $\sigma((A + B_s + \varepsilon P'' B_s)|_{H''})$ can differ at most in a countable number of isolated eigenvalues. Hence, if the intersection $\sigma((A + B_s + \varepsilon P'' B_s)|_{H''}) \cap (0, +\infty)$ is nonempty, then it contains only some eigenvalues of $(A + B_s + \varepsilon P'' B_s)|_{H''}$. Thus, to verify (iii), it is sufficient to show that for $\varepsilon > 0$ small enough no such eigenvalues exist. Suppose the opposite, i.e. that there exist sequences ε_n, ζ_n and u_n such that ε_n is monotonically decreasing and tends to zero, ζ_n is a positive eigenvalue of the operator $A + B_s + \varepsilon_n P'' B_s$ and $u_n \in H''$ is a corresponding eigenfunction. The functions u_n can be chosen so that $\|u_n\|_0 = 1$.

The boundedness of ζ_n is obvious. Multiplying the equation $Au_n + B_s u_n + \varepsilon_n P'' B_s u_n = \zeta_n u_n$ by u_n and using condition (i), we obtain: $\zeta_n \leqslant (1 + \varepsilon_n) c_1 \|u_n\|_1 - \|u_n\|_1^2$, which implies: $\zeta_n \leqslant (1/4)(1 + \varepsilon_1)^2 c_1^2$.

Let ζ_0 be a cluster point of the sequence ζ_n. Then $\zeta_0 \geqslant 0$. In order not to complicate the notation, a subsequence of ζ_n which tends to ζ_0 as $n \to +\infty$ will also be denoted by ζ_n in the following. The equation $Au_n + B_s u_n + \varepsilon_n P'' B_s u_n = \zeta_n u_n$ can be rewritten as $(A + B_s - \zeta_0 I) u_n = (\zeta_n - \zeta_0) u_n + \varepsilon_n P'' B_s u_n$. As the right hand side tends to the zero element of H'' if $n \to +\infty$, ζ_0 belongs to $\sigma((A + B_s)|_{H''})$. However, this spcetrum coincides with the interval $(-\infty, 0]$ and so $\zeta_0 = 0$.

Because $A + B_s + \varepsilon P'' B_s = A + (1 + \varepsilon) B_s - \varepsilon P' B_s$ and $P' B_s u_n = (B_s u_n, u_0)_0 u_0$, the equation $(A + B_s + \varepsilon P'' B_s) u_n = \zeta_n u_n$ can be rewritten as

$$Au_n + (1 + \varepsilon_n) B_s u_n - \zeta_n u_n = \varepsilon_n (B_s u_n, u_0)_0 u_0. \tag{31}$$

Since u_n is orthogonal to u_0, the integral $\int_0^{+\infty} u_n(s)\, u_0(s)\, ds$ is equal to zero.

Suppose that $(B_s u_n, u_0)_0 = \pi^2 \int_0^1 u_n(s)\, u_0(s)\, ds = 0$ at first. Then the integral $\int_1^{+\infty} u_n(s)\, u_0(s)\, ds$ also equals zero. Equation (31) implies that $u_n(x) = c_{33}\, e^{-\sqrt{\zeta_n} x}$ for $x \in (1, +\infty)$ and it can easily be verified that $c_{33} \neq 0$. Therefore

$$\int_1^{+\infty} u_n(s)\, u_0(s)\, ds = \frac{c_{33}\, c_{32}}{\sqrt{\zeta_n} + \sqrt{\lambda_0}}\, e^{-\sqrt{\zeta_n} - \sqrt{\lambda_0}}$$

and this is obviously different from zero. Hence the equality $(B_s u_n, u_0)_0 = 0$ can be excluded.

Put $v_n = u_n / (B_s u_n, u_0)_0$. Then $(B_s v_n, u_0)_0 = 1$. Substituting this to (31) and using also the concrete forms of u_0 on the intervals $(0, 1)$ and $(1, +\infty)$, we can rewrite (31) in the form

$$(32) \qquad \begin{cases} v_n'' + (1 + \varepsilon_n)\, \pi^2 v_n - \zeta_n v_n = \varepsilon_n c_{31} \sin\left(\sqrt{\pi^2 - \lambda_0}\, x\right) & \text{on } (0, 1), \\ v_n'' - \zeta_n u = \varepsilon_n c_{32}\, e^{-\sqrt{\lambda_0} x} & \text{on } (1, +\infty). \end{cases}$$

The solution v_n of this problem can be explicitly expressed. However, the integration of $v_n(x) \cdot u_0(x)$ shows that

$$\int_0^{+\infty} v_n(x)\, u_0(x)\, dx = \frac{\varepsilon_n}{\lambda_0} \int_0^{+\infty} u_0^2(x)\, dx + o(\varepsilon_n + \zeta_n) \qquad \text{for } n \to +\infty,$$

which means that the integral on the left hand side is positive for n large enough and so $v_n \notin H''$ and $u_n \notin H''$. Thus, condition (iii) is fulfilled.

We shall now verify condition (iv). Elementary estimates show that $\sigma(L)$ is a subset of the parabolic region $\{z \equiv \zeta + i\eta \in \mathbb{C};\ \zeta \leqslant a/2 - \eta^2 / a^2\}$. If $\lambda \equiv \zeta + i\eta \in \mathbb{C}$, $\zeta < -\eta^2 / a^2$ then λ is an eigenvalue of operator L.

Suppose that $\delta \in (0, \lambda_0)$. Let us work with $\underline{\lambda \in G \cap \varrho(L)}$ where $G = \{z \equiv \zeta + i\eta \in \mathbb{C};\ \zeta \in (-\delta, a/2 + 1),\ |\eta| < a\sqrt{(1/2)\, a + \delta + 1}\}$ firstly. The function $y_\lambda \equiv R_\lambda(L)\, u_0$ is the solution of the equation $(L - \lambda I)\, y_\lambda = u_0$, i.e.

$$(33) \qquad (y_\lambda)_{xx} + U(x)(y_\lambda)_x + [U_x(x) - \lambda]\, y_\lambda = u_0(x)$$

and $y_\lambda(0)=0$. Denote by y_λ^I, y_λ^{II} a fundamental system of solutions of the corresponding homogeneous equation. Then y_λ can be expressed in the form

$$y_\lambda(x)=y_\lambda^I(x)\left[C^I-\int_0^x \frac{y_\lambda^{II}(s)\,u_0(s)}{\Delta_\lambda(s)}\,\mathrm{d}s\right]+y_\lambda^{II}(x)\left[C^{II}+\int_0^x \frac{y_\lambda^{I}(s)\,u_0(s)}{\Delta_\lambda(s)}\,\mathrm{d}s\right]$$

where $\Delta_\lambda=y_\lambda^I\cdot(y_\lambda^{II})_x-y_\lambda^{II}\cdot(y_\lambda^I)_x$. ($\Delta_\lambda$ is the Wronski determinant of system y_λ^I, y_λ^{II}.) It can be derived that

$$\Delta_\lambda(s)=\Delta_\lambda(1)\cdot\exp\left[-\int_1^s U(\tau)\,\mathrm{d}\tau\right]=\begin{cases}\Delta_\lambda(1)\,\mathrm{e}^{a/2(1-s^2)} & \text{for } x\in[0,1],\\ \Delta_\lambda(1)\,\mathrm{e}^{-a(s-1)} & \text{for } x\in(1,+\infty).\end{cases}$$

The homogeneous equation which corresponds to (33) is the equation with constant coefficients in the interval $[1,+\infty)$, so its fundamental system y_λ^I, y_λ^{II} can easily be expressed here. We can choose y_λ^I, y_λ^{II} so that

$$y_\lambda^I(x)=\exp\left[\left(-\frac{1}{2}a+\sqrt{\frac{1}{4}a^2+\lambda}\right)x\right],\quad y_\lambda^{II}(x)=\exp\left[\left(-\frac{1}{2}a-\sqrt{\frac{1}{4}a^2+\lambda}\right)x\right]$$

for $x\in[1,+\infty)$. (We use the symbol $\sqrt{\ }$ for the square root in the complex domain in such a sense that $\sqrt{z}=\sqrt{|z|}\cdot\exp((1/2)\,i\arg z)$ if $z\neq 0$, $\sqrt{z}=0$ if $z=0$.) We can express $\Delta_\lambda(1)$ now: $\Delta_\lambda(1)=$ $=-2\mathrm{e}^{-a}\sqrt{(1/4)\,a^2+\lambda}$. Since $\operatorname{Re}\sqrt{(1/4)\,a^2+\lambda}>\sqrt{(1/4)\,a^2-\lambda_0}>$ $>\sqrt{(1/4)\,a^2-(1/2)\,a+\sigma_0^2}>(1/2)\,\pi$ for all $\lambda\in G$, $1/\Delta_\lambda(1)$ is the analytic function of λ in G. Let us choose

$$C^I=\int_0^{+\infty}\frac{y_\lambda^{II}(s)\,u_0(s)}{\Delta_\lambda(s)}\,\mathrm{d}s=\int_0^1\frac{\mathrm{e}^{-a/2(1-s^2)}}{\Delta_\lambda(1)}\,y_\lambda^{II}(s)\cdot c_{31}\sin\left(\sqrt{\frac{1}{2}a-\lambda_0}\,s\right)\mathrm{d}s+$$

$$+\int_1^{+\infty}\frac{\mathrm{e}^{-a}}{\Delta_\lambda(1)}\exp\left[\left(\frac{1}{2}a-\sqrt{\frac{1}{4}a^2+\lambda}-\sqrt{\lambda_0}\right)s\right]\mathrm{d}s\,.$$

Since $\operatorname{Re}[(1/2)\,a-\sqrt{(1/4)\,a^2+\lambda}-\sqrt{\lambda_0}]<(1/2)\,a-\sqrt{(1/4)\,a^2-\delta}-$ $-\sqrt{\lambda_0}<\sqrt{\delta}-\sqrt{\lambda_0}<0$, the last integral is finite and so $|C^I|<+\infty$.

Thus, we have

$$y_\lambda(x) = y_\lambda^I(x) \int_x^{+\infty} \frac{y_\lambda^{II}(s)\, u_0(s)}{\Delta_\lambda(s)}\, ds + y_\lambda^{II}(x) \left[C^{II} + \int_0^x \frac{y_\lambda^I(s)\, u_0(s)}{\Delta_\lambda(s)}\, ds \right]. \tag{34}$$

Let us choose C^{II} so that $y_\lambda(0) = 0$:

$$C^{II} = -\frac{y_\lambda^I(0)}{y_\lambda^{II}(0)} \int_0^{+\infty} \frac{y_\lambda^{II}(s)\, u_0(s)}{\Delta_\lambda(s)}\, ds .$$

$y_\lambda^I(0)$, $y_\lambda^{II}(0)$ and $\int_0^{+\infty} y_\lambda^{II}(s)\, u_0(s)/\Delta_\lambda(s)\, ds$ are analytic functions of λ not only in $G \cap \varrho(L)$, but in the whole set G. It can be shown by standard numerical methods that

$$|y_\lambda^{II}(0)| \geqslant 11.791 \left| \exp\left[-\frac{1}{2} a - \sqrt{\frac{1}{4} a^2 + \lambda} \right] \right|$$

for all $a \in ((1/4)\pi^2, (9/4)\pi^2)$ and $\lambda \in G$. So C^{II} is an analytic function of λ in G. To emphasize its dependence on λ, we shall denote it by C_λ^{II} in the following.

It is not difficult to show that $\int_0^1 u_\lambda(x) \cdot u_0(x)\, dx$ is the analytic function of λ in G.

If $x > 1$ then we can decompose $\int_0^x$ in (34) to $\int_0^1 + \int_1^x$, we can substitute the concrete forms of y_λ^I, y_λ^{II}, Δ_λ and u_0 to the integrals $\int_x^{+\infty}$ and $\int_1^x$, we can evaluate these integrals and we obtain the expression:

$y_\lambda(x) =$

$$= \frac{e^{-\sqrt{\lambda_0} x}}{2\sqrt{(1/4)a^2+\lambda}} \left[\frac{1}{(1/2)a - \sqrt{(1/4)a^2+\lambda} - \sqrt{\lambda_0}} - \frac{1}{(1/2)a + \sqrt{(1/4)a^2+\lambda} - \sqrt{\lambda_0}} \right] +$$

$$+ (C_\lambda^{II} + M_\lambda) \exp\left[\left(-\frac{1}{2} a - \sqrt{(1/4)\, a^2 + \lambda} \right) x \right] +$$

$$+ \frac{\exp[(-(1/2)\, a - \sqrt{(1/4)\, a^2 + \lambda})(x-1) + \sqrt{\lambda_0}]}{2\sqrt{(1/4)\, a^2 + \lambda}[1/2a + \sqrt{(1/4)\, a^2 + \lambda} - \sqrt{\lambda_0}}]$$

where

$$M_\lambda = -\frac{e^{a/2}}{2\sqrt{(1/4)\,a^2+\lambda_0}} \int_0^1 e^{-as^2/2} y_\lambda^I(s)\, c_{31} \sin\left(\sqrt{\frac{1}{2}a-\lambda_0 s}\right) ds\,.$$

It can be derived from the conditions $0<\delta<\lambda_0$ and $\mathrm{Re}\lambda > -\delta$ that

$$\mathrm{Re}\left(-\frac{1}{2}a-\sqrt{\frac{1}{4}a^2+\lambda}\right) < -\frac{1}{2}\pi^2, \quad \mathrm{Re}\left(\frac{1}{2}a+\sqrt{\frac{1}{4}a^2+\lambda-\sqrt{\lambda_0}}\right) > \frac{1}{2}\pi(\pi-1),$$

$$\mathrm{Re}\left(\frac{1}{2}a-\sqrt{\frac{1}{4}a^2+\lambda-\sqrt{\lambda_0}}\right) < \sqrt{\delta}-\sqrt{\lambda_0} < 0\,.$$

Hence $\int_1^{+\infty} y_\lambda(x)\cdot u_0(x)\,dx$ is the analytic function of λ in G. This means that $(y_\lambda, u_0)_0$ also depends analytically on λ in G and $(y_\lambda, u_0)_0\cdot u_0$ is the analytic continuation of $P'R_\lambda(L)\,u_0$ from $\varrho(L)$ to G. Since $C_+(-\delta) = \varrho(L)\cup G$, condition (iv) is satisfied.

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