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Prime Rings with Hypercommuting Derivations on a Lie Ideal.

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ABSTRACT - Let R be a prime ring with no non-zero nil right ideals, d a non-zero derivation of R , L a non-central Lie ideal of R . If d satisfies $[d(u^m), u^m]_k = 0$ for all $u \in L$, $m = m(u) \geq 1$, $k = k(u) \geq 1$, then R is an order in a simple algebra of dimension at most 4 over its center.

1. Introduction.

The classical hypercenter theorem, proved by I. N. Herstein [10], asserts that for a ring R not containing non-zero nil two-sided ideals, an inner derivation d_a , induced by $a \in R$, satisfying $d_a(x^m) = 0$, $m = m(x) \geq 1$, for all $x \in R$, must vanish identically on the whole ring R , i.e. $a \in Z(R)$.

Moreover in [4, theorem 4] Chuang and Lin proved that for a ring R not containing non-zero nil right ideals, the inner derivation d_a , induced by $a \in R$, satisfying $[d_a(x^m), x^m]_k = 0$, $m = m(x) \geq 1$, $k = k(x) \geq 1$, for all $x \in R$, must vanish identically on the whole ring R .

Later Chuang generalized the results above to arbitrary derivation d , defined in a prime ring R not containing non-zero nil right ideals. He proved in [3, corollary 2] that if $[d(x^m), x^m]_k = 0$, $m = m(x) \geq 1$,

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$k = k(x) \geq 1$, for all $x \in R$, then either d vanishes identically on the whole ring R or R is commutative.

The starting point of this paper is the following result, obtained in [5]:

Let R be a prime ring, with no non-zero nil right ideals, L a non-central Lie ideal of R , d_a an inner derivation induced by $a \in R$. If $[d_a(u^m), u^m]_k = 0$, for all $u \in L$, $m = m(u) \geq 1$, $k = k(u) \geq 1$, then either d_a vanishes identically on the whole ring R , that is $a \in Z(R)$, or R satisfies $S_4(x_1, x_2, x_3, x_4)$, the standard identity of degree 4.

The last conclusion is well known to be equivalent, by Posner's theorem, to saying that R is an order in a simple algebra at most 4-dimensional over its center.

Moreover we remark that if $R = M_2(C)$, the ring of all 2×2 matrices over a commutative ring C , then for any $u \in [R, R]$ one has $u^2 \in Z(R)$ (see example (3) page 12 in [12]), and so our condition $[d(u^m), u^m]_k = 0$ holds for $m = 2$ and $k = 1$.

The purpose of this note is to generalize the result obtained in [5] to arbitrary derivation d . We will prove the following:

THEOREM. *Let R be a prime ring with no non-zero nil right ideal, L a non-central Lie ideal of R , d a non-zero derivation of R satisfying $[d(u^m), u^m]_k = 0$, for all $u \in L$, $m = m(u) \geq 1$, $k = k(u) \geq 1$. Then R satisfies $S_4(x_1, x_2, x_3, x_4)$, the standard identity of degree 4.*

2. Preliminaries.

The proof of our theorem is based upon two results obtained respectively in [3] and [5].

The first one is a result of Chuang [3, proposition 2] concerning a careful analysis of derivations satisfying a particular property on semiprime rings.

The related objects we need to mention are the left Utumi quotient ring U , and also the two-sided Utumi quotient ring Q of a ring R (sometimes, as in [1], U and Q are called the maximal left ring of quotients and the symmetric ring of quotients respectively).

The definitions, the axiomatic formulations and the properties of these quotient rings U , Q can be found in [15], [8], [1].

For instance U , the left Utumi quotient ring of R , exists if and only if R is right faithful, that is for any $a \in R$, $Ra = 0$ implies $a = 0$.

In the same way we can define Q if R is both right and left faithful.

In any case, when R is a prime ring, all that we need here about these objects is that

$$1) R \subseteq Q \subseteq U.$$

$$2) U \text{ and } Q \text{ are prime rings [8, page 74].}$$

3) For all $q \in Q$ there exists a dense left ideal M of R such that $Mq \subseteq R$, moreover if $Mq = 0$, for some dense left ideal M of R , then $q = 0$.

$$4) \text{ The center of } U, \text{ denoted by } C, \text{ coincides with the center of } Q.$$

C is a field which is called the extended centroid of R [1, pages 68-70].

Moreover if R is a prime P.I. ring then, by Posner's theorem [9, theorem 1.4.3 page 40], C is the quotient field of $Z(R)$ and

$$RZ^{-1} = \{rz^{-1} : r \in R, z \in Z(R) - \{0\}\} = RC$$

is a simple algebra finite dimensional over its center.

In this case it is easy to see that $RC = Q = U$.

Finally we recall that a map $d: R \rightarrow R$ is a derivation if, for any $x, y \in R$, $d(x + y) = d(x) + d(y)$ and $d(xy) = d(x)y + xd(y)$. Each derivation of a prime ring R can be uniquely extended to a derivation of its Utumi quotient ring U and thus all derivations of R will be implicitly assumed to be defined on the whole U (see [15, page 101] or [16, lemma 2]).

Now we are ready to state the result of Chuang ([3, proposition 1, page 46]) for prime rings. In this case as we said above U and Q are prime too and so any central idempotent is trivial.

Hence, for any $a \in U - \{0\}$, the norm $\|a\|$ of a , defined in [3, page 39, (7)], is always 1 and of course $\|0\| = 0$.

PROPOSITION 2.1. *Let R be a prime ring with extended centroid C . Let d a derivation of U , the left Utumi quotient of R , satisfying $d(a)(1 + a)^{-1} \in C$, for any $a \in R$, with $a^2 = 0$. Let Q be the two-sided Utumi quotient ring of R . Then*

either the ring Q is reduced, that is Q does not have any non-zero nilpotent element

or $Q = U = M_2(C)$, the ring of all two by two matrices over C

or the derivation d is the inner derivation induced by a square-zero

element $c \in U$, satisfying the property that, for any $x, y \in Q$ with $xy = 0$, we also have $xc = cy = 0$.

The second result concerns the generalized hypercentralizer of a non-central Lie ideal of R .

More precisely the generalized hypercentralizer of an arbitrary subset S of R is the following subring of R :

$$H_R(S) = \{a \in R: \text{ for all } s \in S \text{ there exist } n = n(a, s) \geq 1,$$

$$k = k(a, s) \geq 1 \text{ such that } [a, s^n]_k = 0\}.$$

In [5] we proved that if L is a non-central Lie ideal of a prime ring R with no non-zero nil right ideal, then either $H_R(L) = Z(R)$ or R satisfies $S_4(x_1, x_2, x_3, x_4)$.

Finally we remark that an important tool in our proof will be the theory of differential identities initiated by Kharchenko [13].

3. Proof of the theorem.

Throughout this paper we will use the following notation:

R will always be an associative prime ring, with no non-zero nil right ideal, L will be a non-central Lie ideal of R and d will be a derivation of R satisfying $[d(u^m), u^m]_k = 0$, for any $u \in L$, $m = m(u) \geq 1$, $k = k(u) \geq 1$. U will be the left Utumi quotient ring of R , and Q will be the two-sided Utumi quotient ring of R .

We start with an easy remark:

REMARK 1. *If R has characteristic $p \neq 0$ then for all $u \in L$ there exists $n = n(u) \geq 1$ such that $d(u^n) = 0$.*

PROOF. Let $u \in L$ be arbitrarily given. There exist $m = m(u) \geq 1$, $k = k(u) \geq 1$ such that $[d(u^m), u^m]_k = 0$. Pick an integer $t \geq 1$ such that $p^t \geq k$. Then

$$0 = [d(u^m), u^m]_{p^t - k} = [d(u^m), u^m]_{p^t}.$$

Since R is of characteristic $p > 0$, we have

$$0 = [d(u^m), u^m]_{p^t} = [d(u^m), u^{mp^t}]$$

and this implies immediately $[d(u^{mp^t}), u^{mp^t}] = 0$. Let $a = u^{mp^t}$. We obtain $[d(a), a] = 0$.

Using the commutativity we have $d(a^p) = pa^{p-1}d(a) = 0$, that is

$$d(u^{mp^{t+1}}) = 0$$

and so we have shown that for all $u \in L$ there exists $l = l(u) \geq 1$ such that $d(u^l) = 0$. ■

Notice that in this case, if $p \neq 2$, then our result follows immediately by theorem 2 of [7].

Now we make some other reductions.

It is well known that if L is a non-central Lie ideal of a prime ring R then either R satisfies $S_4(x_1, x_2, x_3, x_4)$ or there exists a non-zero two-sided ideal I of R such that $[I, R] \subseteq L$ and $[I, R] \not\subseteq Z(R)$.

Therefore we will assume, in all that follows, that $L = [I, R]$ for some non-zero two-sided ideal I of R (see for instance Lemma 2 and Proposition 1 in [6]).

In this case L is invariant under any inner automorphism induced by an invertible (or quasi-invertible) element of R .

Moreover, if $Z(R) \neq 0$, then we can consider

$$\bar{R} = \{rz^{-1} : r \in R, z \in Z(R) - \{0\}\}$$

$$\bar{L} = \{uz^{-1} : u \in L, z \in Z(R) - \{0\}\}$$

which are the localizations at $Z(R)$ of R and L respectively. Since $L = [I, R]$ is a $Z(R)$ -submodule of R , we have

REMARK 2.

- 1) $\bar{L}$ is a non-central Lie ideal of the prime ring $\bar{R}$
- 2) the derivation d extends uniquely to a derivation on $\bar{R}$ as follows

$$d(rz^{-1}) = (d(r)z - rd(z))z^{-2}$$

- 3) the derivation d , defined on $\bar{R}$, satisfies our assumptions on $\bar{L}$, that is for any $\bar{u} \in \bar{L}$, there exist $n = n(\bar{u}) \geq 1$, $k = k(\bar{u}) \geq 1$ such that $[d(\bar{u}^n), \bar{u}^n]_k = 0$.

LEMMA 3.1. *Let $a \in R$. If a is invertible then $d(a)a^{-1} \in H_R(L)$, if a is quasi-invertible then $d(a)(1+a)^{-1} \in H_R(L)$.*

PROOF. First we assume that a is invertible, then, as we said above, $aLa^{-1} = L$, hence for any $u \in L$ there exist $m, k, n, h \geq 1$ such that

$$0 = [d((a^{-1}ua)^m), (a^{-1}ua)^m]_k = 0$$

and

$$[d(u^n), u^n]_h = 0.$$

Hence for $s = nm$ and $t = \max\{h, k\}$ we also have:

$$[d((a^{-1}ua)^s), (a^{-1}ua)^s]_t = 0 = [d(u^s), u^s]_t.$$

It follows that:

$$\begin{aligned} 0 &= [d(a^{-1}u^s a), a^{-1}u^s a]_t = \\ &= [d(a^{-1})u^s a + a^{-1}d(u^s)a + a^{-1}u^s d(a), a^{-1}u^s a]_t = \\ &= [-a^{-1}d(a)a^{-1}u^s a + a^{-1}u^s d(a), a^{-1}u^s a]_t + [a^{-1}d(u^s)a, a^{-1}u^s a]_t = \\ &= -a^{-1}[d(a)a^{-1}u^s - u^s d(a)a^{-1}, u^s]_t a = -a^{-1}[d(a)a^{-1}, u^s]_{t+1} a. \end{aligned}$$

Hence $[d(a)a^{-1}, u^s]_{t+1} = 0$ that is $d(a)a^{-1} \in H_R(L)$.

A similar proof holds if a is a quasi-invertible element of R . ■

We remark that any square-zero element a of R is quasi-invertible with quasi-inverse $-a$.

Therefore, by [5], either R satisfies $S_4(x_1, x_2, x_3, x_4)$ and we are done or $d(a)a^{-1} \in Z(R)$, that is the derivation d satisfies the hypothesis of Chuang's result. In this case one of the three conclusions of the Proposition 2.1 must hold.

Now we treat each case separately.

Of course if $U = M_2(C)$, the ring of all 2×2 matrices over C , then it satisfies the standard identity $S_4(x_1, x_2, x_3, x_4)$ and we are done again, since $R \subseteq U$.

In the second case we have:

PROPOSITION 3.1. *If the derivation d is the inner derivation defined by a square-zero element c in U , satisfying $xc = cy = 0$ for any $x, y \in Q$, with $xy = 0$, then d vanishes identically on R .*

PROOF. Since c is an element of the left Utumi quotient ring of R , there exists a left dense ideal M of R such that $Mc \subseteq R$ (see proposition 2.1.7 in [1]).

Moreover, since R is a prime ring, IM is again a left dense ideal of R and, of course, $IMc \subseteq IR \subseteq I$.

In other words we can assume that there exists M left dense ideal of R such that $Mc \subseteq I$ and so $[Mc, Mc] \subseteq L$. Therefore for any $x, y \in M$ there exist $m = m(c, x, y) \geq 1$, $k = k(c, x, y) \geq 1$ such that

$$[d([xc, yc]^m), [xc, yc]^m]_k = 0.$$

Moreover $d([xc, yc]^m) = [c, [xc, yc]^m] = c[xc, yc]^m$. Therefore

$$\begin{aligned} 0 &= [c[xc, yc]^m, [xc, yc]^m]_k = \\ &= \sum_{h=0}^k \binom{k}{h} (-1)^h [xc, yc]^{mh} (c[xc, yc]^m)[xc, yc]^{m(k-h)} = c[xc, yc]^{m(k+1)}. \end{aligned}$$

Thus $[xc, yc][xc, yc]^{m(k+1)} = 0$, that is $[xc, yc]$ is a nilpotent quasi-invertible element. By lemma 3.1 and the main theorem in [5], either R satisfies $S_4(x_1, x_2, x_3, x_4)$ or $d([xc, yc]) = \alpha(1 + [xc, yc])$, where $\alpha \in Z(R)$.

In the first case R is a prime PI ring and so, by Posner's theorem, $RC = S = Q = U$ is a central simple algebra finite dimensional over its center C .

Since U satisfies $S_4(x_1, x_2, x_3, x_4)$, if c is a non-zero square-zero element, then we have $U = M_2(C)$, the ring of 2×2 matrices over C .

Since $xc = cy = 0$, for any $x, y \in Q = U$ such that $xy = 0$, then $e_{11}c = e_{21}c = e_{22}c = e_{12}c = 0$, that is $M_2(C)c = 0$ and so $c = 0$, a contradiction.

In the second case we know that $d([xc, yc]) = [c, [xc, yc]] = c[xc, yc]$.

$$\text{So } \alpha^2(1 + [xc, yc])^2 = (\alpha(1 + [xc, yc]))^2 = (d([xc, yc]))^2 = (c[xc, yc])^2 = 0.$$

Since $[xc, yc]$ is quasi-invertible, then $\alpha = 0$. Thus $0 = d([xc, yc]) = c[xc, yc]$. Hence, for any x, y in M , $c[xc, yc] = 0$.

Let x, y, z be in R , t in M . Since M is a left dense ideal of R , we have that xt, yt, zt fall in M and so $ztc[xtc, ytc] = 0$, that is R is GPI [2].

In this case, by Martindale's result the central closure $S = RC$ is a primitive ring, containing a minimal right ideal eS , such that eSe is a division algebra finite dimensional over C , for any minimal idempotent e of S [9, theorem 1.3.2].

If $e = 1$ then S is a finite dimensional division algebra over C . Therefore S is PI and so R is PI too. As we said above in this case $S = Q = U$ and so $c \in S$ which is a division ring. Hence $c = 0$ and consequently $d = d_c = 0$.

Now we may suppose $e \neq 1$. We known that $xc = cy = 0$, for any x, y in Q , with $xy = 0$.

Let $x^2 = 0$. Since $xc = cx = 0$ then $d(x) = [c, x] = 0$, that is c commutes with every square-zero element x in Q .

Let A be the subring generated by the elements of square zero. A is invariant under all automorphisms of Q . By our assumption there are non-trivial idempotent in the prime ring Q and so A contains a non-zero ideal J of Q by [11].

Now, since $0 = d(A) \supseteq d(J) \supseteq d(JQ) = Jd(Q)$, by the primeness of Q we obtain $d(Q) = 0$, that is $d = 0$ in Q and so in R too. ■

REMARK. The last case is the one in which Q is a reduced ring. Since Q is also a prime ring then it must be a domain. In fact, let $x, y \in Q$ be such that $xy = 0$ and $y \neq 0$. Then, for any $z \in Q$, we have $(yzx)^2 = yzxyzx = 0$ and so $yzx = 0$, that is $yQx = 0$ and $x = 0$ because Q is prime.

DEFINITION. For $a \in R$ let

$$H(a) = \{r \in R : [r, a]_m = 0 \text{ for some integer } m = m(r) \geq 1\}.$$

Of course $H(a)$ is a subring of R .

We also have:

LEMMA 3.2. *Let R be a domain of characteristic zero and let d be the derivation satisfying our assumption. If a is an element of I such that*

$$[d(a), a]_l = 0 \text{ for some } l = l(a) \geq 1$$

then $H(a)$ is invariant under the derivation d and moreover $d(a)$ is in the center of $H(a)$.

PROOF. By localizing at non-zero integers we may assume that R is an algebra over the field of the rational numbers.

By [3, assertion 2] it follows that $H(a)$ is invariant under d . Now, we put $\delta = d_a$, the inner derivation induced by a .

Of course the derivation δ restricted to $H(a)$, which we also denote δ ,

is nil and hence for any integer λ , the derivation $\lambda\delta$ is also nil on $H(a)$.

Since $H(a)$ is an algebra over the field of the rational numbers, the map $\exp(\lambda)$ is an automorphism of $H(a)$ (see [3, proposition 2]), hence the map $d_\lambda = \exp(\lambda\delta) d \exp(-\lambda\delta)$ is a derivation of $H(a)$.

Obviously $I \cap H(a)$ is a two-sided ideal of $H(a)$ which is invariant under the action of $\exp(\lambda\delta)$.

Hence $L_1 = [I \cap H(a), H(a)] \subseteq [I, R] = L$ is a Lie ideal of $H(a)$, moreover, for any $u \in L_1$, there exist some integers $n = n(u) \geq 1$, $k = k(u) \geq 1$ such that $[d_\lambda(u^n), u^n]_k = 0$.

Now, given $u \in L_1$, there exist integers $n = n(u) \geq 1$, $m = m(u) \geq 1$, $k = k(u) \geq 1$, $h = h(u) \geq 1$ such that

$$[d(u^m), u^m]_h = 0 = [d_\lambda(u^n), u^n]_k$$

hence, as in the proof of lemma 3.1, for $s = nm$ and $t = \max\{h, k\}$ we also have

$$[d(u^s), u^s]_t = [d_\lambda(u^s), u^s]_t$$

that is $[(d_\lambda - d)(u^s), u^s]_t = 0$.

By [3, proposition 2, (3)] the derivation $d_\lambda - d$ is the inner derivation induced by the element $b_\lambda = \sum_{n \geq 1} ([\delta(\lambda a), \lambda a]_{n-1})/n!$, and so b_λ is in the generalized hypercentralizer of L_1 in $H(a)$.

If $I \cap H(a)$ is the zero ideal of $H(a)$, then $a = 0$ since it is in $I \cap H(a)$ and of course $d(a) = 0 \in Z(H(a))$.

If $I \cap H(a)$ is non-zero then, by [5, proposition 4.1], either $H_{H(a)}(L_1) = Z(H(a))$ or $H(a)$ satisfies $S_4(x_1, x_2, x_3, x_4)$.

In the first case we may conclude, by a Vandermonde determinant argument, $d(a) \in Z(H(a))$.

In the other case, by localizing at the center of $H(a)$, we may assume that $H(a)$ is a division algebra of dimension at most 4 over its center $Z(H(a))$. It follows that there exists $m \geq 1$ such that $\delta^m(r) = 0$, for any $r \in H(a)$, that is δ is a nil of bounded index on $H(a)$.

By [9, lemma 1.1.9] there exists $z \in Z(H(a))$ such that $a - z$ is nilpotent and so $a - z = 0$, because $H(a)$ is a division ring. Hence $a \in Z(H(a))$. Therefore, for any $r \in H(a)$, $0 = d([r, a]) = d(ra - ar) = [r, d(a)]$, that is $d(a) \in Z(H(a))$. ■

LEMMA 3.3. Let R be a domain. For any $x, y \in I$ there exists $m = m(x, y) \geq 1$ such that $C_R([x, y]^m) = \{r \in R : [[x, y]^m, r] = 0\}$ is invariant under derivation d , that is $d(C_R([x, y]^m)) \subseteq C_R([x, y]^m)$.

PROOF. If $\text{char. } R > 0$ then, as we said in Remark 1, our assumption about the derivation d implies that for any $x, y \in R$ there exists $m = m(x, y) \geq 1$ such that $d([x, y]^m) = 0$. For any $r \in C_R([x, y]^m)$ we have

$$0 = d([[x, y]^m, r]) = [[x, y]^m, d(r)]$$

that is $d(r) \in C_R([x, y]^m)$.

Now let $\text{char. } R = 0$. For any $r \in C_R([x, y]^m)$ one has

$$0 = d([[x, y]^m, r]) = [d([x, y]^m), r] + [[x, y]^m, d(r)].$$

Since by previous lemma $d([x, y]^m) \in Z(H([x, y]^m))$ then $[[x, y]^m, d(r)] = 0$, that is $d(r) \in C_R([x, y]^m)$. ■

The last step in our proof is the following:

PROPOSITION 3.2. Let Q be a domain, then R satisfies $S_4(x_1, x_2, x_3, x_4)$.

PROOF. First we show that for all $x, y \in I$ one has:

$$[[[x, y], d([x, y])]^2, [x, y]] = 0.$$

In fact given $x, y \in I$, by previous lemma there exists an integer $m = m(x, y) \geq 1$ such that $d(C_R([x, y]^m)) \subseteq C_R([x, y]^m)$, and of course we can assume $[x, y] \neq 0$. We denote $A = C_R([x, y]^m)$, therefore $[x, y]^m$ is a non-zero element of $Z(A)$ and $I \cap A$ is a non-zero two-sided ideal of A . By localizing A at $Z(A)$ we obtain a domain D whose center is a field containing $[x, y]^m$, moreover $D = \{rz^{-1} : r \in A, z \in Z(A) - \{0\}\}$. As we said in Remark 2 d extends uniquely to a derivation on D , which we will also denote d and moreover d satisfies our assumption on D with respect to the Lie ideal $\bar{L}$ which is the localization of $[I \cap A, A] \subseteq [I, R] = L$.

Of course $[x, y]$ is invertible in D , therefore by lemma 3.1 and main result in [5], either $d([x, y]) = \alpha[x, y]$, for some $\alpha \in Z(A)$ or D satisfies $S_4(x_1, x_2, x_3, x_4)$.

In the first case $[[x, y], d([x, y])] = 0$ and a fortiori

$$[[[x, y], d([x, y])]^2, [x, y]] = 0.$$

In the second case D is a division algebra of dimension at most 4 over its center. Moreover we know that in this case, for any $a, b \in D$, $[a, b]^2 \in Z(D)$.

This implies $[[x, y], d([x, y])]^2 \in Z(A)$, because $[x, y] \in A \subseteq D$ and $d([x, y]) \in A \subseteq D$.

In particular the following holds

$$[[x, y], d([x, y])]^2, [x, y] = 0.$$

Therefore, in any case, we have

$$[[x, y], [d(x), y] + [x, d(y)]]^2, [x, y] = 0$$

for all $x, y \in I$.

In other words

$$\phi(x_1, x_2, d(x_1), d(x_2)) = [[x_1, x_2], [d(x_1), x_2] + [x_1, d(x_2)]]^2, [x_1, x_2]$$

is a differential identity for I .

Because any non-zero two-sided ideal of a prime ring R is also a dense (or rational, see [8] page 50) R -submodule of U , then, by [16, theorem 2], $\phi(x_1, x_2, d(x_1), d(x_2))$ is a differential identity for U .

By theorem 1 of [16] (or theorem 2 in [14]) it follows that either d is an inner derivation of U or U satisfies the polynomial identity

$$\phi(z_1, z_2, z_3, z_4) = [[z_1, z_2], [z_3, z_2] + [z_1, z_4]]^2, [z_1, z_2].$$

If d is an inner derivation induced by some $q \in U$ then

$$[[x, y], [q, [x, y]]]^2, [x, y] = 0$$

for all $x, y \in U$.

In particular this one holds in R and so R is a GPI-ring [2], its central closure $S = RC$ is a primitive ring having minimal right ideal, moreover, for any minimal idempotent $e = e^2 \neq 0$, eSe is a division algebra finite dimensional over its center $eCe \cong C$ [9, theorem 1.3.2].

Because $S = RC \subseteq Q$ and Q is a domain then S is a domain and so any idempotent element e of S is trivial.

This implies that S is a division algebra finite dimensional over C , that is R is a PI-ring and C is the quotient field of $Z(R)$.

It follows that $RC = S = Q = U$.

Moreover $RC = S = \overline{R} = \{rz^{-1} : r \in R, z \in Z(R) - \{0\}\}$ by Posner's theorem and so, for any $u \in \overline{L} = \{uz^{-1} : u \in L, z \in Z(R) - \{0\}\}$, there exist integers m, k such that $[d(u^m), u^m]_k = 0$ (see Remark 2).

Because d is the inner derivation in U induced by $q \in U$, we obtain that $q \in H_U(\overline{L})$, that is either $q \in Z(U)$ or U satisfies $S_4(x_1, x_2, x_3, x_4)$. In this last case we are done because $R \subseteq U$. If $q \in Z(U)$ then $d = 0$ in U , and this is a contradiction. Now we have to analyze the only case in which $\phi(z_1, z_2, z_3, z_4)$ is a polynomial identity of U . In this case $R \subseteq U$ satisfies the blended component $[[z_1, z_2], [z_3, z_2]]^2, [z_1, z_2]$ of the polynomial identity $\phi(z_1, z_2, z_3, z_4)$.

Since R is prime there exists a field F such that R and $M_k(F)$, the ring of all $k \times k$ matrices over F , satisfy the same polynomial identities (see [12]).

Suppose $k \geq 3$. Let e_{ij} the matrix unit with 1 in (i, j) entry and 0 elsewhere.

Let $z_1 = e_{13} + e_{22}$, $z_2 = e_{21} + e_{33}$, $z_3 = e_{32} + e_{31}$. By calculation we obtain

$$[z_3, z_2] = -e_{32}$$

$$[z_1, z_2] = e_{21} - e_{23} + e_{13}$$

$$[[z_1, z_2], [z_3, z_2]] = e_{22} - e_{12} + e_{31} - e_{33}$$

$$([z_1, z_2], [z_3, z_2])^2 = e_{22} - e_{12} - e_{32} - e_{31} + e_{33}$$

$$([([z_1, z_2], [z_3, z_2])^2, [z_1, z_2]) = e_{12} - e_{31} \neq 0$$

and this is a contradiction. So $k \leq 2$ and R satisfies $S_4(x_1, x_2, x_3, x_4)$. ■

At this point the proof of our theorem is complete and we state it here again for sake of clearness:

THEOREM 3.1. *Let R be a prime ring with no non-zero nil right ideals, d a non-zero derivation of R , L a non-central Lie ideal of R . If d satisfies $[d(u^m), u^m]_k = 0$ for all $u \in L$, $m = m(u) \geq 1$, $k = k(u) \geq 1$, then R satisfies $S_4(x_1, x_2, x_3, x_4)$.*

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