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Prime Graph Components of Finite Almost Simple Groups.

MARIA SILVIA LUCIDO (*)

ABSTRACT - In this paper we describe the almost simple groups G such that the prime graph $\Gamma(G)$ is not connected. We construct the prime graph $\Gamma(G)$ of a finite group G as follows: its vertices are the primes dividing the order of G and two vertices p, q are joined by an edge, and we write $p \sim q$, if there is an element in G of order pq.

1. Prime graph of almost simple groups.

If G is a finite group, we define its prime graph, $\Gamma(G)$, as follows: its vertices are the primes dividing the order of G and two vertices p, q are joined by an edge, and we write $p \sim q$, if there is an element in G of order pq.

We denote the set of all the connected components of the graph $\Gamma(G)$ by $\{\pi_i(G)$, for $i = 1, 2, \ldots, t(G)\}$ and, if the order of G is even, we denote the component containing 2 by $\pi_1(G) = \pi_1$. We also denote by $\pi(n)$ the set of all primes dividing n, if n is a natural number, and by $\pi(G)$ the set of vertices of $\Gamma(G)$.

The concept of prime graph arose during the investigation of certain cohomological questions associated with integral representations of finite groups. It turned out that $\Gamma(G)$ is not connected if and only if the augmentation ideal of G is decomposable as a module. (see [4]). In addition, nonconnectedness of $\Gamma(G)$ has relations also with the existence of isolated subgroups of G. A proper subgroup H of G is *isolated* if $H \cap$

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$\cap H^g = 1$ or H for every $g \in G$ and $C_G(h) \leqslant H$ for all $h \in H$. It was proved in [10] that G has a nilpotent isolated Hall π-subgroup whenever G is non-soluble and $\pi = \pi_i(G)$, $i > 1$. We have in fact the following equivalences:

THEOREM [8]. *If G is a finite group, then the following are equivalent:*

(1) *the augmentation ideal of G decomposes as a module,*

(2) *the group G contains an isolated subgroup,*

(3) *the prime graph of G has more than one component.*

It is therefore interesting to know when the prime graph of a group G is not connected, i.e. has more than one component. The first classification is a result of Gruenberg and Kegel.

THEOREM A [8]. *If G is a finite group whose prime graph has more than one component, then G has one of the following structures:*

(a) *Frobenius or 2-Frobenius;*

(b) *simple;*

(c) *an extension of a π_1-group by a simple group;*

(d) *simple by π_1;*

(e) *π_1 by simple by π_1.*

The case of solvable groups has been completely determined by Gruenberg and Kegel:

COROLLARY [8]. *If G is solvable with more than one prime graph component, then G is either Frobenius or 2-Frobenius and G has exactly two components, one of which consists of the primes dividing the lower Frobenius complement.*

Also the case (b) of a simple group has been described by Williams in [10], by Kondratev in [8] and by Iiyori and Yamaki in [7]. A complete list of the simple groups with more than one component can also be found in [9].

In this paper we determine the case (d). The case (d) is in fact the case of an almost simple group with more than one component. A group G is *almost simple* if there exists a finite simple non abelian group S such that $S \leqslant G \leqslant Aut(S)$. First we observe that if $\Gamma(G)$ is not connected then also $\Gamma(S)$ is not connected. Then, using the Classification of Finite

Simple Groups and the results of Williams and Kondratev, respectively in [10] and [8], we consider the various cases.

The main results are then Lemma 2 and Theorem 3, concerning sporadic and alternating groups, and Theorem 5, concerning finite simple groups of Lie type. In particular we observe that almost simple groups which are not simple have at most 4 components (see Table IV).

REMARK 1. *If $\Gamma(G)$ is not connected, and G has a non-nilpotent normal subgroup N, then $\Gamma(N)$ is not connected.*

PROOF. We suppose that $\Gamma(N)$ is connected. As $\Gamma(G)$ is not connected, there must be p in $\pi(G)$ such that $p \not\sim q$ for any q in $\pi(N)$. Let P be a p-Sylow subgroup of G. If we consider $K = NP$, where P acts on N by conjugation, then K is a Frobenius group with kernel N, that must be nilpotent, against our hypothesis. ■

In order to describe almost simple groups with non-connected prime graph it is therefore enough to consider groups G such that $G \leq Aut(S)$ and S is a simple group with non-connected prime graph. A complete description of the simple groups with non-connected prime graph can be found in [10] and [8]. We suppose $S < G$.

We use the Classification of Finite Simple Groups. Before beginning with a general study we want to treat a particular case.

LEMMA 2. *If $S = A_6$, there are four groups $G_1 = S_6$, G_2, G_3, $G_4 = Aut(A_6)$ such that $S < G_i \leq Aut(S)$. Then*

$$\Gamma(G_1) = \overset{2}{\bullet}\!-\!\overset{3}{\bullet}\quad \overset{5}{\bullet}; \qquad \Gamma(G_2) = \overset{2}{\bullet}\!-\!\overset{5}{\bullet}\quad \overset{3}{\bullet}; \qquad \Gamma(G_3) = \overset{2}{\bullet}\quad \overset{3}{\bullet}\quad \overset{5}{\bullet};$$

$$\Gamma(G_4) = \overset{3}{\bullet}\!-\!\overset{2}{\bullet}\!-\!\overset{5}{\bullet};$$

PROOF. Since $Aut(S)/Inn(S)$ is isomorphic to the Klein group with 4 elements, there are three almost simple groups over S of order $2|S|$. Let G_1, G_2, G_3 be such groups. Then $G_1 \cong S_6$ and $2 \sim 3$ in $\pi(S_6)$. Let θ be an outer automorphism of S_6 of order 2. Since $Aut(A_6) = Aut(S_6)$, we can consider G_2, the subgroup of $Aut(A_6)$ generated by $Inn(A_6)$ and θ. Then G_2 is a splitting extension of $Inn(A_6)$. Moreover θ centralizes an element of order 5 of A_6, and θ can not centralize any element of order 3 of A_6 because θ exchange the two conjugacy classes of elements of order 3 of A_6. Therefore $2 \sim 5$ in $\Gamma(G_2)$.

Let G_3 be the other subgroup of $Aut(A_6)$ of order $2\,|A_6|$. Then G_3 is a non-split extension of A_6 and there is not any involution outside $Inn(A_6)$. Therefore its prime graph is the same as that of A_6. Let G_4 be $Aut(A_6)$, then $\Gamma(G_4)$ is connected. From these observations we can deduce the structure of the prime graph of the groups G_i, $i = 1, 2, 3, 4$. ■

THEOREM 3. *If $G = Aut(S)$ with S an alternating group or a sporadic group, $S \neq A_6$, then $\Gamma(G)$ is not connected if and only if G is one of the following groups and the connected components are as follows*:

$S = A_n$, $n = p, p+1, p$ prime	$\pi_1(G) = \{2, 3, \ldots, q\}$ $q < n-1$	$\pi_2(G) = \{p\}$
$S = M_{12}$	$\pi_1(G) = \{2, 3, 5\}$	$\pi_2(G) = \{11\}$
$S = M_{22}$	$\pi_1(G) = \{2, 3, 5, 7\}$	$\pi_2(G) = \{11\}$
$S = J_3$	$\pi_1(G) = \{2, 3, 5, 17\}$	$\pi_2(G) = \{19\}$
$S = HS$	$\pi_1(G) = \{2, 3, 5, 7\}$	$\pi_2(G) = \{11\}$
$S = Sz$	$\pi_1(G) = \{2, 3, 5, 7, 11\}$	$\pi_2(G) = \{13\}$
$S = He$	$\pi_1(G) = \{2, 3, 5, 7\}$	$\pi_2(G) = \{17\}$
$S = O'N$	$\pi_1(G) = \{2, 3, 5, 7, 11, 19\}$	$\pi_2(G) = \{31\}$
$S = Fi_{22}$	$\pi_1(G) = \{2, 3, 5, 7, 11\}$	$\pi_2(G) = \{13\}$
$S = Fi'_{24}$	$\pi_1(G) = \{2, 3, 5, 7, 11, 13, 17, 23\}$	$\pi_2(G) = \{29\}$
$S = HN$	$\pi_1(G) = \{2, 3, 5, 7, 11\}$	$\pi_2(G) = \{19\}$

PROOF. If $S = A_n$, the alternating group on n letters, $n \neq 6$, we know that $G = S_n$. From [10] we observe that $\Gamma(S)$ is not connected if and only if $n = p, (p+1), (p+2)$ for some prime p. If $n = p+2$ the element $(1, 2, \ldots, p)$ commutes with the element $(p+1, p+2)$ in S_n and then $p \sim 2$ in $\Gamma(S_n)$. On the other hand if $n = p$, $p+1$ the centralizers of all the elements of order p are exactly the cyclic subgroups of order p that they generate.

If S is a sporadic group, then $|G/S| = 2$ and the result easily follows from the Atlas [2]. ■

We now suppose that $S = {}^dL_l(\overline{q})$ is a finite simple group of Lie type over the field with $\overline{q} = p^f$ elements. We recall that, in this case, the connected components $\pi_i(S)$ for $i > 1$ of $\Gamma(S)$ are exactly sets of type $\pi(|T|)$ for some maximal torus T such that T is isolated (see Lemma 5 of [10]).

We also observe that $\pi_1(S)$ is obviously contained in $\pi_1(G)$.

We use the notation and theorems of [3]. We consider Inndiag(S): it is also a group of Lie type $\tilde{S}$, in which the maximal tori $\tilde{T}$ have order $|T|d$, if $T = \tilde{T} \cap S$ and $d = |\mathrm{Inndiag}(S)/S|$. Then d divides $\bar{q}^2 - 1$ and, if $S \neq A_1(q)$, $A_2(4)$, then $\pi(\bar{q}^2 - 1) \subseteq \pi_1(S)$.

As $\tilde{T}$ is abelian, it is therefore clear that if $t \in \pi(|T|) = \pi_i(S)$ for $i > 1$, we have $t \sim s$ for $s \in \pi(\bar{q}^2 - 1)$.

If $S \neq A_1(q)$, $A_2(4)$ and G contains an element of Inndiag$(S) \setminus S$, then by the above argument we can conclude that $\Gamma(G)$ is connected.

We can now consider G such that Inndiag$(S) \cap G = S$. Let α be in $G \setminus S$, then α does not belong to Inndiag(S); we denote by π_α the set of primes dividing $|C_S(\alpha)|$. We also recall from paragraph 9 of [3] that $p \in \pi_\alpha$ for any $\alpha \in G \setminus S$ and therefore $\pi_\alpha \subseteq \pi_1(G)$ (if $S \neq A_1(q), A_2(4)$). Therefore, if r is a prime dividing $|G/S|$, then r divides $|\alpha|$ for some $\alpha \in G \setminus S$ and we have $r \sim p$, and so $r \in \pi_1(G)$.

Let $\pi_i = \pi(|T|)$ be a component of $\Gamma(S)$, for some isolated torus T. Then π_i remains a component of $\Gamma(G)$ if and only if $(|T|, |C_S(\gamma)|) = 1$ for any $\gamma \in G \setminus S$.

Then we only have to consider $\pi(|T|)$ and check if $\pi(|T|) \subseteq \subseteq \pi_1(G)$.

If $\alpha \in G \setminus S$ and $|\alpha| = r$ a prime, then, by Theorem 9.1 of [3], α is a field, a graph-field or a graph automorphism (in the sense of paragraph 7 of [3]); in the same theorem $C_S(\alpha)$ is described for α a field or a graph-field automorphism. When α is a graph automorphism, $C_S(\alpha)$ is described in the paper [1], if S is over a field of even characteristic, and in the paper [5], if S is over a field of odd characteristic. If $(|T|, |C_S(\alpha)|) \neq 1$, then $r \sim t$ for some prime $t \in \pi(|T|)$ and therefore $\pi_i(S) = \pi(|T|) \subseteq \pi_1(G)$.

We now consider $\gamma \in G \setminus S$ and m a positive integer such that $\gamma^m = \alpha$ is an automorphism of order a prime r. If $(|T|, |C_S(\alpha)|) = 1$, then $C_S(\alpha) \cap T = 1$ and, as $C_S(\gamma) \leq C_S(\alpha)$, we also have $C_S(\gamma) \cap T = 1$. Thus γ does not centralize any element x in T. If for any $\alpha \in G \setminus S$ of order a prime r we have that $(|T|, |C_S(\alpha)|) = 1$, then, for any $\gamma \in G \setminus S$, we have $C_S(\gamma) \cap T = 1$. We conclude that, in this case, $\pi_i(S) = \pi(|T|)$ is a connected component $\pi_j(G)$ for some $j > 1$. We want to state now a number theoretical lemma:

LEMMA 4. *Let m, n, f be positive integers, p, r prime numbers and $q = p^f$. Then:*

i) $(q^m - 1, q^n - 1) = q^{(m, n)} - 1$;

ii) $((q^r - 1)/(q - 1), q - 1) = (r, q - 1)$ *and* $((q^r - 1)/(q - 1)(r, q - 1), q - 1) = 1$.

PROOF.

i) See Hilfsatz 2 a) of [6].

ii) Let t be a prime dividing $q - 1$, then

$$\frac{(q^r - 1)}{(q - 1)} = q^{r-1} + q^{r-2} + \ldots + q + 1 \equiv r \equiv 0 \ (t) \Leftrightarrow t = r .$$

For the second statement we observe that, if $(r, q - 1) = 1$, we can conclude by applying the first statement. Otherwise, as r is a prime, $(r, q - 1) = r$ and therefore if $q = 1 + (r)m$, for a positive integer m, we have

$$\frac{q^{r-1} + q^{r-2} + \ldots + q + 1}{l+1} = \frac{(1 + (r)\, m)^l + \ldots + (1 + (r)\, m) + 1}{r} =$$

$$= \frac{(r) + (r)\, m \left(\sum_{j=1}^{l} \binom{j}{1} \right) + (r)^2 s}{(r)} =$$

$$= 1 + mr(r-1)/2 + rs \not\equiv 0 \quad (r),$$

where s is a positive integer. ■

THEOREM 5. *Let $S < G \leqslant Aut(S)$ with S a finite simple group of Lie type, then $\Gamma(G)$ is not connected if and only if G is one of the groups described in the Tables* I, II, III, IV.

PROOF. The proof is made by a case by case analysis. For the connected components of $\Gamma(S)$, S a finite simple group of Lie type, we refer, without further reference, to [10] and [8]. From the remarks preceding Lemma 4, it is therefore enough to consider automorphisms α of S of order a prime r.

Type A_l

If $S = A_l(\overline{q})$ with $l > 1$, $S \neq A_2(2), A_2(4)$, then $\Gamma(S)$ is not connected if and only if:

TABLE I. – *Connected components not containing 2 of $\Gamma(G)$ with $t(G)=2$ and $S<G\leqslant Aut(S)$, S finite simple group of Lie type, not twisted, $G/S\cong\langle\gamma\rangle\times\langle\beta\rangle$ with γ a field automorphism and β a graph automorphism.*

S	$\lvert\gamma\rvert$	$\lvert\beta\rvert$	$\pi_2(G)$
$A_l(\overline{q})$, $l+1$ an odd prime, $\overline{q}=q^n$, $n=(l+1)^s$	n	1	$\pi((\overline{q}^l+\overline{q}^{l-1}+\ldots+1)/(\overline{q}-1,\,l+1))$
$A_l(\overline{q})$, $l+1$ an odd prime	1	2	$\pi((\overline{q}^l+\overline{q}^{l-1}+\ldots+1)/(\overline{q}-1,\,l+1))$
$A_l(\overline{q})$, $l+1$ an odd prime, $\overline{q}=q^n$, $n=(l+1)^s$	n	2	$\pi((\overline{q}^l+\overline{q}^{l-1}+\ldots+1)/(\overline{q}-1,\,l+1))$
$A_l(\overline{q})$, l an odd prime, $(\overline{q}-1)\mid(l+1)$, $\overline{q}=q^n$, $n=l^s$	n	1	$\pi(\overline{q}^{l-1}+\overline{q}^{l-2}+\ldots+1)$
$A_l(\overline{q})$, l an odd prime, $(\overline{q}-1)\mid(l+1)$	1	2	$\pi(\overline{q}^{l-1}+\overline{q}^{l-2}+\ldots+1)$
$A_l(\overline{q})$, l an odd prime, $(\overline{q}-1)\mid(l+1)$, $\overline{q}=q^n$, $n=l^s$	n	2	$\pi(\overline{q}^{l-1}+\overline{q}^{l-2}+\ldots+1)$
$A_2(4)$	2	1	$\{5\}$
$A_1(\overline{q})$, $\overline{q}=q^n$, $n=2^s$	n	1	$\pi((\overline{q}+1)/(2,\overline{q}-1))$
$A_1(\overline{q})$, $\overline{q}=2^f$ or 3^f, f an odd prime	f	1	$\pi((\overline{q}-1)/(2,\overline{q}-1))$
$B_l(\overline{q})$, $l=2^m$, $\overline{q}=q^n$, $n=2^s$	n	1	$\pi((\overline{q}^l+1)/(2,\overline{q}-1))$
$D_l(\overline{q})$, l an odd prime, $\overline{q}=2,3,5$	1	2	$\pi((\overline{q}^l-1)/(4,\overline{q}^l-1))$
$E_6(\overline{q})$, $\overline{q}=q^n$, $n=3^s$	n	1	$\pi((\overline{q}^6+\overline{q}^3+1)/(\overline{q}-1,\,3))$
$E_6(\overline{q})$,	1	2	$\pi((\overline{q}^6+\overline{q}^3+1)/(\overline{q}-1,\,3))$
$E_6(\overline{q})$, $\overline{q}=q^n$, $n=3^s$	n	2	$\pi((\overline{q}^6+\overline{q}^3+1)/(\overline{q}-1,\,3)$
$E_8(\overline{q})$, $\overline{q}=q^n$, $n=2^s3^t5^v$	n	1	$\pi(\overline{q}^8+\overline{q}^7-\overline{q}^5-\overline{q}^4-\overline{q}^3+\overline{q}+1)$
$F_4(\overline{q})$, $\overline{q}$ odd, $\overline{q}=q^n$, $n=2^u3^w$, $u,\,w\geqslant 0$, $u+w>0$	n	1	$\pi(\overline{q}^4-\overline{q}^2+1)$
$F_4(\overline{q})$, $\overline{q}=2^m=q^n$, $n=2^u3^t$, $u\geqslant 0$	n	1	$\pi(\overline{q}^4-\overline{q}^2+1)$
$F_4(\overline{q})$, $\overline{q}=2^m$, m odd	1	2	$\pi(\overline{q}^4+1)$
$G_2(\overline{q})$, $\overline{q}=q^n$, $n=3^t$, $\overline{q}\equiv 1\ (3)$	n	1	$\pi(\overline{q}^2-\overline{q}+1)$
$G_2(\overline{q})$, $\overline{q}=q^n$, $n=3^t$, $\overline{q}\equiv -1\ (3)$	n	1	$\pi(\overline{q}^2+\overline{q}+1)$
$G_2(\overline{q})$, $\overline{q}=q^n$, $n=2^s3^u$, $u\geqslant 0$, $\overline{q}\equiv 0$ or $1\ (3)$	n	1	$\pi(\overline{q}^2-\overline{q}+1)$
$G_2(\overline{q})$, $\overline{q}=3^m$, m odd	1	2	$\pi(\overline{q}^2+\overline{q}+1)$

we always suppose $s,\,t,\,v>0$.

TABLE II. – *Connected components not containing* 2 *of* $\Pi(G)$ *with* $t(G) = 2$ *and* $S < G \leqslant Aut(S)$, *S finite simple group of Lie type, twisted,* $G/S \cong \langle\gamma\rangle$.

S	$\lvert\gamma\rvert$	$\pi_2(G)$
${}^2A_l(\overline{q}^2)$, $l+1$ a prime, $\overline{q}=q^n$, $n=2^s(l+1)^t$	n	$\pi((-\overline{q})^l+(-\overline{q})^{l-1}+\ldots+1)/(\overline{q}+1,l+1))$
${}^2A_l(\overline{q}^2)$, l an odd prime, $(\overline{q}+1)\mid(l+1)$, $S\neq{}^2A_5(4)$, ${}^2A_3(9)$, $\overline{q}=q^n$, $n=2^s l^t$	n	$\pi((-\overline{q})^{l-1}+(-\overline{q})^{l-2}+\ldots+1))$
${}^2A_3(2^2)$	2	$\{5\}$
${}^2D_l(\overline{q}^2)$, $l=2^n$, $\overline{q}=q^n$, $n=2^s$	n	$\pi((\overline{q}^l+1)/(\overline{q}^l+1,4))$
${}^2D_l(3^2)$, l an odd prime	2	$\pi((3^l+1)/4)$
${}^2E_6(\overline{q}^2)$, $\overline{q}=q^n$, $n=2^s3^t$	n	$\pi((\overline{q}^6-\overline{q}^3+1)/(\overline{q}+1,3))$
${}^2F_4(2)'$	2	$\{13\}$
${}^3D_4(\overline{q}^2)$, $\overline{q}=q^n$, $n=2^s3^t$	n	$\pi(\overline{q}^4-\overline{q}^2+1)$

we always suppose $s, t \geqslant 0$ and $s+t>0$.

TABLE III. – *Connected components not containing* 2 *of* $\Gamma(G)$ *with* $t(G)=2$ *and* $S<G\leqslant\leqslant Aut(S)$, *S finite simple group of Lie type,* $G/S\cong\langle\gamma\rangle$.

S	$\lvert\gamma\rvert$	
$E_8(\overline{q}),\ \overline{q}=q^n,\ n=2^s3^t$	n	$\pi_2(G)=\pi(\overline{q}^8-\overline{q}^4+1)$ $\pi_3(G)=\pi(\overline{q}^8+\overline{q}^7-\overline{q}^5-\overline{q}^4-\overline{q}^3+\overline{q}+1)$
$E_8(\overline{q}),\ \overline{q}=q^n,\ n=2^s5^t$	n	$\pi_2(G)=\pi(\overline{q}^8+\overline{q}^7-\overline{q}^5-\overline{q}^4-\overline{q}^3+\overline{q}+1)$ $\pi_3(G)=\pi(\overline{q}^8-\overline{q}^6+\overline{q}^4-\overline{q}^2+1)$
$E_8(\overline{q}),\ \overline{q}=q^n,\ n=3^s5^t$	n	$\pi_2(G)=\pi(\overline{q}^8+\overline{q}^7-\overline{q}^5-\overline{q}^4-\overline{q}^3+\overline{q}+1)$ $\pi_3(G)=\pi(\overline{q}^8-\overline{q}^7+\overline{q}^5-\overline{q}^4+\overline{q}^3-\overline{q}+1)$
$E_8(\overline{q}),\ \overline{q}=q^n,\ n=5^s,\ \overline{q}\equiv 2,3\ (5)$	n	$\pi_2(G)=\pi(\overline{q}^8+\overline{q}^7-\overline{q}^5-\overline{q}^4-\overline{q}^3+\overline{q}+1)$ $\pi_3(G)=\pi(\overline{q}^8-\overline{q}^7+\overline{q}^5-\overline{q}^4+\overline{q}^3-\overline{q}+1)$
$F_4(\overline{q}),\ \overline{q}=2^m=q^n$, m even, $n=2^s$	n	$\pi_2(G)=\pi(\overline{q}^4-\overline{q}^2+1)$ $\pi_3(G)=\pi(\overline{q}^4+1)$
$G_2(\overline{q}),\ \overline{q}=3^m=q^n$, m even, $n=3^s$	n	$\pi_2(G)=\pi(\overline{q}^2-\overline{q}+1)$ $\pi_3(G)=\pi(\overline{q}^2+\overline{q}+1)$
${}^2A_3(3^2)$	2	$\pi_2(G)=\{5\}$ $\pi_3(G)=\{7\}$
${}^2A_5(2^2)$	2	$\pi_2(G)=\{7\}$ $\pi_3(G)=\{11\}$
${}^2F_4(\overline{q}^2),\ \overline{q}=q^n,\ n=3^s$	n	$\pi_2(G)=\pi(\overline{q}^4-\sqrt{2}\overline{q}^3+\overline{q}^2-\sqrt{2}\overline{q}+1)$ $\pi_3(G)=\pi(\overline{q}^4+\sqrt{2}\overline{q}^3+\overline{q}^2+\sqrt{2}\overline{q}+1)$
${}^2G_2(\overline{q}^2),\ \overline{q}=q^n,\ n=3^s$	n	$\pi_2(G)=\pi(\overline{q}^2-\sqrt{3}\overline{q}+1)$ $\pi_3(G)=\pi(\overline{q}^2+\sqrt{3}\overline{q}+1)$

we always suppose $s,t>0$.

TABLE IV. – *Connected components not containing* 2 *of* $\Gamma(G)$ *with* $t(G)=2$ *and* $S<G\leqslant Aut(S)$, *S finite simple group of Lie type,* $G/S\cong\langle\gamma\rangle$.

S	$\lvert\gamma\rvert$	
$E_8(\overline{q}),\ \overline{q}=q^n,\ n=2^s$	n	$\pi_2(G)=\pi(\overline{q}^8-\overline{q}^4+1)$ $\pi_3(G)=\pi(\overline{q}^8+\overline{q}^7-\overline{q}^5-\overline{q}^4-\overline{q}^3+\overline{q}+1)$ $\pi_4(G)=\pi(\overline{q}^8-\overline{q}^6+\overline{q}^4-\overline{q}^2+1)$
$E_8(\overline{q}),\ \overline{q}=q^n,\ n=3^s$	n	$\pi_2(G)=\pi(\overline{q}^8-\overline{q}^4+1)$ $\pi_3(G)=\pi(\overline{q}^8+\overline{q}^7-\overline{q}^5-\overline{q}^4-\overline{q}^3+\overline{q}+1)$ $\pi_4(G)=\pi(\overline{q}^8-\overline{q}^7+\overline{q}^5-\overline{q}^4+\overline{q}^3-\overline{q}+1)$
$E_8(\overline{q}),\ \overline{q}=q^n,\ n=5^s$	n	$\pi_2(G)=\pi(\overline{q}^8+q^7-\overline{q}^5-\overline{q}^4-\overline{q}^3+\overline{q}+1)$ $\pi_3(G)=\pi(\overline{q}^8-\overline{q}^7+\overline{q}^5-\overline{q}^4+\overline{q}^3-\overline{q}+1)$ $\pi_4(G)=\pi(\overline{q}^8-\overline{q}^6+\overline{q}^4-\overline{q}^2+1)$ if $\overline{q}\equiv 0,1,4\ (5)$

we suppose $s>0$.

i) $l+1$ is a prime and in this case $\pi_2(S) = \pi(|T|) = \pi((\overline{q}^l + \overline{q}^{l-1} + \ldots + 1)/d)$ where $d = (\overline{q}-1, l+1)$;

ii) l is an odd prime and $(\overline{q}-1) \mid (l+1)$ and in this case

$$\pi_2(S) = \pi(|T|) = \pi(\overline{q}^{l-1} + \overline{q}^{l-2} + \ldots + 1).$$

We study the different automorphisms of A_l.

i) If α is a field automorphism, then by Theorem 9.1 of [3], $\pi_\alpha = \pi(|A_l(q)|)$, where $\overline{q} = q^r$.

If $r \neq l+1$, then r and $l+1$ are two distinct primes and therefore

$$\frac{(q^{l+1}-1)}{(q-1)(q-1, l+1)} \text{ divides } \frac{(q^{r(l+1)}-1)}{(q^r-1)\, d} = \frac{(\overline{q}^l + \overline{q}^{l-1} + \ldots + \overline{q} + 1)}{d}.$$

This proves that $\Gamma(G)$ is connected.

If $r = l+1$, then $(|T|, |A_l(q)|) = 1$. In fact by Lemma 4 i), we know that, for any $i \leqslant r$,

$$\left(\frac{(q^{rl} + q^{r(l-1)} + \ldots + 1)}{d} = \frac{(q^{(l+1)^2}-1)}{(q^{l+1}-1)d}, q^i - 1\right) = 1.$$

Since $(q^{(l+1)^2}-1)/(q^{l+1}-1)\, d = (\overline{q}^{l+1}-1)/(\overline{q}-1)(l+1, \overline{q}-1)$ by Lemma 4 ii), we have

$$\left(\frac{(\overline{q}^{l+1}-1)}{(\overline{q}-1)(l+1, \overline{q}-1)}, \overline{q}-1\right) = 1.$$

We can conlcude that $\pi_2(S) = \pi_2(G)$.

If α is a graph-field automorphism, then by Theorem 9.1 of [3], $\pi_\alpha = \pi(|{}^2A_l(q)|)$, where $\overline{q} = q^r$ and $r = 2$. Then $(q^{l+1}+1)/(q+1)(q+1, l+1)$ divides both $|{}^2A_l(q)|$ and $|T| = (q^{2(l+1)}-1)/(q^2-1)(q^2-1, l+1)$ and so $\Gamma(G)$ is connected.

If α is a graph automorphism, then $r = 2$ and by Theorems 19.9 of [1] and 4.27 of [5], $\pi_\alpha = \pi(|B_m(\overline{q})|) = \pi(\overline{q}(\overline{q}^2-1)(\overline{q}^4-1)\ldots(\overline{q}^{2m}-1))$ if $l+1 = 2m+1$.

We already know that $|T| = (\overline{q}^{l+1}-1)/(\overline{q}-1)(\overline{q}-1, l+1)$ is coprime with all the primes in π_α because π_α is contained in $\pi_1(S)$. So in this case we have that $\pi_2(S) = \pi_2(G)$.

ii) The proof is similar to the one of i). We obtain that $\Gamma(G)$

is connected, except in the cases in which α is a field automorphism and $r = l$, or α is a graph automorphism of order 2.

$S = A_2(2)$ admits only a graph automorphism α and $\pi_\alpha = \{2, 3\}$ and then $\pi_1(G) = \{2, 3\}$, $\pi_2(G) = \pi(2^2 + 2 + 1) = \{7\}$. $S = A_2(4)$: if α is a diagonal automorphism then $G \geqslant PGL(3, 4)$ and in this case $\Gamma(G)$ is connected.

If α is a field automorphism, then $r = 2$ and $\pi_\alpha = \{2, 3, 7\}$ and so $\pi_1(G) = \{2, 3, 7\}$, $\pi_2(G) = \{5\}$.

If α is a graph-field or a graph automorphism, then $r = 2$ and $\pi_\alpha = \{2, 3, 5\}$ and so $\pi_1(G) = \{2, 3, 5\}$, $\pi_2(G) = \pi((4^2 + 4 + 1)/3) = \{7\}$. $S = A_1(\overline{q})$: if α is a diagonal automorphism of order 2, then $\overline{q}$ is odd. If $G = PGL(2, \overline{q})$ then $\pi_1(G) = \pi(G) \backslash \{p\}$, $\pi_2(G) = \{p\}$, because 2 divides the order of every maximal torus T of G.

If α is a field automorphism, $\overline{q} = q^r$, $\pi_\alpha = \pi(q(q^2 - 1))$. If $(q - 1)/(2, q - 1) \neq 1$ then

$$1 \neq \frac{q-1}{(2, q-1)} \text{ divides } \frac{\overline{q}-1}{(2, \overline{q}-1)}.$$

If $(q - 1)/(2, q - 1) = 1$, then $q = 2$ or 3, and in this case $\pi(|T_2|) = \pi(\overline{q} - 1)/(2, \overline{q} - 1) = \pi_2(G)$. Moreover if $r \neq 2$, then

$$1 \neq \frac{q+1}{(2, q-1)} \text{ divides } \frac{\overline{q}+1}{(2, \overline{q}-1)}.$$

Therefore if $q \neq 2, 3$ and $r \neq 2$, $\Gamma(G)$ is connected, while if $r = 2$ we have $\pi_1(G) = \pi(q(q^2 - 1))$ and $\pi_2(G) = \pi((q^2 + 1)/(2, q - 1))$.

If α is a graph automorphism, then $r = 2$ and $\pi_\alpha = \{2\}$, so that $\pi_i(G) = \pi_i(S)$ for $i = 1, 2, 3$.

Type B_l

If $S = B_l(\overline{q})$, then $\Gamma(S)$ is not connected if and only if:

i) l is an odd prime and $\overline{q} = 2, 3$; in this case $\pi_2(S) = \pi(|T|) = \pi(\overline{q}^{l-1} + \overline{q}^{l-2} \ldots + 1)$.

ii) $l = 2^n$; in this case $\pi_2(S) = \pi(|T|) = \pi((\overline{q}^l + 1)/d)$ where $d = (\overline{q} - 1, 2)$.

i) In this case $Aut(S) = \text{Inndiag}(S)$ and so there is nothing else to prove.

ii) If α is a field automorphism, then $\pi_\alpha = \pi(|B_l(q)|)$ and $\overline{q} = q^r$.

If r is odd, $(q^l+1)/d$ divides both $|C_S(\alpha)|$ and $(\bar{q}^l+1)/d$ and then $\Gamma(G)$ is connected.

If $r=2$, $(\bar{q}^l+1)/d=(q^{2l}+1)/d$ is coprime with all the primes in π_α, because π_α is contained in $\pi_1(S)$. So in this case we have that $\pi_2(S)=\pi_2(G)$.

If $l=2$ and $p=2$, then α can also be a graph automorphism of order 2. Then $\pi_\alpha=\pi(|{}^2B_2(\bar{q})|)=\pi(\bar{q}(\bar{q}^2-1)(\bar{q}^4+1))$ and then $\Gamma(G)$ is connected.

By proposition 19.5 of [1], we have thus described the centralizers of all $\alpha\in G\backslash S$.

Type D_l

If $S=D_l(\bar{q})$, then $\Gamma(S)$ is not connected if and only if:

i) l is an odd prime and $\bar{q}=2,3,5$ and in this case

$$\pi_2(S)=\pi(|T|)=\pi((\bar{q}^l-1)/(4,\bar{q}^l-1))\,;$$

ii) $l-1$ is an odd prime and $\bar{q}=2,3$ and in this case

$$\pi_2(S)=\pi(|T|)=\pi((\bar{q}^{l-1}-1)/(2,\bar{q}-1))\,.$$

If $l\neq 4$, then the only automorphism α that we have to consider is a graph automorphism of order 2, then $\pi_\alpha=\pi(|B_{l-1}(\bar{q})|)=\pi(\bar{q}(\bar{q}^2-1)\dots(\bar{q}^{2(l-1)}-1))$. Therefore in case i) we have that $\pi_2(S)=\pi_2(G)$ and in case ii) $\Gamma(G)$ is connected.

If $l=4$, we have to consider also a graph automorphism of order 3. In this case, by Theorem 9.1(3) of [3], in $Aut(S)$ there are two conjugacy classes of subgroups of order 3 generated by a graph automorphism. We denote these two graph automorphisms by α and β. Then β is obtained from α by multiplying it with an element of order 3 of S, that is $\beta=g\alpha$, $g\in S$. Therefore, as $\pi_\beta\subseteq\pi_\alpha=\pi(|G_2(\bar{q})|)=\pi(\bar{q}(\bar{q}^6-1))$ and $\pi_2(S)=\pi(\bar{q}^3-1)$, we have that, in this case, $\Gamma(G)$ is connected.

Type E_6

If $S=E_6(\bar{q})$, then $\Gamma(S)$ is not connected and $\pi_2(S)=\pi(|T|)=\pi((\bar{q}^6+\bar{q}^3+1)/d)$ where $d=(\bar{q}-1,3)$.

If α is a field automorphism, then $\bar{q}=q^r$ and $\pi_\alpha=\pi(|E_6(q)|)$ ([3], Theorem 9.1). If $r \neq 3$, then

$$\frac{(q^6+q^3+1)}{d} = \frac{(q^9-1)}{d(q^3-1)} \text{ divides } \frac{(q^{9r}-1)}{d(q^{3r}-1)} = \frac{(\overline{q}^9-1)}{d(\overline{q}^3-1)} = \frac{(\overline{q}^6+\overline{q}^3+1)}{d}.$$

Therefore $\Gamma(G)$ is connected.

If $r=3$, then $\pi_\alpha = \pi(q(q^5-1)(q^8-1)(q^9-1)(q^{12}-1))$. As $(q^{18}+q^9+1)/d$ divides $(q^{27}-1)/(q^9-1)$, by Lemma 4 i) it is clear that $((q^{18}+q^9+1)/d, (q^5-1)(q^8-1)) = 1$. Moreover, if we apply Lemma 4 ii) to q^9, we have that $(q^{18}+q^9+1)/d$ is coprime with q^9-1. Finally $(q^{18}+q^9+1)/d = (\overline{q}^9-1)/(\overline{q}^3-1)\, d$, $\overline{q}^4-1 = q^{12}-1$ and $((\overline{q}^9-1)/(\overline{q}^3-1)\, d, \overline{q}^4-1) = 1$ again by Lemma 4 i). So in this case we have that $\pi_2(S) = \pi_2(G)$. If α is a graph-field automorphism, then $r=2$, $\overline{q} = q^2$ and $\pi_\alpha = \pi(|{}^2E_6(\overline{q})|)$ ([3], Theorem 9.1). We observe that $|T| = (q^{12}+q^6+1)/d = (q^6+q^3+1)(q^6-q^3+1)/d$ and (q^6-q^3+1) divides $|{}^2E_6(q^2)|$. Then $\Gamma(G)$ is connected. If α is a graph automorphism, by Lemma 4.25 c) of [5] and 19.9 iii) of [1], we have that $\pi_\alpha \subseteq \pi(\overline{q}(\overline{q}^8-1)(\overline{q}^{12}-1))$. Since $|T|$ is coprime with all the primes in π_α, we have that $\pi_2(S) = \pi_2(G)$.

Type E_7

If $S = E_7(\overline{q})$, then $\Gamma(S)$ is not connected if and only if $\overline{q} = 2, 3$ and in this case S admits only a diagonal automorphism of order 2, and then there is nothing else to prove.

Type E_8

If $S = E_8(\overline{q})$, then $\Gamma(S)$ is not connected and

$$\pi_2(S) = \pi(|T_0|) = \pi(x(\overline{q}) = \overline{q}^8 - \overline{q}^4 + 1),$$
$$\pi_3(S) = \pi(|T_1|) = \pi(y_1(\overline{q}) = \overline{q}^8 + \overline{q}^7 - \overline{q}^5 - \overline{q}^4 - \overline{q}^3 + \overline{q} + 1),$$
$$\pi_4(S) = \pi(|T_2|) = \pi(y_2(\overline{q}) = \overline{q}^8 - \overline{q}^7 + \overline{q}^5 - \overline{q}^4 + \overline{q}^3 - \overline{q} + 1),$$

moreover, if $\overline{q} \equiv 0, 1, 4\ (5)$,

$$\pi_5(S) = \pi(|T_3|) = \pi(z(\overline{q}) = \overline{q}^8 - \overline{q}^6 + \overline{q}^4 - \overline{q}^2 + 1).$$

In this case α can only be a field automorphism ([3], Theorem 9.1), $\overline{q} = q^r$ and

$$\pi_\alpha = \pi(|E_8(q)|) = \pi(q(q^{14}-1)(q^{18}-1)(q^{20}-1)(q^{24}-1)(q^{30}-1)).$$

We observe that

$$x(\overline{q}) = \frac{(\overline{q}^{12}+1)}{(\overline{q}^4+1)}; \qquad y_1(\overline{q}) = \frac{(\overline{q}^{10}-\overline{q}^5+1)}{(\overline{q}^2-\overline{q}+1)};$$

and

$$y_2(\overline{q}) = \frac{(\overline{q}^{10}+\overline{q}^5+1)}{(\overline{q}^2+\overline{q}+1)}; \qquad z(\overline{q}) = \frac{(\overline{q}^{10}+1)}{(\overline{q}^2+1)}.$$

If $r \neq 2, 3, 5$ we prove that $\Gamma(G)$ is connected. In fact

$$r \neq 2, 3 \Rightarrow \frac{(q^{12}+1)}{(q^4+1)} \quad \text{divides } x(\overline{q});$$

$$r \neq 2, 5 \Rightarrow \frac{(q^{10}+1)}{(q^2+1)} \quad \text{divides } z(\overline{q});$$

$$r \neq 2, 3 \Rightarrow \frac{(q^{15}+1)}{(q^5+1)} = q^{10}-q^5+1 \quad \text{divides } \overline{q}^{10}-\overline{q}^5+1.$$

Since $y_1(\overline{q}) = (\overline{q}^{10}-\overline{q}^5+1)/(\overline{q}^2-\overline{q}+1)$ we have to prove that $\overline{q}^2-\overline{q}+ +1 = q^{2r}-q^r+1$ is coprime with $(q^{10}-q^5+1)/(q^2-q+1)$. In fact

$$r \neq 5 \Rightarrow \left(\frac{(q^{3r}+1)}{(q^r+1)}, q^{15}+1\right) = (q^{2r}-q^r+1, q^3+1) = q^2-q+1 \Rightarrow$$

$$\left(q^{2r}-q^r+1, \frac{(q^{15}+1)}{q^2-q+1}\right) = (q^{2r}-q^r+1, q+1) = (3, q+1).$$

Finally, since 3 does not divide $(q^{10}-q^5+1)/(q^2-q+1)$, we have proved the above statement and also that $(q^{10}-q^5+1)/(q^2-q+1)$ divides $y_1(\overline{q})$.

We can prove in a similar way that $(q^{10}+q^5+1)/(q^2+q+1)$ divides $y_2(\overline{q})$: in this case it is enough $r \neq 3, 5$. We can conclude that $\Gamma(G)$ is connected.

We suppose now that $r = 2$. Then $y_2(\overline{q}) = (q^{20}+q^{10}+1)/(q^4+q^2+ +1)$ divides $q^{30}-1$ and therefore $\pi_4(S) \subseteq \pi_1(G)$.

We want to prove that $x(\overline{q})=(q^{24}+1)/(q^8+1)$ is coprime with $|C_S(\alpha)|$. We observe that $(x(\overline{q}),\, q^{24}-1)=1$ and therefore

$$(x(\overline{q}),(q^{14}-1)(q^{18}-1)(q^{30}-1)(q^{20}-1))=1\,.$$

In a similar way we can prove that $z(\overline{q})=(q^{20}+1)/(q^4+1)$ is coprime with $|C_S(\alpha)|$.

Now we consider $y_1(\overline{q})$ which is a divisor of $(q^{30}+1)/(q^{10}+1)$. We observe that

$$(y_1(\overline{q}),(q^{14}-1)(q^{18}-1)(q^{30}-1)(q^{20}-1))=1\,.$$

Moreover $(y_1(\overline{q}),\, q^{24}-1)=(y_1(\overline{q}),\, q^6+1)$ and therefore, since

$$y_1(\overline{q})=(q^{30}+1)/(q^{10}+1)(q^4-q^2+1)=$$
$$=(q^{30}+1)/(q^6+1)\,s, \qquad s \text{ a positive integer},$$

we have that $(y_1(\overline{q}),\, q^6+1)=1$. We have thus proved that $y_1(\overline{q})$ is coprime with $|C_S(\alpha)|$. Therefore for $r=2$, we have that $\pi_2(G)=\pi_2(S)$, $\pi_3(G)=\pi_3(S)$. Moreover if $r=2$, then $\overline{q}\equiv 0, 1, 4$ (5) and therefore $\pi_4(G)=\pi_5(S)$.

The proof for the cases $r=3$ and 5 are similar to the previous one.

Type F_4

If $S=F_4(\overline{q})$, then $\Gamma(S)$ is not connected and:

i) if $\overline{q}$ is odd, then $\pi_2(S)=\pi(|T|)=\pi(\overline{q}^4-\overline{q}^2+1)$;

ii) if $\overline{q}$ is even, then $\pi_2(S)=\pi(|T|)=\pi(\overline{q}^4-\overline{q}^2+1)$ and $\pi_3(S)=$ $=\pi(|T_1|)=\pi(\overline{q}^4+1)$.

i) α must be a field automorphism, $\overline{q}=q^r$ and $\pi_\alpha=\pi(|F_4(q)|)=$ $=\pi(q(q^8-1)(q^{12}-1))$. We observe that $\overline{q}^4-\overline{q}^2+1=(q^{6r}+1)/(q^{2r}+$ $+1)$.

If $r\neq 2, 3$, then $(q^6+1)/(q^2+1)$ divides $\overline{q}^4-\overline{q}^2+1$. So in this case $\Gamma(G)$ is connected.

If $r=2$ or 3, then $\pi_\alpha\cap\pi_2(S)$ is empty and therefore $\pi_2(S)=$ $=\pi_2(G)$.

ii) If α is a field automorphism, $\overline{q}=q^r$ and $\pi_\alpha=\pi(|F_4(q)|)=$ $=\pi(q(q^8-1)(q^{12}-1))$ and if $r\neq 2$, then q^4+1 divides $q^{4r}+1$ and so $\pi_3(S)\subseteq\pi_1(G)$.

For the component $\pi_2(S)$, the proof is exactly the same as in part i).

If $r=2$, then $(\overline{q}^4+1)=q^8+1$ is coprime with all the primes in π_α; so in this case $\pi_3(S) = \pi_3(G)$. By Proposition 19.5 of [1], it is now enough to consider α a graph-field automorphism, $\overline{q} = q^2 = 2^m$ with m odd and $\pi_\alpha = \pi(|{}^2F_4(\overline{q})|) = \pi(\overline{q}(\overline{q}^3+1)(\overline{q}^4-1)(\overline{q}^6+1))$. As $\overline{q}^4 - \overline{q}^2 + 1$ divides $\overline{q}^6 + 1$, we have that $\pi_2(S) \subseteq \pi_1(G)$, while $(\overline{q}^4+1) = q^8+1$ is coprime with all the primes in π_α; so in this case $\pi_3(S) = \pi_2(G)$.

Type G_2

If $S = G_2(\overline{q})$, then $\Gamma(S)$ is not connected and:

i) if $\overline{q} \equiv 1$ (3), then $\pi_2(S) = \pi(|T|) = \pi(\overline{q}^2 - \overline{q} + 1)$;

ii) if $\overline{q} \equiv -1$ (3), then $\pi_2(S) = \pi(|T|) = \pi(\overline{q}^2 + \overline{q} + 1)$;

iii) if $\overline{q} \equiv 0$ (3), then $\pi_2(S) = \pi(|T|) = \pi(\overline{q}^2 - \overline{q} + 1)$ and $\pi_3(S) = \pi(|T|) = \pi(\overline{q}^2 + \overline{q} + 1)$. If α is a field automorphism, $\overline{q} = q^r$ and $\pi_\alpha = \pi(|G_2(q)|) = \pi(q(q^6-1))$. If $r \neq 2, 3$ then $(q^3+1)/(q+1)$ divides $(q^{3r}+1)/(q^r+1) = (\overline{q}^2 - \overline{q} + 1)$ and $(q^3-1)/(q-1)$ divides $(q^{3r}-1)/(q^r-1) = (\overline{q}^2 + \overline{q} + 1)$ and so $\Gamma(G)$ is connected, in any of the three cases.

i) Let α be a field automorphism. If α has order $r = 2$, $\overline{q} = q^2$ and $\overline{q}^2 - \overline{q} + 1 = q^4 - q^2 + 1$ divides $q^6 + 1$ and is therefore coprime with $|G_2(q)|$. Similarly if α has order 3. So if $r = 2, 3$ we have that $\pi_2(S) = \pi_2(G)$.

ii) Let α be a field automorphism. If α has order $r = 2$, $\overline{q} = q^2$ and $\overline{q}^2 + \overline{q} + 1 = q^4 + q^2 + 1$. Since $q^2 + q + 1$ divides both $q^4 + q^2 + 1$ and $|G_2(q)|$, we have $\pi_2(S) \subseteq \pi_1(G)$. If α has order $r = 3$, we use the same argument of i) and conclude that $\pi_2(S) = \pi_2(G)$.

iii) Let α be a field automorphism. If α has order $r = 2$, as in case i), we have that $\pi_2(S) = \pi_2(G)$, while, as in case ii), $\pi_3(S) \subseteq \pi_1(G)$. If $r = 3$, we use the same argument of i) and conclude that $\pi_2(S) = \pi_2(G)$ and $\pi_3(S) = \pi_3(G)$.

If α is a graph-field automorphism of order $r = 2$, then $\overline{q} = q^2 = 3^n$, n an odd integer and $\pi_\alpha = \pi(|{}^2G_2(q^2)|) = \pi(q(q^6+1)(q^2-1))$. As $\overline{q}^2 - \overline{q} + 1 = q^4 - q^2 + 1$ divides $q^6 + 1$, we have $\pi_2(S) \subseteq \pi_1(G)$, while $\pi_3(S) = \pi_2(G)$.

By Lemma 4.22 of [5] and 19.2 of [1], we have thus examined the centralizers of all automorphisms of S. We now consider the twisted finite simple groups of Lie type. By the hypothesis that $G \cap \mathrm{Inndiag}(S) = 1$, we obtain that, in this case, $G/S \cong \langle \gamma \rangle$ and therefore we consider again an automorphism α of order a prime r. We suppose $S \neq {}^3D_4(\overline{q})$; if $r \neq 2$, α

is a field automorphism, if $r=2$ then α is a graph automorphism (in the sense of paragraph 7 of [3]). The same is true for $S={}^3D_4(\bar{q})$, substituting the prime 3 to the prime 2.

Type ${}^2A_l$

If $S={}^2A_l(\bar{q}^2)$ with $l>1$, $S\neq{}^2A_3(2^2)$, ${}^2A_3(3^2)$, ${}^2A_5(2^2)$, then $\Gamma(S)$ is not connected if and only if

i) $l+1$ is a prime and in this case

$$\pi_2(S)=\pi(|T|)=\pi(((-\bar{q})^l+(-\bar{q})^{l-1}+\ldots-\bar{q}+1)/d)$$

where $d=(\bar{q}+1,\, l+1)$;

ii) l is an odd prime and $(\bar{q}+1)\,|\,(l+1)$ and in this case

$$\pi_2(S)=\pi(|T|)=\pi((-\bar{q})^{l-1}+(-\bar{q})^{l-2}+\ldots-\bar{q}+1)\,.$$

i) If $r\neq 2$, then $\bar{q}=q^r$ and $\pi_\alpha=\pi(|{}^2A_l(q^2)|)$ (see Theorem 9.1 of [3]). If $r\neq l+1$, then r and $l+1$ are two distinct primes and therefore

$$\frac{(q^{l+1}+1)}{(q+1)(q+1,\,l+1)}\ \text{divides}\ \frac{(q^{r(l+1)}+1)}{(q^r+1)\,d}=$$

$$=\frac{(-\bar{q})^l+(-\bar{q})^{l-1}+\ldots-\bar{q}+1)}{d}\,.$$

Therefore in this case $\Gamma(G)$ is connected.

If $r=l+1$, the proof is similar to the one of A_l.

If $r=2$, then by Theorems 19.9 of [1] and 4.27 of [5] we have that $\pi_\alpha=\pi(|B_m(\bar{q})|)=\pi(\bar{q}(\bar{q}^2-1)(\bar{q}^4-1)\ldots(\bar{q}^{2m}-1))$, where $l+1=2m+1$.

We know that $|T|=(\bar{q}^{l+1}+1)/(\bar{q}+1)(\bar{q}+1,\,l+1)$ is coprime with all the primes in π_α because π_α is contained in $\pi_1(S)$. So in this case $\pi_2(S)=\pi_2(G)$.

ii) The proof is similar to the one of i) and so $\Gamma(G)$ is connected, except when $r=l,\,2$.

$S={}^2A_3(2^2)$: it is enough to consider the automorphism of order 2 and so, as before, we have that $\pi_2(S)=\pi_2(G)=\{5\}$.

$S={}^2A_3(3^2)$: it is enough to consider the automorphism of order 2 and so, as before, we have that $\pi_2(S)=\pi_2(G)=\{5\}$ and $\pi_3(S)=\pi_3(G)=\{7\}$.

$S = {}^2A_5(2^2)$: it is enough to consider the automorphism of order 2 and so, as before, we have that $\pi_2(S) = \pi_2(G) = \{7\}$ and $\pi_3(S) = \pi_3(G) = \{11\}$.

Type ${}^2B_2$

If $S = {}^2B_2(\overline{q}^2)$, then $\Gamma(S)$ is not connected and

$$\pi_2(S) = \pi(|T|) = \pi(\overline{q}^2 - 1),$$
$$\pi_3(S) = \pi(|T_1|) = \pi(\overline{q}^2 - \sqrt{2}\overline{q} + 1),$$
$$\pi_4(S) = \pi(|T_2|) = \pi(\overline{q}^2 + \sqrt{2}\overline{q} + 1).$$

We only have to consider the case in which r is an odd prime and $\overline{q}^2 = q^{2r} = 2^m$, m an odd integer. Then $\pi_\alpha = \pi(|{}^2B_2(q^2)|) = \pi(q(q^2-1)(q^4-1))$ ([3], Theorem 9.1). As $q^2 - 1$ divides $q^{2r} - 1$, it is clear that $\pi_2(S) \subseteq \pi_1(G)$.

We observe that $q^4 + 1$ divides $q^{4r} + 1 = (\overline{q}^2 - \sqrt{2}\overline{q} + 1)(\overline{q}^2 + \sqrt{2}\overline{q} + 1)$ because r is odd.

It can be proved that, if $r \equiv 1, 7$ (8), $(q^2 - \sqrt{2}q + 1)$ divides $(\overline{q}^2 - \sqrt{2}\overline{q} + 1)$ and $(q^2 + \sqrt{2}q + 1)$ divides $(\overline{q}^2 + \sqrt{2}\overline{q} + 1)$;

or, if $r \equiv 3, 5$ (8), then $(q^2 - \sqrt{2}q + 1)$ divides $(\overline{q}^2 + \sqrt{2}\overline{q} + 1)$ and $(q^2 + \sqrt{2}q + 1)$ divides $(\overline{q}^2 - \sqrt{2}\overline{q} + 1)$.

So, in any case, we have that $\Gamma(G)$ is connected.

Type ${}^2D_l$

If $S = {}^2D_l(\overline{q}^2)$, then $\Gamma(S)$ is not connected if and only if

i) $l = 2^n$ and in this case $\pi_2(S) = \pi(|T|) = \pi((\overline{q}^l + 1)/d)$ where $d = (\overline{q}^l + 1, 4)$;

ii) $\overline{q} = 2$ and $l = 2^n + 1$ and in this case $\pi_2(S) = \pi(|T|) = \pi(2^{l-1} + 1)$;

iii) $\overline{q} = 3$ and

- $l = 2^n + 1$ and l is not a prime and in this case $\pi_2(S) = \pi(|T|) = \pi((3^{l-1} + 1)/2)$;
- $l \neq 2^n + 1$ and l is a prime and in this case $\pi_2(S) = \pi(|T_1|) = \pi((3^l + 1)/4)$;
- $l = 2^n + 1$ and l is a prime and in this case $\pi_2(S) = \pi(|T|) = \pi((3^{l-1} + 1)/2)$ and $\pi_3(S) = \pi(|T_1|) = \pi((3^l + 1)/4)$.

i) If $r \neq 2$, then $\bar{q} = q^r$, $\pi_\alpha = \pi(|{}^2D_l(q^2)|)$ ([3], Theorem 9.1) and $(q^l+1)/d$ divides $(\bar{q}^l+1)/d$; therefore $\Gamma(G)$ is connected.

If $r = 2$, then $\pi_\alpha = \pi(|C_{l-1}(\bar{q})|) = \pi(\bar{q}(\bar{q}^2-1)\ldots(\bar{q}^{2(l-1)}-1))$. Therefore, as π_α is contained in $\pi_1(G)$, we have that $\pi_2(S) = \pi_2(G)$.

ii) We only have to consider an automorphism of order $r = 2$. Then $\pi_\alpha = \pi(|B_{l-1}(\bar{q})|) = \pi(\bar{q}(\bar{q}^2-1)\ldots(\bar{q}^{2(l-1)}-1))$ and, as $(2^{l-1}+ +1)$ divides $|B_{l-1}(2)|$, we can conclude that $\Gamma(G)$ is connected.

iii) As in case ii), we only have to consider the case $r = 2$. Then $\pi_\alpha = \pi(|B_{l-1}(\bar{q})|) = \pi(\bar{q}(\bar{q}^2-1)\ldots(\bar{q}^{2(l-1)}-1))$ and $(3^{l-1}+1)$ divides $|B_{l-1}(3)|$, while $(3^l+1)/4$ is coprime with $|B_{l-1}(3)|$ by Lemma 4 i), when l is a prime. Therefore, when $l = 2^n+1$, $\pi(|T|) \subseteq \pi_1(G)$ and, when l is a prime, $\pi(|T_1|) = \pi_2(G)$.

Type ${}^2E_6$

If $S = {}^2E_6(\bar{q}^2)$, then $\Gamma(S)$ is not connected and:

i) if $\bar{q} = 2$ then $\pi_2(S) = \pi(|T|) = \pi((2^6-2^3+1)/3) = \{19\}$, $\pi_3(S) = \pi(|T_1|) = \{17\}$ and $\pi_4(S) = \pi(|T_2|) = \{13\}$.

ii) if $\bar{q} \neq 2$ then $\pi_2(S) = \pi(|T|) = \pi((\bar{q}^6-\bar{q}^3+1)/d)$ where $d = = (\bar{q}+1, 3)$.

i) We only have to consider the case $r = 2$. Then, by 19.9 iii) of [1], we have $\pi_\alpha = \pi(2(2^8-1)(2^{12}-1)) = \{2, 3, 5, 17, 13, 7\}$ and then $\pi_3(S) \subseteq \pi_1(G)$, $\pi_4(S) \subseteq \pi_1(G)$, while $\pi_2(S) = \pi_2(G)$.

ii) If $r \neq 2$, then $\bar{q} = q^r$ and $\pi_\alpha = \pi(|{}^2E_6(q^2)|)$ (see Theorem 9.1 of [3]). If $r \neq 3$, then $(q^6-q^3+1)/d$ divides $(\bar{q}^6-\bar{q}^3+1)/d$ and therefore $\Gamma(G)$ is connected.

If $r = 3$, then $\pi_\alpha = \pi(q(q^5+1)(q^8-1)(q^9+1)(q^{12}-1))$. It can be proved that $\pi_\alpha \cap \pi_2(S)$ is empty and therefore $\pi_2(S) = \pi_2(G)$.

If $r = 2$, then by Lemma 4.25 c) of [5] and 19.9 iii) of [1], we have $\pi_\alpha \subseteq \subseteq \pi(\bar{q}(\bar{q}^8-1)(\bar{q}^{12}-1))$. As $(\bar{q}^6-\bar{q}^3+1)/d = (\bar{q}^9+1)/(\bar{q}^3+1)\,d$ by Lemma 4 we can conclude that $|T|$ is coprime with all the primes in π_α; so in this case we have that $\pi_2(S) = \pi_2(G)$.

Type $^2F_4$

If $S={}^2F_4(2)'$, then $\Gamma(S)$ is not connected and $G={}^2F_4(2)$ and $\pi_2(G)=\{13\}$ (see [2]).

If $S={}^2F_4(\bar{q}^2)$, then $\Gamma(S)$ is not connected and

$$\pi_2(S)=\pi(|T_1|)=\pi(\bar{q}^4-\sqrt{2}\bar{q}^3+\bar{q}^2-\sqrt{2}\bar{q}+1),$$

$$\pi_3(S)=\pi(|T_2|)=\pi(\bar{q}^4+\sqrt{2}\bar{q}^3+\bar{q}^2+\sqrt{2}\bar{q}+1).$$

We only have to consider the case in which r is an odd prime and $\bar{q}^2=q^{2r}=2^m$, m an odd integer. Then $\pi_\alpha=\pi(|{}^2F_4(q^2)|)=$ $=\pi(q(q^8-1)(q^6+1)(q^{12}+1))$ ([3], Theorem 9.1). We observe that

$$(\bar{q}^{12}+1)/(\bar{q}^4+1)=(\bar{q}^8-\bar{q}^4+1)=$$

$$=(\bar{q}^4-\sqrt{2}\bar{q}^3+\bar{q}^2-\sqrt{2}\bar{q}+1)(\bar{q}^4+\sqrt{2}\bar{q}^3+\bar{q}^2+\sqrt{2}\bar{q}+1).$$

If $r=3$, $(\bar{q}^8-\bar{q}^4+1)=q^{24}-q^{12}+1=(q^{36}+1)/(q^{12}+1)$ and it is therefore coprime with $(q^{36}-1)$. Moreover $(\bar{q}^8-\bar{q}^4+1,\ q^8-1)=(\bar{q}^8-\bar{q}^4+$ $+1,\ q^4+1)$ and (q^4+1) divides $(q^{12}+1)$; $(\bar{q}^8-\bar{q}^4+1,\ q^{12}+1)=$ $=(3,\ q^{12}+1)=1$. Therefore, in this case we have $\pi_2(S)=\pi_2(G)$ and $\pi_3(S)=\pi_3(G)$.

We can now suppose that $r\neq 3$. It can be proved that if $r\equiv$ $\equiv 1, 7, 17, 23\ (24)$, then

$$(q^4-\sqrt{2}q^3+q^2-\sqrt{2}q+1) \quad \text{divides} \quad (\bar{q}^4-\sqrt{2}\bar{q}^3+\bar{q}^2-\sqrt{2}\bar{q}+1)$$

and

$$(q^4+\sqrt{2}q^3+q^2+\sqrt{2}q+1) \quad \text{divides} \quad (\bar{q}^4+\sqrt{2}\bar{q}^3+\bar{q}^2+\sqrt{2}\bar{q}+1);$$

or, if $r\equiv 5, 11, 13, 19\ (24)$, then

$$(q^4-\sqrt{2}q^3+q^2-\sqrt{2}q+1) \quad \text{divides} \quad (\bar{q}^4+\sqrt{2}\bar{q}^3+\bar{q}^2+\sqrt{2}\bar{q}+1)$$

and

$$(q^4+\sqrt{2}q^3+q^2+\sqrt{2}q+1) \quad \text{divides} \quad (\bar{q}^4-\sqrt{2}\bar{q}^3+\bar{q}^2-\sqrt{2}\bar{q}+1).$$

So, if $r\neq 3$, we have that $\Gamma(G)$ is connected.

Type $^2G_2$

If $S = {}^2G_2(\bar{q}^2)$, then $\Gamma(S)$ is not connected and:

$$\pi_2(S) = \pi(|T_1|) = \pi(\bar{q}^2 - \sqrt{3}\bar{q} + 1),$$
$$\pi_3(S) = \pi(|T_2|) = \pi(\bar{q}^2 + \sqrt{3}\bar{q} + 1).$$

We only have to consider the case in which r is an odd prime and $\bar{q}^2 = q^{2r} = 3^m$, m an odd integer. Then $\pi_\alpha = \pi(|{}^2G_2(q^2)|) = \pi(q(q^2-1)(q^6+1))$ ([3], Theorem 9.1). We observe that $(\bar{q}^2 - \sqrt{3}\bar{q} + 1)(\bar{q}^2 + \sqrt{3}\bar{q} + 1) = (\bar{q}^4 - \bar{q}^2 + 1) = (\bar{q}^6 + 1)/(\bar{q}^2 + 1)$.

If $r = 3$, $(\bar{q}^4 - \bar{q}^2 + 1) = q^{12} - q^6 + 1 = (q^{18} + 1)/(q^6 + 1)$ and it is therefore coprime with $(q^2 - 1)$ and also with $(q^6 + 1)$. Therefore, in this case we have $\pi_2(S) = \pi_2(G)$ and $\pi_3(S) = \pi_3(G)$.

If $r \neq 3$, the proof is similar to the one of $^2B_2$

Type $^3D_4$

If $S = {}^3D_4(\bar{q}^3)$, then $\Gamma(S)$ is not connected and $\pi_2(S) = \pi(\bar{q}^4 - \bar{q}^2 + 1)$. If $r \neq 3$, then $\bar{q} = q^r$ and $\pi_\alpha = \pi(|{}^3D_4(q)|) = \pi(q(q^2-1)(q^8+q^4+1)$ ([3], Theorem 9.1). If $r \neq 2$, then $q^4 - q^2 + 1$ divides both $\bar{q}^4 - \bar{q}^2 + 1$ and $q^8 + q^4 + 1 = (q^4 - q^2 + 1)(q^4 + q^2 + 1)$, and then $\Gamma(G)$ is connected.

If $r = 2$, $\pi_2(S) = \pi(q^8 - q^4 + 1)$ and $(q^8 - q^4 + 1)$ is coprime with $(q^2 - 1)(q^8 + q^4 + 1)$. So in this case we have that $\pi_2(S) = \pi_2(G)$.

If $r = 3$, then by Theorem 9.1(3) of [3], in $Aut(S)$ there are two conjugacy classes of subgroups generated by automorphisms of order 3. We denote these two automorphisms by α and β. Then β is obtained from α by multiplying it with an element of order 3 of S, that is $\beta = g\alpha$, $g \in S$. Therefore, as $\pi_\beta \subseteq \Pi_\alpha = \pi(|G_2(\bar{q})|) = \pi(\bar{q}(\bar{q}^6 - 1))$ and $(\bar{q}^4 - \bar{q}^2 + 1)$ divides $\bar{q}^6 + 1$, we can conclude that $\pi_2(S) = \pi_2(G)$. ■

We have thus examined all the almost simple groups.

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