RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

V. V. BELYAEV
M. KUZUCUOĞLU
E. SEÇKIN

Totally inert groups

Rendiconti del Seminario Matematico della Università di Padova, tome 102 (1999), p. 151-156
<http://www.numdam.org/item?id=RSMUP_1999__102__151_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 102 (1999)

Totally Inert Groups.

V. V. BELYAEV (*) - M. KUZUCUOĞLU (**) - E. SEÇKIN (***)

Let G be any group and H be a subgroup of G. Then H is called an *inert subgroup* of G if $|H: H \cap H^g| < \infty$ for all $g \in G$.

Every finite subgroup, every normal subgroup and the group itself are the trivial examples of inert subgroups in a group. Moreover if $G = GL(n, \mathbb{Q})$, then $SL(n, \mathbb{Z})$ is inert in G see [7, page 55]. In a barely transitive group G the stabilizer H of a point and any group containing H are inert subgroups of G see [5].

A group G is called *totally inert group* (*TIN-group*) if every subgroup of G is inert.

Clearly every FC-group is a TIN-group. But there exist totaly inert non-FC- groups. The groups constructed by Olsanskiĭ are the examples of non-FC TIN-groups. The following is an easier example of a non-FC, TIN-group.

Let A be an infinite abelian $2'$-group. Let t be an involutory automorphism of A such that $a^t = a^{-1}$ for all $a \in A$. Then $G = A \rtimes \langle t \rangle$ is not an FC-group but it is a TIN-group. Because for any $H < G$ the group $H \cap A$ is a normal subgroup of G.

Clearly the property of being a TIN-group is a natural generalization of being an FC-group. The above example shows that the class of TIN-groups is larger than the class of FC-groups.

One may think that every FC by finite group is TIN-group. But the following example shows that this is not true.

(*) Indirizzo dell'A.: Krasnoyarsk State Academy of Architechture and Civil Engineering. Svobodny av., 82, Krasnoyarsk, 660041, Russia.
E-mail: belyaev@home.krasnoyarsk.su

(**) Department of Mathematics, Middle East Technical University, 06531 Ankara, Turkey. E-mail matmah@rorqual.cc.metu.edu.tr

(***) Department of Mathematics, Middle East Technical University, 06531 Ankara, Turkey. E-mail elifh@rorqual.cc.metu.edu.tr

Let $G = A \rtimes \langle t \rangle$ where $A = \langle a_1 \rangle \times \langle a_1^t \rangle \times \langle a_2 \rangle \times \langle a_2^t \rangle \times \ldots$ and the order of t is 2. Then for the subgroups $C = \langle a_1 \rangle \times \langle a_2 \rangle \times \ldots$ and $C^t = \langle a_1^t \rangle \times \times \langle a_2^t \rangle \times \ldots$ we have $C \cap C^t = 1$. This group G is an FC by finite group but it is not a TIN-group.

It is clear that any subgroup of a TIN-group is a TIN-group and any homomorphic image of a TIN-group is a TIN-group. But the following example shows that direct product of two TIN-groups is not necessarily a TIN-group.

Let A be an infinite abelian $2'$-group and t be an involutory automorphism of A such that $a^t = a^{-1}$ for all $a \in A$. Let $G = A \times (A \rtimes \langle t \rangle)$ and $H = \{(a, a): a \in A\}$. Let $(1, t)$ be an element of G. Then $H^{(1,t)} = \{(a, a^t): a \in A\} = \{(a, a^{-1}): a \in A\}$. Then $x \in H \cap H^{(1,t)}$ implies that $x = (a, a) = (a, a^{-1})$. Hence $a^2 = 1$, but this implies $a = 1$ as A is an abelian $2'$-group. Hence $x = 1$ and H is not an inert subgroup of G.

What are the structures of locally finite *TIN*-groups?

The groups constructed by Olsanskiĭ [6] are examples of simple *TIN*-groups. But of course they are not locally finite. Does there exist an infinite locally finite simple TIN-group? In this article we answer this question negatively.

THEOREM 1. *There exists no infinite simple locally finite TIN-group.*

Basic Properties of Inert Subgroups.

LEMMA 1. (i) *If H is an infinite simple inert subgroup of a group G, then $H \trianglelefteq G$.*

(ii) *Let G be a simple group and H be a proper inert subgroup of G. Then H is residually finite.*

(iii) *Homomorphic image of a TIN-group is a TIN-group.*

(iv) *If H is an inert subgroup of G and $N \trianglelefteq G$, then HN is inert in G.*

PROOF. (i) Trivial.

(ii) $\bigcap_{g \in G} H^g \lhd G$. Since G is a simple group we have $\cap_{g \in G} H^g = 1$. Hence the result follows.

(iii) and (iv) are trivial.

LEMMA 2. *Let P be a locally and residually finite p-group for some prime p. If P/P' is finite, then P is a finite group.*

PROOF. Since P/P' is finite, there exists a finite subset $K \subseteq G$ such that $\langle K, P' \rangle = P$. Assume if possible that $\langle K \rangle \neq P$. Let $g \in P - \langle K \rangle$. Then $\langle K, g \rangle$ is finite and there exists a normal subgroup $N \lhd P$ such that $|P\colon N| < \infty$ and $N \cap \langle K, g \rangle = 1$. Let $\overline{P} = P/N, \overline{K} = KN/N$. Since $\langle K, P' \rangle = P$ we get $\langle \overline{K}, \overline{P'} \rangle = \overline{P}$, but for a finite p-group the commutator subgroup lies inside the Frattini subgroup therefore it is a non generator. Hence $\langle \overline{K} \rangle = \overline{P}$. But $\bar{g} = gN \notin \overline{K}$. So we get $P = \langle K \rangle$. It follows that P is a finite group.

DEFINITION 1. *Let X and Y be two subgroups of a group G. We say that X and Y are* commensurable *if $|X\colon X \cap Y| < \infty$ and $|Y\colon Y \cap X| < \infty$.*

LEMMA 3. *If X is an inert subgroup of a group G and X is commensurable with the subgroup Y of G, then Y is an inert subgroup of G.*

PROOF. Let $g \in G$. By assumption $|X\colon X \cap X^g| < \infty$ and $|Y\colon X \cap \cap Y| < \infty$ and $|X\colon X \cap Y| < \infty$. Then $|X^g\colon X^g \cap X^{g^2}| < \infty$. It follows that

$$|X \cap X^g\colon X \cap X^g \cap X^{g^2}| < \infty .$$

Since the group X is inert we have $|X \cap Y\colon X \cap X^g \cap Y| < \infty$. Hence $|Y\colon X \cap X^g \cap Y| < \infty$ and $|X\colon X \cap X^g \cap Y| < \infty$. This gives $|X^g\colon X^g \cap \cap X^{g^2} \cap Y^g| < \infty$. Then $|X \cap X^g\colon X \cap X^g \cap X^{g^2} \cap Y^g| < \infty$. Then we get $|X\colon X \cap X^g \cap X^{g^2} \cap Y^g| < \infty$ and $|X\colon X \cap X^g \cap Y| < \infty$. Moreover $|X\colon X \cap X^g \cap X^{g^2} \cap Y \cap Y^g| < \infty$. Then $|Y\colon Y \cap X| < \infty$ implies $|Y\colon Y \cap Y^g| < \infty$.

LEMMA 4 [1, Corollary 2.6]. *Let G be a simple TIN-group and $1 \neq \neq K \lhd H$ be subgroups of G. Then H/K is an FC-group.*

LEMMA 5. *Let G be a simple TIN-group. Then*

(i) *for all non identity elements x and y in G, the groups $C_G(x)$ and $C_G(y)$ are commensurable.*

(ii) *if $C_G(x)$ is infinite for a non identity torsion element x in G, then G is locally finite.*

(iii) *if G is locally finite, $H < G$, then either H is an FC-group or $F(H) = 1$ where $F(H)$ is the Hirsh-Plotkin radical of H.*

PROOF. (i). Let $1 \neq x$ be an element of G. Then $N = \{y \in G\colon |C_G(x)\colon C_G(x) \cap C_G(y)| < \infty\}$ is a normal subgroup of G. It is clear that product of two elements are in N. So we show the normality of N. Let $y \in N$ and $t \in G$. Then $|C_G(x)\colon C_G(x) \cap C_G(y)| < \infty$. It follows that $|C_G(x)^t\colon C_G(x)^t \cap C_G(y)^t| < \infty$. Then $|C_G(x) \cap C_G(x)^t\colon C_G(x) \cap C_G(y)^t \cap C_G(x)^t| < \infty$. Since $C_G(x)$ is an inert subgroup of G we have $|C_G(x)\colon C_G(x) \cap C_G(y^t) \cap C_G(x^t)| < \infty$. Hence $|C_G(x)\colon C_G(x) \cap C_G(y^t)| < \infty$. But G is simple and $x \in N$ implies that $N = G$. Hence for all x and y in G the groups $C_G(x)$ and $C_G(y)$ are commensurable.

It follows that for any $x \in G$ the group $C_G(x)$ is an FC-group.

(ii) Let $1 \neq x$ be a fixed element of G such that $C_G(x)$ is infinite. Let $T(G) = \{g \in G\colon \text{order of } g \text{ is finite}\}$ be the set of torsion elements of G. Then $T(G)$ is a normal subgroup of G. Indeed if K is a finite subset of $T(G)$, then by (i) $|C_G(x)\colon C_G(x) \cap C_G(K)| < \infty$. Hence $C_G(K)$ is an infinite group. Let $a \in C_G(K)$. Then $K \leqslant C_G(a)$ and $C_G(a)$ is an FC-group. Then by Dietzmann Lemma $\langle K \rangle \leqslant \langle K^{C_G(a)} \rangle$ is finite. Hence $\langle K \rangle$ is a finite subgroup. Hence $T(G) = G$ and G is locally finite.

(iii) Let x and y be nonidentity elements of H. Then $K_1 = \langle x^H \rangle$ and $K_2 = \langle y^H \rangle$ are commensurable.

Indeed, for any $i = 1, 2$ by Lemma 4 H/K_i are FC-groups. Let $\bar{y} = yK_1 \in H/K_1$. Then $\langle \bar{y}^H \rangle = K_2K_1/K_1 \cong K_2/(K_1 \cap K_2)$ which is finite. Similarly $K_1/(K_1 \cap K_2)$ is finite.

Let $F(H)$ be the Hirsh-Plotkin radical. Assume if possible that H is not an FC-group. Then by [2, Theorem 1.4] $F(H)$ is a p-group and by [2, Corollary 2.2 and Theorem 4.5] $FC(F(H)) = 1$. Since there exists no FC-element every conjugacy class is infinite. Let $x \in F(H) \backslash \{1\}$ and $\langle x^{F(H)} \rangle = L$. Since the group L is an infinite residually finite p-group we get $L > L'$. Let $y \in L'$. Then $\langle y^{F(H)} \rangle \lhd L'$ and by Lemma 2 $|L\colon \langle y^{F(H)} \rangle|$ is infinite. But any two normal subgroups of $F(H)$ are commensurable by the above paragraph. Hence we obtain a contradiction.

LEMMA 6. *Let G be a locally finite group. Let A be a normal infinite elementary abelian p-subgroup of G. Assume that every subgroup of A is inert in G. Then for any $x \in G$, there exists a subgroup B of finite index in A such that, for all $b \in B$, $b^x \in \langle b \rangle$.*

PROOF. Assume on the contrary that, there exists $1 \neq x \in G$ such that if $|A\colon B| < \infty$, then there exists $a_1 \in B$ such that $a_1^x \notin \langle a_1 \rangle$. Then $\langle a_1^x \rangle \cap \langle a_1 \rangle = 1$.

Let $A_1 = \langle a_1 \rangle \times \langle a_1^x \rangle$. Then A_1 is finite and $A = A_1 \times A_1'$.

$$\bigcap_{n \in \mathbb{N}} (A_1')^{x^n} = \bigcap_{g \in \langle x \rangle} (A_1')^g = B_1$$

B_1 is $\langle x \rangle$-invariant and has finite index in A.

By a similar argument there exists $a_2 \in B_1$ such that $\langle a_2^x \rangle \cap \langle a_2 \rangle = 1$. Let $A_2 = \langle a_2^x \rangle \times \langle a_2 \rangle$ and $B_1 = A_2 \times A_2'$

$$\bigcap_{g \in \langle x \rangle} (A_2')^g = B_2.$$

The group B_2 has finite index in B_1 and B_2 is $\langle x \rangle$-invariant. There exists $a_3 \in B_2$ such that $\langle a_3^x \rangle \cap \langle a_3 \rangle = 1$.

Let $A_3 = \langle a_3^x \rangle \times \langle a_3 \rangle$. Continuing like this we obtain an infinite subgroup of A namely, $\langle a_1 \rangle \times \langle a_1^x \rangle \times \langle a_2 \rangle \times \langle a_2^x \rangle \times \ldots$

Let $C = \langle a_1 \rangle \times \langle a_2 \rangle \times \langle a_3 \rangle \times \ldots$ Then $C^x \cap C = 1$. Hence C is not inert in G which is a contradiction.

LEMMA 7. *Let G be a locally finite group and A be a normal infinite elementary abelian p-subgroup of G. If every subgroup of A is inert in G, then for all $x \in G'$, $|[A, x]| < \infty$. I.e G' acts as finitary linear group on A.*

PROOF. Let $x \in G'$. Then $x = x_1 x_2 \ldots x_n$ where $x_i = [y_i, z_i]$ for some $y_i, z_i \in G$, $i = 1, 2, \ldots n$. Clearly $[A, x] \leq [A, x_1][A, x_2] \ldots [A, x_n]$. Hence it is enough to prove that $[A, x_i]$ is finite for all $i = 1, 2, \ldots n$.

By Lemma 6, for y_i and z_i, there exist B_i and B_i' such that $|A : B_i| < \infty$ and $|A : B_i'| < \infty$ and for all $b \in B_i$, $b^{y_i} \in \langle b \rangle$ and for all $b \in B_i'$, $b^{z_i} \in \langle b \rangle$. Let $B = B_i \cap B_i'$. Then $|A : B| < \infty$ and for all $b \in B$, $b^{y_i} \in \langle b \rangle$ and $b^{z_i} \in \langle b \rangle$. This implies $b^{y_i} = b^{n_i}$ and $b^{z_i} = b^{m_i}$ for some n_i and m_i in $\mathbb{Z}$. Since the automorphism group of a cyclic group is abelian we get $b^{[y_i, z_i]} = b$ for all $b \in B$. So $|A : C_A(x_i)| < \infty$ since $B \leq C_A(x_i)$. Hence $[A, x_i] < \infty$.

PROOF OF THE THEOREM 1. Asume that G is a simple locally finite *TIN*-group. By [4] Theorem 4.4 every locally finite simple group has a local system consisting of countably infinite locally finite simple subgroups. But by Lemma 1 in a simple locally finite *TIN*-group, an infinite simple subgroup is a normal subgroup, so we may assume that G itself is countable.

By [3, Theorem B Page 190] a countable non-finitary locally finite simple group has maximal subgroups. Let M be a maximal subgroup of G. Then for all $g \in G$ $|M: M \cap M^g| < \infty$ and $|M^g: M \cap M^g| < \infty$. It follows from [1] Lemma 5 that $|\langle M, M^g \rangle: M \cap M^g| < \infty$. As M is a maximal subgroup of G we get $M = M^g$ for all $g \in G$. Hence M is a normal subgroup of G. But G is simple, hence we may assume that G is a countable, finitary locally finite simple group. By [3, page 216 Theorem 8], there exists a prime p such that if H is any proper inert subgroup of G, then $H/O_p(H)$ is locally normal. Consequently if $F(H) = 1$, then H is locally normal. If $F(H) \neq 1$, then by Lemma 5 (iii) H is locally normal. By [3, page 216 Theorem 6] G is isomorphic to an alternating group of finitary permutations on some set Ω. But a stabilizer G_α of $\alpha \in \Omega$ is an infinite proper simple subgroup of G, and this is impossible by Lemma 1 (i).

REFERENCES

[1] V. V. BELYAEV, *Locally finite groups with Černikov Sylow p-subgroups*, Algebra i Logika, **20** (1981), pp. 605-619. Translation *Algebra and Logic*, **20** (1981), pp. 393-402.

[2] V. V BELYAEV, *Inert subgroups in infinite simple groups*, Sibirskii Mathematicheskii Zhurnal, **34** (1993), pp. 17-23. English Translation Siberian, *Math. J.*, **34** (1993), pp. 218-232.

[3] B. HARTLEY *et al*, *Finite and Locally finite Groups*, NATO ASI series C, **471**, Kluwer Academic Publishers (1995).

[4] O. H. KEGEL - B. A. F. WEHRFRITZ, *Locally finite Groups*, North Holland, Amsterdam (1973).

[5] M. KUZUCUOĞLU, *Barely Transitive permutation groups*, Arch. Math., **55** (1990), pp. 521-532.

[6] A. JU. OL'SANSKII, *An infinite group with subgroups of prime orders*, Izv. Akad. Nauk. USSR. Ser. Mat., **44**, 2 (1980), pp. 309-321.

[7] G. SHIMURA, *Introduction to the Arithmetic Theory of Automorphic Functions*, Iwanami Shoten, Princeton University.

Manoscritto pervenuto in redazione l'1 dicembre 1997.