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A priori Inequalities in $L^\infty(\Omega)$ for Solutions of Elliptic Equations in Unbounded Domains.

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ABSTRACT - We prove some a priori inequalities in $L^\infty(\Omega)$ for subsolutions of elliptic equations in divergence form, with Dirichlet's boundary conditions, in unbounded domains.

1. Introduction.

In an open subset $\Omega$ of $\mathbb{R}^n$, not necessarily bounded, we consider a linear uniformly elliptic second order operator in variational form with discontinuous coefficients, associated to the bilinear form

$$ a(u, v) = \int_{\Omega} \left\{ \sum_{i,j=1}^{n} a_{ij} u_{x_i} v_{x_j} + \sum_{i=1}^{n} (b_i u_{x_i} v + d_i uv_{x_i}) + cuv \right\} \, dx $$

If $u \in H^1(\Omega)$ is a solution of the inequality

$$ a(u, v) \leq \int_{\Omega} \left\{ f_0 v + \sum_{i=1}^{n} f_i v_{x_i} \right\} \, dx \quad \forall v \in C_0^1(\Omega), \; v \geq 0 \; \text{in} \; \Omega, $$

we can consider the problem of determining the minimal hypotheses on the coefficients $b_i$, $d_i$, $c$ of the bilinear form (1) and on the known functions $f_i$ ($i = 0, 1, \ldots, n$) for the subsolution $u$ to be (essentially) bounded from above in $\Omega$. Such a problem was already studied e.g. in [2] and [3],

where an inequality of the kind

\[
\text{ess sup}_\Omega u \leq \max (0, \max_\Omega u) + K_1 \left\{ \| f_0 \|_{L^p(\Omega)} + \sum_{i=1}^n \| f_i \|_{L^p(\Omega)} \right\} + K_2 \| u \|_{L^q(\Omega)}
\]

was proved, by supposing \( \Omega \) bounded and \( f_i, d_i \in L^p(\Omega) \) (\( i = 1, 2, \ldots, n \)), \( f_0, c \in L^{n^2}(\Omega), p > n \).

The aim of the present work is to extend these results first of all allowing the set \( \Omega \) to be unbounded and relaxing the hypotheses on the functions \( f_0, f_i, b_i, d_i, c \) (\( i = 1, 2, \ldots, n \)). Finally, the constants in the a priori inequality (3) are explicitly evaluated.

2. Notations and Hypotheses.

Let \( \Omega \) be an open subset (bounded or unbounded) of \( \mathbb{R}^n \). Let \( a_{ij} \in L^\infty(\Omega) \) (\( i, j = 1, 2, \ldots, n \)), \( \sum_{i,j=1}^n a_{ij} t_i t_j \geq \nu |t|^2 \) \( \forall t \in \mathbb{R}^n \) a.e. in \( \Omega \), where \( \nu \) is a positive constant. Let \( c^+ := \max (c, 0), c^- := \min (c, 0) \) and suppose that \( c^+ \in L^{2n/(n+2)}(\Omega') \) for any \( \Omega' \) bounded, \( \Omega' \subset \Omega \). Let us define the spaces

\[
X^p(\Omega) := \{ f \in L^p_{\text{loc}}(\Omega): \omega(f, p, \delta) < +\infty \ \forall \delta > 0 \}
\]

\[
X_0^p(\Omega) := \{ f \in X^p(\Omega): \lim_{\delta \to 0^+} \omega(f, p, \delta) = 0 \}
\]

where

\[
\omega(f, p, \delta) := \sup \{ \| f \|_{L^p(E)}: E \text{ measurable}, \ E \subset \Omega, \text{ meas}(E) \leq \delta \}.
\]

**Remark 1.** If \( f \in L^p_{\text{loc}}(\Omega) \), we define, for \( k > 0 \),

\[
\phi(f, p, k) := \inf \{ \text{meas}(E): E \text{ measurable}, \ E \subset \Omega, \| f \|_{L^p(E)} \geq k \},
\]

and we have

\[
f \in X^p(\Omega) \quad \text{if and only if} \quad \exists k_0 > 0 \text{ such that } \phi(f, p, k_0) > 0,
\]

\[
f \in X_0^p(\Omega) \quad \text{if and only if} \quad \phi(f, p, k) > 0 \quad \forall k > 0.
\]

**Remark 2.** If \( G \) is a measurable subset of \( \Omega \) such that \( \text{meas}(G) \leq \phi(f, p, k) \), then it turns out that \( \| f \|_{L^p(G)} \leq k \). In fact, if not there would
exist a subset $G_0$ of $G$ with positive measure but so small that
\[ \|f\|_{L^p(G \setminus G_0)} > k \]
which is in contradiction with the definition of $\phi$, since $\text{meas}(G \setminus G_0) < \text{meas}(G)$.  

**Remark 3.** If $1 \leq q < p$ it turns out $X^p(\Omega) \subset X^q(\Omega)$. In fact, if $E \subset \Omega$, $\text{meas}(E) \leq \delta$, $f \in X^p(\Omega)$ we have
\[ \|f\|_{L^q(E)} \leq \|f\|_{L^p(E)} \left( \text{meas}(E) \right)^{(p-q)/pq} \leq \omega(f, p, \delta)\delta^{(p-q)/pq} \]
whence
\[ \omega(f, q, \delta) \leq \omega(f, p, \delta)\delta^{(p-q)/pq}. \]

We denote by $S$ the constant in the Sobolev inequality
\[ \|g\|_{L^{2n/(n-2)}(\mathbb{R}^n)} \leq S\|g\|_{L^2(\mathbb{R}^n)} \quad \forall g \in C^1_0(\mathbb{R}^n). \]

It is a well known fact (see e.g. [4]) that $S$ is given by the following formula:
\[ S = \left[ n(n-2) \pi \right]^{-1/2} \Gamma(n)^{1/n} \Gamma(n/2)^{-1/n}. \]

**Lemma.** Let $u \in H^1_0(\Omega)$, $B \subset \Omega$, $u = 0$ in $B$. Then there exists a sequence $\{u_j\}_{j \in \mathbb{N}} \subset H^1_0(\Omega)$ such that $u_j = 0$ in $B$, $u_j$ has compact support in $\Omega$ ($j = 1, 2, \ldots$), $\lim_j \|u - u_j\|_{H^1(\Omega)} = 0$.

**Proof.** It follows from the results of [3] that $u^+ := \max(u, 0)$, $u^- := \min(u, 0)$ both belong to $H^1_0(\Omega)$, therefore we may assume without loss of generality that $u \geq 0$ in $\Omega$. By definition of $H^1_0(\Omega)$, there exists a sequence $\{\phi_j\}_{j \in \mathbb{N}} \subset C^1_0(\Omega)$ such that $\lim_j \|u - \phi_j\|_{H^1(\Omega)} = 0$; we may assume $\phi_j \geq 0$ in $\Omega$ ($j = 1, 2, \ldots$). Consider the functions $u_j := \min(u, \phi_j)$ ($j = 1, 2, \ldots$). These functions are in $H^1_0(\Omega)$ and they vanish on $B$ and where $\phi_j = 0$. Furthermore it is easy to verify that $|\left(u - u_j\right)_x| \leq |(u - \phi_j)_x|$ where all the derivatives exist (i.e. almost everywhere in $\Omega$), whence
\[ \|u - u_j\|_{L^2(\Omega)} \leq \|u - \phi_j\|_{L^2(\Omega)} \quad (j = 1, 2, \ldots). \]

Therefore the sequence $\{u_j\}_{j \in \mathbb{N}}$ has the required properties.  

3. Main result.

**Theorem.** In addition to the hypotheses mentioned above, we assume: $p > n$, $c^{-} \in X_{0}^{np/(n+p)}(\Omega)$, $b_i \in X_{0}^{n}(\Omega)$, $d_i \in X_{0}^{p}(\Omega)$, $f_i \in X^{p}(\Omega)$ ($i = 1, 2, \ldots, n$), $f_0 \in X_{0}^{np/(n+p)}(\Omega)$, $u \in H_{0}^{1}(\Omega)$,

\[ a(u, v) \leq \int_{\Omega} f_0 v + \sum_{i=1}^{n} f_i v_{x_i} \, dx \quad \forall v \in C_{0}^{1}(\Omega), \quad v \geq 0 \text{ in } \Omega. \]

Furthermore suppose that there exists a nonnegative real number $m$ such that $\max(u - m, 0) \in H_{0}^{1}(\Omega)$.

Then there exist constants $K_1$, $K_2$, $K_3$, depending on the coefficients of $a(\cdot, \cdot)$, on $n$ and $p$, such that

\[ \text{ess sup}_{\Omega} u \leq K_1 \| \max(u - m, 0) \|_{L^{2}(\Omega)} + 2^{np/(p-n)} m + K_2 \left\{ S \omega(f_0, np/(p + n), K_3) + \sum_{i=1}^{n} \omega(f_i, p, K_3) \right\} \]

where:

- $S$ is the Sobolev constant (10),
- $K_1 = (4/3)^{np/(p-n)} + 2^{np/(p-n)} K_3^{-1/2}$,
- $K_2 = (3S/\nu)[2^{np/(p-n)} - 1]$,
- $K_3 = \min \{ 1, \phi(b_i, n, \nu/(6Sn)), \phi(d_i, p, \nu/(6Sn)), \phi(c^{-}, np/(p+n), \nu/(6S^2)) \}$ ($i = 1, 2, \ldots, n$).

**Proof.** First of all we notice that if $t \geq m$ obviously the function $u_t := \max(u - t, 0)$ is in $H_{0}^{1}(\Omega)$ as well. Moreover, it is easy to check that (12) is verified also by nonnegative functions $v \in H_{0}^{1}(\Omega)$ with compact support contained in $\Omega$. In fact, let $A$ be an open bounded set containing the support of $v$, such that $\overline{A} \subset \Omega$. It is easy to find a sequence $\{ u_j \}_{j \in N} \subset C_{0}^{1}(A)$ which converges to $v$ in the norm of $H^{1}(A)$. We can write (12) with $u_j$ instead of $v$ and let $j$ go to infinity, taking into account Hölder's and Sobolev's inequalities and the fact that $u \in H^{1}(A)$ by hypothesis (and also $u \in L^{2n/(n-2)}(A)$). So, (12) is true if $v \in H_{0}^{1}(\Omega)$ with compact support contained in $\Omega$. Then from the lemma above we can find a sequence of functions $\{ u_j \}_{j \in N} \subset H_{0}^{1}(\Omega)$ having compact support in $\Omega$, vanishing where $u_t = 0$ (i.e. where $u \leq t$), and converging to $u_t$ in the norm of $H^{1}(\Omega)$. As before, we can write (12) with $u_j$ instead of $v$ and let $j$ go to infinity, because $u_t$ and $u_j$ are different from zero only in a (fixed) set of finite measure, in which $u = u_t + t$, thus allowing again the use of Hölder's
and Sobolev’s inequalities. We conclude that (12) can be written with \( v \) replaced by \( u_t \) (where it is always \( t \geq m \)). Let us denote for brevity

\[ \Omega_t := \{ x \in \Omega : u(x) > t \}. \]

By using Hölder’s and Sobolev’s inequalities, and taking into account our previous hypotheses, we deduce

\[ \nu \|(u_t)_x\|_{L^2(\Omega_t)}^2 \leq \sum_{i=1}^n a_{ij} (u_t)_x \, dx, \]

\[ \left| \int_{\Omega} \sum_{i=1}^n b_i u_{x_i} \, u_t \, dx \right| \leq \sum_{i=1}^n \int_{\Omega_t} |b_i(u_t)_x| u_t \, dx \leq S \sum_{i=1}^n \|b_i\|_{L^\infty(\Omega_t)} \|(u_t)_x\|_{L^2(\Omega_t)}^2, \]

\[ \left| \int_{\Omega} \sum_{i=1}^n d_i u_t \, dx \right| \leq \sum_{i=1}^n \int_{\Omega_t} |d_i u_t| \, dx + t \sum_{i=1}^n \int_{\Omega_t} |d_i(u_t)_x| \, dx \leq \]

\[ \leq S \sum_{i=1}^n \|d_i\|_{L^p(\Omega_t)} \left( \text{meas } \Omega_t \right)^{(p - n)/np} \|(u_t)_x\|_{L^2(\Omega_t)}^2 + \]

\[ + t \sum_{i=1}^n \|d_i\|_{L^p(\Omega_t)} \left( \text{meas } \Omega_t \right)^{(p - 2)/2p} \|(u_t)_x\|_{L^2(\Omega_t)}, \]

\[ \left| \int_{\Omega} c - uu_t \, dx \right| \leq \int_{\Omega_t} |c - u_t^2| \, dx + t \int_{\Omega_t} |c - u_t| \, dx \leq \]

\[ \leq S^2 \|c - \|_{L^{np(n + p)}(\Omega_t)} \left( \text{meas } \Omega_t \right)^{(p - n)/np} \|(u_t)_x\|_{L^2(\Omega_t)}^2 + \]

\[ + tS \|c - \|_{L^{np(n + p)}(\Omega_t)} \left( \text{meas } \Omega_t \right)^{(p - 2)/2p} \|(u_t)_x\|_{L^2(\Omega_t)}, \]

\[ \left| \int_{\Omega} f_0 u_t \, dx \right| \leq S \|f_0\|_{L^{np(n + p)}(\Omega_t)} \left( \text{meas } \Omega_t \right)^{(p - 2)/2p} \|(u_t)_x\|_{L^2(\Omega_t)}, \]

\[ \left| \int_{\Omega} \sum_{i=1}^n f_i(u_t)_x \, dx \right| \leq \sum_{i=1}^n \|f_i\|_{L^p(\Omega_t)} \left( \text{meas } \Omega_t \right)^{(p - 2)/2p} \|(u_t)_x\|_{L^2(\Omega_t)}. \]

Therefore it follows easily from (12)

\[ (14) \quad \nu \|(u_t)_x\|_{L^2(\Omega_t)}^2 \leq \]

\[ \leq t \left[ \sum_{i=1}^n \|d_i\|_{L^p(\Omega_t)} + S \|c - \|_{L^{np(n + p)}(\Omega_t)} \right] \left( \text{meas } \Omega_t \right)^{(p - 2)/2p} \|(u_t)_x\|_{L^2(\Omega_t)} + \]
For brevity, let us denote \( \alpha(t) := \text{meas}(\Omega_t) \). Then we get

\[
\{ \nu - S \left[ \sum_{i=1}^{n} \| b_i \|_{L^n(\Omega_t)} + \sum_{i=1}^{n} \| d_i \|_{L^p(\Omega_t)} (\alpha(t))^{(p-n)/np} + \right.
\]

\[
\left. + S \| c^- \|_{L^{n+p}(\Omega_t)} (\alpha(t))^{(p-n)/np} \right\} \| (u_t)_x \|_{L^2(\Omega_t)} \leqslant
\]

\[
\leqslant \left[ S \| f_0 \|_{L^{n+p}(\Omega_t)} + \sum_{i=1}^{n} \| f_i \|_{L^p(\Omega_t)} (\alpha(t))^{(p-2)/2p} + \right.
\]

\[
\left. + t \left[ \sum_{i=1}^{n} \| d_i \|_{L^p(\Omega_t)} + S \| c^- \|_{L^{n+p}(\Omega_t)} \right] (\alpha(t))^{(p-2)/2p} \right].
\]

We notice that, when \( t \geq m \), we have

\[
\int_{\Omega_m} (u - m)^2 \, dx \geq \int_{\Omega_t} (u - m)^2 \, dx \geq (t - m)^2 \alpha(t)
\]

that is:

\[
\alpha(t) \leq \frac{\| u_t \|_{L^2(\Omega_m)}^2}{(t - m)^2}, \quad \forall t > m.
\]

Now we define (see (7))

\[
\delta_0 := \min \{ 1, \phi(b_i, n, \nu/(6Sn)), \phi(d_i, p, \nu/(6Sn)), \phi(c^-, np/(n+p), \nu/(6S^2)) \}
\]

\[
t_0 := m + \frac{\| u_m \|_{L^2(\Omega)}}{\delta_0^{1/2}}
\]

(please note that \( \delta_0 > 0 \) because of our previous hypotheses and remark 1).

Then if \( t \geq t_0 \) we have

\[
\alpha(t) \leq \alpha(t_0) \leq \frac{\| u_m \|_{L^2(\Omega)}^2}{(t_0 - m)^2} = \delta_0
\]
therefore by the definition of $\phi$ and remark 2 we deduce

\begin{equation}
\sum_{i=1}^{n} \|b_i\|_{L^p(\Omega_t)} \leq \nu/(6S),
\end{equation}

\begin{equation}
\sum_{i=1}^{n} \|d_i\|_{L^p(\Omega_t)} \leq \nu/(6S),
\end{equation}

\begin{equation}
\|c^-\|_{L^{np/(n+p)}(\Omega_t)} \leq \nu/(6S^2).
\end{equation}

From (16), (17), (19) it follows $\alpha(t) \leq 1$; then from (15), (20), (21), (22) when $t \geq t_0$ we get

\begin{equation}
(1/2)\|u_t\|_{L^2(\Omega_t)} \leq [\alpha(t)]^{(p-2)/2p} \left( t \left( \sum_{i=1}^{n} \|d_i\|_{L^p(\Omega_t)} + S\|c^-\|_{L^{np/(n+p)}(\Omega_t)} \right) + S\|f_0\|_{L^{np/(n+p)}(\Omega_t)} + \sum_{i=1}^{n} \|f_i\|_{L^p(\Omega_t)} \right).
\end{equation}

Let us denote, for brevity,

\begin{equation}
K_4 := (2S/\nu) \left( \sum_{i=1}^{n} \|d_i\|_{L^p(\Omega_{t_0})} + S\|c^-\|_{L^{np/(n+p)}(\Omega_{t_0})} \right),
\end{equation}

\begin{equation}
K_5 := (2S/\nu) \left( \sum_{i=1}^{n} \|f_i\|_{L^p(\Omega_{t_0})} + S\|f_0\|_{L^{np/(n+p)}(\Omega_{t_0})} \right)
\end{equation}

and apply Hölder's and Sobolev's inequalities to (23), thus obtaining

\begin{equation}
\|u_t\|_{L^{1}(\Omega_t)} \leq [\alpha(t)]^{(2+n)/2n} \|u_t\|_{L^{2n/(n-2)}(\Omega_t)} \leq [\alpha(t)]^{1+(p-n)/np} (K_4 t + K_5)
\end{equation}

Now we follow a procedure of [1]. Define

\begin{equation}
\beta(t) := \|u_t\|_{L^{1}(\Omega_t)}, \quad t \geq t_0
\end{equation}

and note that it turns out $\beta(t) = \int_{t_0}^{\infty} \alpha(s) \, ds$. Therefore

\begin{equation}
\beta'(t) = -\alpha(t) \leq 0 \quad \text{a.e. in } [t_0, +\infty).
\end{equation}

From (26), (28) we get the differential inequality

\begin{equation}
\beta(t) \leq (K_4 t + K_5)[-\beta'(t)]^{1+(p-n)/np} \quad \text{a.e. in } [t_0, +\infty)
\end{equation}

Suppose now, by contradiction, that $\beta(t) > 0 \, \forall t \geq t_0$ (i.e., by definition of
\(\beta(t), \text{ess sup } u = +\infty\). Then in (29) we can divide by \(\beta(t)\) obtaining
\[\begin{equation}
-\beta'(t) [\beta(t)]^{-np/(np+p-n)} \geq (K_4 t + K_5)^{-np/(np+p-n)}.
\end{equation}\]

Integrating (30) between \(t_0\) and \(t^* > t_0\) (suppose for the moment \(K_4 > 0\)), we obtain
\[\begin{equation}
K_4 [\beta(t_0)]^{(p-n)/(np+p-n)} - K_4 [\beta(t^*)]^{(p-n)/(np+p-n)} \geq (K_4 t^* + K_5)^{np/(np+p-n)} - (K_4 t_0 + K_5)^{np/(np+p-n)}
\end{equation}\]
which gives a contradiction when \(t^*\) tends to \(+\infty\).

Then it must be \(\text{ess sup } u < +\infty\). We can rewrite (31) with \(t_0 < t^* < \text{ ess sup } u\); by letting \(t^*\) tend to \(\text{ ess sup } u\) we get
\[\begin{equation}
(K_4 \text{ ess sup } u + K_5)^{np/(np+p-n)} \leq (K_4 t_0 + K_5)^{np/(np+p-n)} + K_4 [\beta(t_0)]^{np/(np+p-n)}
\end{equation}\]

Please note that the constant \(K_4\) is not greater than \(2/3\) because of (21), (22). From (32) by easy calculations we get
\[\begin{equation}
\text{ess sup } u \leq (4/3)^{np/(p-n)} \|u_0\|_{L^p(\Omega)} + 2^{np/(p-n)} t_0 + (3/2) [2^{np/(p-n)} - 1] K_5
\end{equation}\]
whence, by recalling the definition of \(t_0\) (18) and \(K_5\) (25) one can write
\[\begin{equation}
\text{ess sup } u \leq 2^{np/(p-n)} m + [(4/3)^{np/(p-n)} + 2^{np/(p-n)} \delta_0^{-1/2}] \|u_m\|_{L^2(\Omega)} + (3S/v [2^{np/(p-n)} - 1] [S \|f_0\|_{L^{np/(p+n)}(\Omega)} + \sum_{i=1}^{n} \|f_i\|_{L^p(\Omega)}].
\end{equation}\]

Finally, by taking into account (19), the definition of \(\delta_0\) (see (17)) and the functions \(\phi, \omega\), we conclude
\[\begin{equation}
\text{ess sup } u \leq 2^{np/(p-n)} m + [(4/3)^{np/(p-n)} + 2^{np/(p-n)} \delta_0^{-1/2}] \|u_m\|_{L^2(\Omega)} + (3S/v) [2^{np/(p-n)} - 1] [S \omega(f_0, np/(p+n), \delta_0) + \sum_{i=1}^{n} \omega(f_i, p, \delta_0)]
\end{equation}\]
with \(\delta_0\) given by (17).

**Remark 4.** If we suppose, in addition to the hypotheses of the previous theorem, that there exists \(q \geq 1\) such that \(u_m \in L^q(\Omega)\), then we can
write, instead of (16) and (18)

\[
\alpha(t) \leq \|u_m\|_{L^q(\Omega_m)}(t-m)^{-q} \quad \forall t > m ,
\]

(16')

\[
t_0 := m + \|u_m\|_{L^q(\Omega)} \delta_0^{-1/q}
\]

and proceeding as before we get to the conclusion in the form

(35')

\[
\text{ess sup }_{\Omega} u \leq 2^{np(p-n)}m + [(4/3)^{np/(p-n)} + 2^{np(p-n)} \delta_0^{-1/q}]\|u_m\|_{L^q(\Omega)} +
\]

\[
+ (3S/V)[2^{np(p-n)} - 1]\left[S\omega(f_0, np/(p+n), \delta_0) + \sum_{i=1}^n \omega(f_i, p, \delta_0)\right]
\]

where \(\delta_0\) is always given by (17).

**Remark 5.** Suppose the coefficients \(d_i\) and \(c_i\) of the bilinear form \(a(\cdot, \cdot)\) to be identically zero. Then the constant \(K_4\) defined in (24) vanishes, and by integrating (30) we get, more simply,

(36)

\[
\text{ess sup }_{\Omega} u \leq t_0 + (np + p - n)/(p - n) \delta_0^{-1/2} \|u_m\|_{L^2(\Omega)}^{np/(np+p-n)}
\]

whence, by taking into account the definitions of \(t_0, \delta_0, \ldots\), and Young's inequality, we deduce

(37)

\[
\text{ess sup }_{\Omega} u \leq m + (\delta_0^{-1/2} + 1)\|u_m\|_{L^2(\Omega)} + [np/(p-n)]K_5 .
\]

This inequality is of the same kind of (35), but the coefficient of \(m\) in it is now 1.

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