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Semicontinuous differential inclusions

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Semicontinuous Differential Inclusions.

TZANKO DONCHEV (*)

ABSTRACT - Almost upper and almost lower semicontinuous differential inclusions in a Banach space with uniformly convex dual are considered. We suppose that the right-hand side is a sum of one-side Lipschitz multifunction and a multimap which satisfies compactness type conditions. A relaxation theorem stating that the solution set of the original problem is dense in the set of convexified upper semicontinuous regularization one is proved.

1. - Introduction.

We consider the following differential inclusion:

$$\dot{x} \in F(t, x) + G(t, x) \qquad \text{a.e. on } I, \qquad x(0) = x_0 \tag{1}$$

in a Banach space E with uniformly convex dual E^*. Here $I = [0, 1]$, F and G are multifunctions (multimaps). The first one is almost Upper SemiContinuous (USC), the second one is almost Lower SemiContinuous (LSC). The existence of solutions in USC and LSC cases under additional compactness assumptions is considered in great number of papers (see e.g. [7] and references given there). Here we examine the existence of solutions and prove that the solution set of (1) is connected. For one-side Lipschitz F and G we approximate solution set of the relaxed problem (2) (below) by the solution set of the corresponding discretized inclusion. With the help of this approximation using Γ^M continous selections (see [4], [5], [6]) the existence result without compactness assumptions is proved. Using refined version of the main idea of [12] we also prove new

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relaxation theorem (Theorem 1). Afterwards the case of sum of one-side Lipschitz and a multimap satisfying compactness conditions is considered (Theorem 2).

Now we recall some definitions and notations. All the concepts not discussed in the sequel can be found in [7].

Given metric spaces Y and X, the multimap $F\colon Y \to 2^X$ (nonempty compact subsets of X) is said to be Hausdorff upper semicontinuous-USC [Hausdorff lower semicontinuous-LSC] iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that $F(x) \subset B_\varepsilon(F(y))$ $[F(y) \subset B_\varepsilon(F(x))]$ for all $x \in B_\delta(y)$, where $B_\delta(y)$ is the open ball centered at y with radius δ. For $A \subset E$, $\overline{A}$ (coA) is the closed (convex) hull of A. For bounded sets $D_H(A, B) = \max\left\{\sup_{a \in A} \varrho(a, B), \sup_{b \in B} \varrho(b, A)\right\}$, where $\varrho(a, B) = \inf_{b \in B} |a - b|$, is the Hausdorff distance. The multimap $F\colon I \times Y \to 2^X$ is called A(lmost)USC (LSC) iff there exists a sequence $\{J_m\}_{m=1}^{\infty}$ of compact mutually disjoint subsets of I such that $\operatorname{meas}\left(I \setminus \bigcup_{i=1}^{\infty} J_i\right) = 0$ and F is USC (LSC) on $J_i \times Y$ for every i. Given $M > 0$ we define the cone:

$$\Gamma^M := \{(t, x) \in I \times E\colon t \geqslant 0;\ |x| \leqslant Mt\}.$$

Let $A \subset I \times E$. The map $f\colon A \to E$ is said to be Γ^M continuous at (t_0, x_0) if to $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(t, x) - f(t_0, x_0)| < \varepsilon\,,$$

whenever

$$(t, x) \in B_\delta(t_0, x_0) \cap A \cap [(t_0, x_0) + \Gamma^M]\,.$$

For $A \in 2^E$, $\sigma(x, A) = \sup_{a \in A} \langle x, a\rangle$, where $x \in E^*$ is the support function. We denote

$$\beta(A) =$$

$$= \inf\{R > 0\colon A \textit{ can be covered by finitely many balls with radius} \leqslant r\}\,.$$

For $x \in E$ define $J(x) = \{x^* \in E^*\colon |x|^2 = |x^*|^2 = \langle x^*, x\rangle\}$. $J(\cdot)$ is called duality map. When E^* is uniformly convex it is well known that J is single valued and uniformly continuous on the bounded sets. (see [7] for instance). Define

$$f_J(\varepsilon, K) := \sup\{|J(x) - J(y)|\colon |x|,\ |y| \leqslant K;\ |x - y| \leqslant \varepsilon\}\,.$$

The indicator function

$$\chi_A(x) := \begin{cases} 1 & \text{for } x \in A\,, \\ 0 & \text{elsewhere}\,. \end{cases}$$

2. – Main results.

In this section we prove our main results. Denote

$$H(t, x) := \overline{\text{co}} \bigcap_{\varepsilon > 0} G(t, B_\varepsilon(x))$$

and

$$R(t, x) = F(t, x) + H(t, x)\,.$$

We will use the following assumptions:

H1. The multimap $F(\cdot, \cdot)$ is AUSC with convex compact values and $G(\cdot, \cdot)$ is closed valued ALSC such that $H(\cdot, \cdot)$ is compact valued and USC. Let also $|F(t, x) + G(t, x)| \leq \lambda(t)\{1 + |x|\}$ for $\lambda(\cdot)$ positive integrable function.

H2 (One-side Lipschitz condition). There exist integrable $M(\cdot)$ and $N(\cdot)$ such that

$$\sigma(J(x-y), F(t, x)) - \sigma(J(x-y), F(t, y)) \leq M(t)\,|x-y|^2\,,$$

$$\sigma(J(x-y), G(t, x)) - \sigma(J(x-y), G(t, y)) \leq N(t)\,|x-y|^2\,.$$

REMARK 1. The last condition-one-side Lipschitz is introduced in [8]. It is generalization of the known one sided Lipschitz condition:

for every x, y *and every* $f_x \in F(t, x)$, $f_y \in F(t, y)$

$$\langle J(x-y), f_x - f_y \rangle \leq L\,|x-y|^2\,.$$

In this case $G(t, x)$ is single valued and if H1 holds, then (1) admits an unique solution. The one sided Lipschitz is an extension of the classical Lipschitz condition in single valued case. Our condition (one-side Lipschitz) is an extension of the following multivalued Lipschitz condition:

$$D_H(F(t, x), F(t, y)) \leq L|x - y| .$$

Moreover $F(t, \cdot)$ is, obviously one-side Lipschitz iff $\overline{\text{co}}\, F(t, \cdot)$ is one side Lipschitz.

THEOREM 1. *Under the assumptions* H1, H2 *the problem* (1) *admits a solution. Moreover the solution set of* (1) *is connected and dense in the solution set of the relaxed problem:*

$$\dot{x}(t) \in R(t, x), \qquad x(0) = x_0. \tag{2}$$

Furthermore the solution set R_{RP} of (2) *is R_δ set.*

PROOF. First note that using standard arguments one can replace $N(\cdot) + M(\cdot)$ and $\lambda(\cdot)$ by 1 if needed, preserving the other hypotheses. Namely define $\varphi(t) = \max\{1, M(t) + N(t), \lambda(t)\} > 0$. The

$$\text{map}\, t \to \int_0^t \varphi(s)\, ds$$

is continuous and strictly increasing. Let $\Phi(\cdot)$ be its inverse, i.e. $\Phi\left(\int_0^t \varphi(s)\, ds\right) = t$, define

$$\tilde{F}(t, x) = \frac{1}{\varphi(\Phi(t))} F(\Phi(t), x)$$

and

$$\tilde{G}(t, x) = \frac{1}{\varphi(\Phi(t))} G(\Phi(t), x) \qquad \text{for } (t, x) \in I \times E .$$

Evidently $\tilde{F}$ and $\tilde{G}$ satisfy all the conditions mentioned above. Moreover the set of trajectories, as curves in the phase space, is preserved.

We divide the proof into four steps.

1. *A priori bounds.* Denote $U_M = \overline{B_M(0)}$. If $\varrho[\dot{x}(t), R(t, x(t) + U_1)] \leq 1$ then $|\dot{x}(t)| \leq |R(t, x(t) + U_1)| + 1 \leq 2\lambda(2 + |x|)$, i.e. $|x(t)| \leq (2 + |x_0|)\exp(\lambda)$ thanks to Gronwall inequality. Therefore there exist

constants L, K such that $|x(t)| \leqslant L$ and $|R(t, x)| \leqslant K$ for every absolutely continuous (AC) $x(\cdot)$ with

$$\dot{x}(t) \in R(I, x(t) + U_1) + U_1 , \qquad x(0) = x_0 .$$

We suppose $|R(t, x)| = |F(t, x) + H(t, x)| \leqslant K$ for all $x \in E$ since we consider only AC $x(\cdot)$, satisfying the differential inclusion above.

2. *Approximation.* Let $\tau_i = ih$; $h = 1/m$ be uniform grid of $[0, 1]$. Consider the discretized inclusion:

$$(3) \quad \dot{x}(t) \in R(t, x(\tau_i)) , \qquad x(0) = x_0 , \qquad t \in [\tau_i, \tau_{i+1}], \ i = 1, \ldots, m$$

Denote by R_{RP} and R_{DI} the solution set of (2.2) and (2.3) respectively.

We claim $\lim\limits_{h \to 0} D_H(R_{DI}, R_{RP}) = 0$.

Let $y(\cdot)$ be solution of (2.2). We get $\dot{x}(t) \in R(t, x(\tau_i))$ such that

$$\langle J(y(t) - x(\tau_i)), \dot{y}(t) - \dot{x}(t)\rangle \leqslant |y(t) - x(\tau_i)|^2.$$

I.e.

$$\langle J(y(t) - x(t)), \dot{y}(t) - \dot{x}(t)\rangle \leqslant |y(t) - x(t)|^2 + 2LKh + 2Kf_J(Kh, 2L).$$

Thus there exists a constant C such that $|x(t) - y(t)| \leqslant C(h + f_J(h, 2L))^{1/2}$. Let $x(\cdot)$ be a solution of (2.3). Consider another uniform grid $\{\tau_j^y\}_{j=1}^n$, $\tau_j^y = jh_y$ of $[0,1]$ such that the elements ih of the first grid are elements of the second one. We let $\dot{y}(t) \in R(t, y(\tau_j^y))$ for $j = 1, \ldots, n$ be such that

$$\langle J(y(\tau_j^y) - x(\tau_i)), \dot{y}(t) - \dot{x}(t)\rangle \leqslant |y(\tau_j^y) - x(\tau_i)|^2 .$$

I.e.

$$\langle J(x(t) - y(t)), \dot{x}(t) - \dot{y}(t)\rangle \leqslant |y(t) - x(t)|^2 + C(h + h_y + \gamma(h) + \gamma(h_y)) ,$$

where we have denoted $\gamma(h) = f_J(Kh, 2L)$. Furthermore $\tau_j^y \in [\tau_i, \tau_{i+1})$. Therefore $|x(t) - y(t)|^2 \leqslant C(h + h_y + \gamma(h) + \gamma(h_y))$. Using similar arguments one can show that $D_H(R_{DI}^1, R_{DI}) \leqslant C(h + h_y + \gamma(h) + \gamma(h_y))^{1/2}$. Since $\lim\limits_{h \to 0} \gamma(h) = 0$ one can conclude that there exists a sequence of subdivisions $\{P_m\}_{m=1}^\infty$ such that $P_m \subset P_{m+1}$ and $D_H(R_{DI}^m, R_{DI}^{m+1}) \leqslant (C(h + \gamma(h))^{1/2})/2^n$. Thus $\{R_{DI}^m\}_{m=1}^\infty$ is a Cauchy sequence in $C(I, E)$, here R_{DI}^m is the solution set of (3) with respect to m. Thus there exists $R = \lim_{m\to\infty} R_{DI}^m$. Obviously $R \equiv R_{RP}$. Using standard construction one can prove that given $\lambda > 0$ there exists a solution $y(\cdot)$ of (2) with $|x(t) - y(t)|^2 \leqslant C(h + \gamma(h)) + \lambda$. Since λ is arbitrary $D_H(R_{DI}, R_{RP}) \leqslant C(h + \gamma(h))^{1/2}$. The claim is proved.

3. *Existence of solutions.* Let $S(\cdot, x)$ be strongly measurable and let $\sigma(l, S(t, \cdot))$ be USC as a real valued function for every $l \in E^*$. Furthermore let $S(t, x) \subset R(t, x)$ be nonempty convex and compact valued. We claim that the solution set R_1 of

$$\dot{x}(t) \in S(t, x), \qquad x(0) = x_0$$

is nonempty compact valued and $\lim_{\varepsilon\to 0} D_H(R_1, R_\varepsilon) = 0$. Here R_ε is the solution set of

$$\dot{x}(t) \in \overline{\mathrm{co}}\, S(t, x + U_\varepsilon) + U_\varepsilon, \qquad x(0) = x_0 .$$

From the previous step we know that the solution set of (2) is nonempty compact. Consider

$$\dot{x}(t) \in S(t, x(\tau_i)), \qquad x(0) = x_0, \qquad x(\tau_i) = \lim_{t\to\tau_i - 0} x(t),$$

with the solution set R_{DI}^S. Obviously $\lim_{h\to 0} \varrho(R_{DI}^S, R_{RP}) = 0$. Therefore one can conclude that $\lim_{h\to 0} \varrho(R_{DI}^S, R_1) = 0$. I.e. R_1 is nonempty compact. The fact that $R_1 = \lim_{\varepsilon\to 0} R_\varepsilon$ is straightforward (see for instance [3, 7]).

Since G is ALSC there exists a sequence of mutualy disjoint compacts A_i such that $I = \bigcup_{i=i}^{\infty} A_i \cup \widetilde{N}$, where $\operatorname{meas}(\widetilde{N}) = 0$ and $G(\cdot, \cdot)$ is LSC on $A_i \times E$ for every i. From theorem 2 of [5] we know that there exists a selection $g(t, x) \in G(t, x)$ which is Γ^{K+1} continuous on every $A_i \times E$. If $h(t, x) = \overline{\mathrm{co}} \bigcap g(t, x + U_\varepsilon)$ then $h(t, x) \subset H(t, x)$. Hence there exists a solution $x(\cdot)$ of $\dot{x}(t) \in F(t, x) + h(t, x)$. It is easy to show that $\dot{x}(t) \in F(t, x) + g(t, x)$ (see the proof of Theorem 6.1 of [7]). Therefore the solution set of (1) is nonempty.

4. *Density.* Let $z(\cdot)$ be a solution of (2). It is easy to see that there exist sequences $\varepsilon_i \to 0$ and x_i such that $|x_i(t) - z(t)| \to 0$, where

$$\dot{x}_i(t) \in G(t, x_i(t) + U_{\varepsilon_i}) + F(t, x_i(t) + U_{\varepsilon_i}) + U_{\varepsilon_i}.$$

If $\dot{x}(t) \in R(t, x)$, $x(0) = x_0$, then $\dot{x} \in F(t, x) + \overline{\text{co}}\, G(t, x + U_\varepsilon)$ for every $\varepsilon > 0$. One can easily show that $(t, x) \to \overline{G(t, x + U_\varepsilon)}$ is ALSC. It is straightforward to prove that for $\{\varepsilon_i\}_{i=1}^{\infty}$ decreasing to zero there exist $\{x_i(\cdot)\}$ such that $\dot{x}_i(t) \in F(t, x + U_{\varepsilon_i}) + G(t, x + U_{\varepsilon_i}) + U_{\varepsilon_i}$ and $\lim_{i \to \infty} x_i(t) = z(t)$. Let $\dot{x}(t) = f(t) + g(t)$, where $f(t) \in F(t, x_i(t) + U_{\varepsilon_i})$ and let $g(t) \in G(t, x_i(t) + U_{\varepsilon_i}) + U_{\varepsilon_i}$ be strongly measurable. Let $\varepsilon > 0$ be given and let

$$\dot{x}(t) \in G(t, x(t) + U_\varepsilon) + F(t, x(t) + U_\varepsilon) + U_\varepsilon, \qquad x(0) = x_0.$$

Fix $\mu > 0$ and define the multimap:

$H_\mu(t, u) :=$

$$:= \text{cl}\,\{v \in G(t, u) : \langle J(x(t) - u), g(t) - v\rangle < (|x(t) - u| + \varepsilon)^2 + 2K\gamma(\varepsilon) + \mu\}.$$

We will prove that $H_\mu(\cdot, \cdot)$ is ALSC with nonempty compact values. Fix t. Let $y \in x(t) + U_\varepsilon$ be such that $g(t) \in G(t, y) + U_\varepsilon$. From H2 there exists $v \in G(t, u)$ such that $\langle J(y - u), \tilde{g}g - v\rangle \le |u - y|^2$, where $\tilde{g} \in G(t, y)$ and $|g(t) - \tilde{g}| < \varepsilon$. Since $|y - x(t)| < \varepsilon$, one has that

$$\langle J(x(t) - u), \tilde{g} - v\rangle \le \langle J(y - u), \tilde{g} - v\rangle + |J(y - u) - J(x - u)|\,|\tilde{g} - v|,$$

$$|y - u|^2 + 2K\gamma(\varepsilon) \le \{|x(t) - u| + \varepsilon\}^2 + 2K\gamma(\varepsilon).$$

I.e. $H_\mu(\cdot, \cdot)$ is nonempty compact valued. Let $G(\cdot, \cdot)$ be jointly LSC and $g(\cdot)$ be continuous on A. Suppose $t_i \to t$ $(t_i \in A)$, $u_i \to u$. Let $v \in G(t, u)$ be such that

$$\langle J(x(t) - u), g - v\rangle = 2K\gamma(\varepsilon) + \{|x(t) - u| + \varepsilon\}^2 + \mu_0,$$

where $\mu_0 < \mu$. Since $G(\cdot, \cdot)$ is LSC there exists $v_i \in G(t_i, u_i)$ such that $v_i \to v$ and $\langle J(x(t_i) - u_i), g - v_i\rangle < \{|x(t_i) - u_i| + \varepsilon\}^2 + 2K\gamma(\varepsilon) + \mu$ because $\langle J(x(t_i) - u_i), g(t_i) - v_i\rangle \to \langle J(x(t) - u), g(t) - v\rangle$ and $|x(t_i) - u_i| \to$ $\to |x(t) - u|$. Thus $H_\mu(\cdot, \cdot)$ is also ALSC. Therefore the differential inclusion:

$$\dot{y}(t) \in H_\mu(t, y) + F(t, y); \qquad y(0) = x_0$$

admits a solution $y(\cdot)$. By standard computations we obtain

$$\langle J(x(t)-y(t)),\dot{x}(t)-\dot{y}(t)\rangle \leqslant 2\,|x(t)-y(t)\,|^2+2KL(\varepsilon+\mu)+2K\gamma(\varepsilon).$$

Thus $|x(t)-y(t)\,| \leqslant C(\varepsilon+\mu+\gamma(\varepsilon))^{1/2}$.

One can prove that the solution set is connected as the corresponding result of [4]. Indeed let u_2, u_2 be two solutions of (1). Let $f_i(\cdot)$, $g_i(\cdot)$ be strongly measurable selections of $F(\cdot, u_i(\cdot))$ and of $G(\cdot, u_i(\cdot))$ respectively $i=1, 2$. For $i=$ $=1, 2$ consider the map

$$G^i(t, x) := \begin{cases} g_i(t), & \text{for } x = u_i(t), \\ G(t, x), & \text{elsewhere}. \end{cases}$$

By Lusin's theorem there exists a sequence of mutually disjoint compacts $J_n \subset I$ with meas $(I \setminus \bigcup J_n) = 0$ such that $\dot{u}_i(\cdot)$ are continuous on J_n and $G(\cdot, \cdot)$ is LSC on $J_n \times E$. Since $G^i(\cdot, \cdot)$ is LSC on $J_n \times E$ it admits Γ^{K+1} continuous selection $g_n^i(\cdot$ $\cdot, \cdot)$. Define $h^i(t, x) = g_n^i(t, x)$, for $t \in J_n$. Set $G_n^i(t, x) = \bigcap_{\varepsilon > 0} \overline{\mathrm{co}}\,\{g_n^i(s, y)$ for $|x-y| < \varepsilon$; $s \in [t, t+\varepsilon) \bigcap J_n\}$. Define also $H^i(t, x) = G_n^i(t, x)$, for $t \in J_n$. I.e. $H^i(\cdot, \cdot)$ is well defined on $I \setminus \widetilde{N}$. For $\lambda \in [0, 1]$ consider:

$$r_\lambda(t, x) = \chi_{[0, \lambda)}(t)\, h^1(t, x) + \chi_{[\lambda, 1]}(t)\, h^2(t, x),$$

$$R_\lambda(t, x) = \chi_{[0, \lambda)}(t)\, H^1(t, x) + \chi_{[\lambda, 1]}(t)\, H^2(t, x).$$

Let S_λ be the solution set of (1) with $R_\lambda(\cdot, \cdot)$ instead of $G(\cdot, \cdot)$. From Theorem 5.2 of [2] we know that S_λ is compact connected. Obviously $\lambda \to S_\lambda$ is USC. Thus $\bigcup_{\lambda \in [0, 1]} S_\lambda \subset S(x)$ is compact connected containing u_1 and u_2. Therefore $S(x)$ is itself connected.

It remains to show that R_{RP} is R_δ set. As in the proof of Theorem 5.2 of [3] consider the sequence of locally Lipschitz $R_n(t, x) \supset R(t, x)$ on $I \times E$. Denote $\widetilde{R}_n(t, x) = \overline{R_n(t, x) + U_{b_n}}$ where $b_n = 2^{-n}$. The solution set R_n^2 of (2.2) with $\widetilde{R}_n$ instead of R is closed contractible. Moreover $R_{RP} = \bigcap_{n \geqslant 1} R_n^2$ and $\lim_{n \to \infty} \beta(R_n^2) = 0$. Therefore R_{RP} is in fact R_δ set (see for details [3]). ■

Corollary 1. *Under the conditions of Theorem 1 there exists a constant C such that* $D_H[R_{RP}, RU_{RP}] \leqslant C\sqrt{\varepsilon + f_J(K\varepsilon, 2L)}$, *where* RU_{RP} *is the solution set of*

$$\dot{x}(t) \in R(t, x + U_\varepsilon) + U_\varepsilon, \qquad x(0) = x_0.$$

Consider the following differential inclusion

$$(4) \qquad \dot{x}(t) \in F(t, x(t)) + V(t), \qquad x(0) = x_0,$$

where $V: I \to 2^E$ is bounded strongly measurable compact valued. Denote by $R_F(V)$ the solution set of (4). From Theorem 1 and Theorem 5.2 of [3] follows:

COROLLARY 2. *If F satisfies* H1 *and* H2 *then $R_F(V)$ is nonempty compact R_δ set.*

PROPOSITION 1. $D_H(R_F(V_1), \ R_F(V_2)) \leq \left[\int_0^t D_H(V_1(s), V_2(s))\, ds\right] \exp(1)$.

PROOF. Let $y(t) \in R_F(V_2)$. It is sufficient to show that there exists $x(t) \in R_F(V_2)$ such that $|x(t) - y(t)| \leq \left[\int_0^t D_H(V_1(s), V_2(s))\, ds\right] \exp(M)$. Consider the multimap

$$G(t, x) = \{u \in F(t, x) + V_1(t) : \langle J(x - y(t)), u - \dot{y}(t)\rangle \leq$$
$$\leq |J(x - y(t))| [\,|x - y(t)| + D_H(V_1(t), V_2(t))]\}$$

Since $F(\cdot, \cdot)$ is compact valued and since $\varrho(\dot{y}(t), F(t, y(t)) + V_1(t)) \leq \leq D_H(V_1(t), V_2(t))$, taking into account H2, we obtain $G(t, x) \neq \emptyset$. It is easy to show that $G(t, x)$ is convex compact valued, $G(\cdot, x)$ is strongly measurable and $\sigma(l, G(t, \cdot)$ is USC for every $l \in E^*$ as a real valued function. Therefore the differential inclusion $\dot{x}(t) \in G(t, x)$; $x(0) = x_0$ admits a solution thanks to Corollary 1 because $G(t, x) \subset F(t, x)$. Consequently $d/dt\, |x(t) - y(t)| \leq |x(t) - y(t)| + D_H(V_1(t), V_2(t))$. ■

The following lemma is proved in [10].

LEMMA 1. *Let X be a Banach space, $\emptyset \neq D \subset X$ is compact convex and $F: D \to 2^D \setminus \emptyset$* USC *with compact R_δ values. Then F has a fixed point.*

The problem of the existence of solutions of (1) when $F(\cdot, \cdot)$ is one-side Lipschitz and $G(\cdot, \cdot)$ satisfies compactness conditions is difficult even when F and G are single valued.

In the sequel suppose E is separable. We need the next condition.

H3. $G(\cdot, \cdot)$ is ALSC closed valued and F is AUSC convex compact valued. Furthermore $\beta(G(t, A)) \leqslant k(t)\beta(A)$ for every bounded $A \subset E$.

LEMMA 2. *Let $l(t)$ be $L_1(I, R)$ function and let $\{v_k\}_{k=1}^{\infty} \subset L_1(I, E)$ satisfy $|v_k(t)| \leqslant l(t)$ for a.e. $t \in I$ and all k, then*

$$\beta\left[R_F\left\{\bigcup_{k=1}^{\infty} v_k\right\}(t)\right] \leqslant \left[\int_0^t \beta\left\{\bigcup_{k=1}^{\infty} v_k(s)\right\} dt\right] \exp(M).$$

The proof of the corresponding result of [2] works also in our case and will be given in the last section.

The following theorem extends Theorem 3.2 of [1] and Theorem 4 of [2].

THEOREM 2. *If F satisfies* H1, H2 *and G satisfies* H1, H3 *then the solution set of the differential inclusion* (1) *is nonempty connected.*

PROOF. First one can suppose $k(t) \equiv M(t) = \lambda = 1$ if needed. Obviously there exist positive constants K and L such that $|x(t)| \leqslant L$ and $|F(t, x) + G(t, x)| \leqslant K$ whenever $\varrho(\dot{x}(t), F(t, x + U_1) + G(t, x + U_1)) \leqslant$ $\leqslant 1$ (see proof of Theorem 1 apriori bounds). Let $g(t, x) \in G(t, x)$ be Γ^{K+1} continuous selection and let $h(t, x)$ be as in the proof of Theorem 1 (existence of solutions). Consider (2) with $H(t, x)$ replaced by $h(t, x)$. For AC x we set $V_x(t) = h(t, x(t))$. Due to H1 there exists a closed convex bounded and equicontinuous set $S_1 \subset C(I, E)$ such that $R_F(S_1) \subset S_1$. Here $R_F(S_1) = \bigcup_{x \in S_1} R_F(V_x)$. We let $S_{n+1} = \overline{\mathrm{co}}\ R_F(S_n)$ for $n \geqslant 1$. Denote $S = \bigcap_{k=1}^{\infty} S_n$. Set $P_n(t) = \beta(S_n(t))$. Therefore $P_n(t) \leqslant$ $\leqslant \beta\{R_F[S_n(t)]\} \exp\{1\}$. Since E is separable one has that there exists a sequence $\{v_k(\cdot)\} \subset S_n$ such that $\beta\{R_F[S_n(t)]\} = \beta\left[R_F\left(\bigcup_{k=1}^{\infty} v_k\right)\right]$. Thus $P_{n+1} \leqslant \int_0^t \beta\left(\left\{\bigcup_{k=1}^{\infty} v_k(s)\right\}\right) ds \leqslant \left[\int_0^t P_n(s)\, ds\right] \exp(1)$ due to Lemma 2.

Therefore $P_\infty(t) \leqslant \exp(1)\left\{\int_0^t P_\infty(s)\, ds\right\}$, and $P_\infty(0) = 0$. Hence $P_\infty(t) \equiv 0$, i.e. $\beta(S) = 0$. Since E is reflexive $S \neq \emptyset$. Furthermore $\lim_{n \to \infty} P_n(t) \geqslant P_\infty(t)$ and hence $\bigcap_{n=1}^{\infty} S_n = S$ is nonempty convex compact in

$C(I, E)$. Moreover $R_F(S) \subset S$, i.e. there exists a fixed point x. This AC $x(\cdot) \in R_F(V_x)$ i.e. $\dot{x}(t) \in F(t, x) + h(t, x)$, $x(0) = x_0$. Now it is obvious to show that $\dot{x}(t) \in F(t, x) + g(t, x)$ a.e. in I. ■

3. – Concluding remarks.

PROPOSITION 2. *Let E be a Banach space and let $E \supset K_n \neq \emptyset$ with $K_n \subset K_{n+1}$ be such that $\beta(K_n \cup A) = 0$ for bounded $A \subset E$ and all n. If $K = \overline{\bigcup_{n \geq 1} K_n}$ and if $B = \{x_k : k \geq 1\} \subset K$ is bounded then*

$$\beta_K(B) = \lim_{n \to \infty} \overline{\lim_{k \to \infty}} \varrho(x_k, K_n),$$

where

$$\beta_K = \inf\{r > 0 : B \text{ can be covered by finitely many balls with radius} \leqslant \leqslant r \text{ with center in } K\}.$$

This is Proposition 3 of [2] and can be proved (with obvious modifications) as Proposition 9.2 of [7]. Moreover it is not difficult to see that this proposition holds also when x_k are replaced by compact sets X_k. One has only to set $\varrho(X_k, K_n) = \sup_{x \in X_k} \inf_{a \in K_n} |x - a|$.

Obviously for all bounded $B \subset K$ the following inequalities are valid:

$$\beta(B) \leqslant \beta_K(B) \leqslant 2\beta(B).$$

PROOF OF LEMMA 2. Let E_n be finite dimensional such that $E = \overline{\bigcup_{n \geq 1} E_n}$. Define $W_n = \{v \in L^1_{E_n}(I) : |v(s)| \leqslant 2l(s) \text{ a.e. on } I\}$ and let $K_n = \{R_F(V)(t) : V \in W_n\}$ for fixed $t \in I$. For $\varepsilon > 0$ there exists $I_\varepsilon \subset I$ with $\operatorname{meas}(I_\varepsilon) \geqslant 1 - \varepsilon$ such that $l(t)$ is bounded on I_ε. Therefore $K_n^\varepsilon = \{R_F(W)(t) : W = V\xi_{I_\varepsilon} \text{ with } V \in K_n\}$ is relatively compact thanks to Theorem 2.1. Since $K_n \subset K_n^\varepsilon + U_\varepsilon$ one has that $\beta(K_n) = 0$. Due to Proposition 2

$$\beta\left(\left[R_F\left\{\bigcup_{k=1}^{\infty} v_k\right\}(t)\right]\right) \leqslant \beta_K\left(\left[R_F\left\{\bigcup_{k=1}^{\infty} v_k\right\}(t)\right]\right) = \lim_{n \to \infty} \overline{\lim_{k \to \infty}} \varrho(R_F(w_k)(t), K_n).$$

Now

$$\varrho(R_F(w_k)(t), K_n) \leqslant \inf\{D_H[R_F(w_k)(t), R_F(w)(t)]: w \in K_n\} \leqslant$$

$$\leqslant \left\{\int_0^t \inf\{w\,|w_k(t) - w(t)\,|: w \in K_n\}\, ds\right\} \exp(t) \leqslant \left\{\int_0^t \varrho(w_k(s), E_n)\, ds\right\} \exp(t).$$

From Fatou's lemma and the dominated convergence theorem we get

$$\beta\left[R_F\left\{\bigcup_{k=1}^{\infty} v_k\right\}(t)\right] \leqslant \int_0^t \beta\left\{\bigcup_{k=1}^{\infty} v_k\right\}(s)\, ds \cdot \exp(t) \text{ on } I. \qquad \blacksquare$$

REMARK 2. Using obvious modifications of the presented proof one can prove Theorem 2 when F satisfies compactness conditions and G is one-side Lipschitz multimap. Moreover Theorems 1 and 2 can be proved when F is not AUSC, but the support function $\sigma(l, F(\cdot, \cdot))$ is AUSC for every $l \in E^*$ as a real valued function. Moreover H2 can be relaxed to

$$\sigma(J(x-y), F(t, x) + G(t, x)) - \sigma(J(x-y), F(t, y) + G(t, y)) \leqslant$$

$$\leqslant w(t, |x-y|)\,|x-y|,$$

where $w(\cdot, \cdot)$ is a Kamke function. One can prove Theorems 1 and 2 also in case of

$$\dot{x}(t) \in F(t, x) + G(t, x) + H(t, x), \qquad x(0) = x_0,$$

where $H(\cdot, \cdot)$ is one sided Lipschitz and $H(\cdot, \cdot)$ is USC with closed bounded convex (non nessesarily compact) values. In this case (2.3) becomes

$$\dot{x}(t) \in R(t, x(\tau_i)) + H(t, x(t)), \qquad x(0) = x_0, \qquad t \in [\tau_i, \tau_{i+1}].$$

Moreover suppose $\lim_{h \to 0} \beta([t, t+h], A) \leqslant u(t, \beta(A))$, where $u(\cdot, \cdot)$ is a Kamke function one can prove Theorem 2 for nonseparable E. If $M(\cdot)$ and $N(\cdot)$ are constants, then the grouth condition in H1 can be replaced by $R(\cdot, \cdot)$ *is bounded on bounded sets*. Indeed from H2 follows

$$\sigma(J(x-y), R(I, x + U_1)) - \sigma(J(x-y), R(t, y + U_1)) \leqslant (N + M)\,|x-y|^2,$$

I.e.

$$\langle J(x(t)), \dot{x}(t)\rangle \leqslant \sigma(J(x), R(I, x + U_1)) + \sigma(J(x), R(I, U_1) + U_1),$$

for every $\dot{x}(t) \in \overline{\mathrm{co}}\ R(I, x+U_1)+U_1$. I.e. $|x(t)| \leqslant \{|x_0|+M'+1\} \exp(M + N)$, where $M' \geqslant |R(I, U_1)|$.

We give two examples for systems which are neither Lipschitz, nor one sided Lipschitz, but satisfy H1 and H2.

EXAMPLE 1. Define $f(y) = -y/\sqrt{|y|}$ for $y \neq 0$ and 0 elsewhere for $y \in R^n$. Consider the system:

$$\dot{x}(t) \in \{-1, 1\}, \quad x(0) = 0,$$
$$\dot{y}(t) = x^2 + f(y), \quad y(0) = 0.$$

Obviously H1, H2 hold if $t \in [0, 1]$. Therefore the solution set of this system is dense in the solution set of the convexified problem:

$$\dot{x}(t) \in [-1, 1], \quad x(0) = 0,$$
$$\dot{y}(t) = x^2 + f(y), \quad y(0) = 0.$$

However if we replace the second equation by

$$\dot{y}(t) = x^2 + f(-y), \quad y(0) = 0,$$

then H2 does not hold and the solution set of (1) is not dense in the solution set of the convexified problem. It is the significant counter example of Plis ([11]).

EXAMPLE 2. Let E be the Hilbert space l_2. Consider the system:

$$\dot{x}(t) \in \sum_{k=1}^{\infty} g(x_i - x) + V, \quad x(0) = 0.$$

Here $\{x_i\}_{i=1}^{\infty}$ is dense in U set, $g(x) = -x/|x|^{2/3}$ for $x \neq 0$, $g(x) = 0$ for $x = 0$ and $V := \{x \in l_2 \colon x_i \in [-1/i, 1/i]\}$. Obviously H1, H2 hold. Here the one-side Lipschitz constant is 0.

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