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Some New Partially Symmetric Designs and their Resolution.

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ABSTRACT - In this note we study a resolution (a generalisation of a parallelism) in the (new) partially symmetric designs of the type $\mathcal{S} = \mathcal{O} \backslash \mathcal{O}'$ where $\mathcal{O}'$ is a tight (Baer) subdesign in the symmetric $2-(v, k, \lambda)$ design $\mathcal{O}$ (with $\lambda > 1$).

1. - Introduction.

Throughout this note let $\mathcal{O}$ be a symmetric $2-(v, k, \lambda)$ design (with $\lambda > 1$) and let $\mathcal{O}'$ be a symmetric $2-(v', k', \lambda')$ subdesign of $\mathcal{O}$. By Jungnickel [4] $\mathcal{O}'$ is a tight subdesign of $\mathcal{O}$ iff each block of $\mathcal{O} \backslash \mathcal{O}'$ meets $\mathcal{O}'$ in a constant number $x$ of points. Furthermore if $\lambda = \lambda'$ (and then $x = 1$) we say $\mathcal{O}'$ is Baer subdesign of $\mathcal{O}$.

By Hughes [2] a square 1-design $\mathcal{S}$ is a partial symmetric design (a PSD) if there exist integers $\lambda_1, \lambda_2 \geq 0$ such that two points are on $\lambda_1$ or $\lambda_2$ common blocks; two blocks of $\mathcal{S}$ contains $\lambda_1$ or $\lambda_2$ common points and all such that $\mathcal{S}$ is connected. We say then $\mathcal{S}$ is PSD for $(v_1, k_1, \lambda_1, \lambda_2)$ (where $v_1$ is the number of points (blocks) in $\mathcal{S}$ and $k_1$ block (point)-size of $\mathcal{S}$).

The concept of a divisibility and resolution (a generalisation of the parallelism) we take as in [3] (pp. 206, 154) and [1] (pp. 45, 39).

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2. – Results.

2.1 Proposition. Let \( \Omega' \) be a Baer subdesign of \( \Omega \). Then \( \mathcal{J} = \Omega \setminus \Omega' \) is a PSD for \( (v_1 = v - v', k_1 = k - 1, \lambda_1 = \lambda - 1, \lambda_2 = \lambda) \) and the relation \( \parallel \) (for arbitrary blocks \( b \) and \( c \) in \( \mathcal{J} \))

\[ b \parallel c \text{ iff } b \text{ and } c \text{ lie on the same point in } \Omega' \]

is an equivalence relation and, in this sense, \( \mathcal{J} \) is divisible.

Proof. It is clear that \( \mathcal{J} \) is a square 1-design with \( v_1 = v - v' \) points (blocks) and with point (block)-size \( k - 1 \). Each point in \( \Omega \setminus \Omega' \) is exactly on one block of \( \Omega' \) and thus two points in \( \Omega \setminus \Omega' \) lie exactly on \( \lambda - 1 \) or \( \lambda \) blocks of \( \Omega \setminus \Omega' \). Further any two blocks of \( \Omega \setminus \Omega' \) lie exactly on 0 or 1 points in \( \Omega' \) and their integers in \( \Omega \setminus \Omega' \) are \( \lambda - 1 \) or \( \lambda \).

The partition of the set of all blocks in \( \mathcal{J} \) onto subsets of blocks passing through some point from \( \Omega' \) is disjoint and therefore the \( \parallel \) is an equivalence relation.

2.2 Remark. By 2.1, \( \mathcal{J} = \Omega \setminus \Omega' \) (where \( \Omega' \) is a Bear subdesign of \( \Omega \)) is a PSD with a divisibility. But, in general, \( \mathcal{J} \) cannot have a resolution. For instance \( \mathcal{J} = \Omega \setminus \Omega' \), where \( \Omega \) is a symmetric 2 - (16, 6, 2) design with a symmetric 2 - (4, 3, 2) subdesign \( \Omega' \).

But we have

2.3 Proposition. Let \( \Omega = PG_2(q) \) (\( q \) a prime power). Then \( \Omega \) has Baer 2 - (\( q + 1, q + 1, q + 1 \)) subdesign \( \Omega' \) and the \( \mathcal{J} = \Omega \setminus \Omega' \) has a strong resolution.

Proof. By [4] \( \Omega' \) exists. Any class of blocks are all blocks in \( \mathcal{J} = \Omega \setminus \Omega' \) through any point in \( \Omega' \). Two different classes are disjoint.

Each of these classes have exactly \( m = q^2 + q + 1 - (q - 1) = q^2 \) blocks and any two blocks in the same class have exactly \( \lambda - 1 \) points in common. Two blocks in the different classes have exactly \( \lambda \) points in common. Finally, each point in \( \Omega \setminus \Omega' \) lies exactly on \( \lambda - 1 \) blocks of any one class. Thus, \( \mathcal{J} \) have a strong resolution.

In general, let \( \Omega = PG_{2d}(2d + 1, q) \) (\( d \geq 2 \)) be the design of points and hyperplanes of the \( (2d + 1) \)-dimensional projective space over \( GF(q) \). Then, by [4], \( \Omega \) has a tight \( (c, c, c) \)-subdesign \( \Omega' \) with \( c = q^d + + \ldots + q + 1 \).

In general, we cannot say anything of the relation \( \parallel \) (as in 2.1 and 2.3) in \( \mathcal{J} = \Omega \setminus \Omega' \). Namely, the partition of \( \mathcal{J} \), corresponding to \( \parallel \), is not
disjoint. Further, we cannot say anything of an inner (outer) constant
(for \( \| \)). But we have

2.4 PROPOSITION. Let \( \mathbb{W} = PG_{2d}(2d + 1, q) \) \((d \geq 2)\) and let \( \mathbb{W}' \) be a
tight \((c, c, c)\)-subdesign. Then \( \mathcal{J} = \mathbb{W} \setminus \mathbb{W}' \) is a PSD having a strong
resolution.

PROOF. The parameters of \( \mathbb{W} \) and \( \mathbb{W}' \) are \( v = q^{2d+1} + \ldots + q + 1, \)
\( k = q^{2d} + \ldots + q + 1, \) \( \lambda = q^{2d-1} + \ldots + q + 1 \) and \( c = q^d + \ldots + q + 1. \)
It is not difficult to check that any point in \( \mathcal{J} = \mathbb{W} \setminus \mathbb{W}' \) is exactly on \( x \)
blocks of \( \mathbb{W}' \) where \( x = q^{d-1} + \ldots + q + 1. \) Thus \( \mathcal{J} \) is a square \( 1-\left(v - c, k - x, k - x \right) \) design. Further we have:

\[
\text{from } 1 + (x(x-1))/\lambda_1 = c \quad \text{we get } \lambda_1 = q^{d-2} + \ldots + q + 1 \; ;
\]

\[
k - c = \ldots = q^{d+1} (q^{d-1} + \ldots + q + 1) = q^{d+1} x
\]

and

\[
\lambda - c = \ldots = q^{d+1} (q^{d-2} + \ldots + q + 1) = q^{d+1} \lambda_1.
\]

Here there is an automorphism \( y \mapsto y + u \) exchanging the points
resp. the blocks (pointwise) in \( \mathbb{W}' \) (Singer cycle in \( \mathbb{W}' \), generated
additively with \( u \) in \( \mathbb{Z}_\nu \)). Thus we conclude that the blocks in \( \mathbb{W}' \)
\((\equiv PG_{d-1}(d, q))\) have \( x \) sets, each of these sets has exactly \( q^{d+1} \)
points (in \( \mathbb{W} \setminus \mathbb{W}' \)) and any two blocks in \( \mathbb{W}' \) have exactly \( \lambda_1 \) sets in common. We
are calling these sets «points». So any two points of \( \mathbb{W} \setminus \mathbb{W}' \) are in one or
in two «points». Therefore through two points in \( \mathbb{W} \setminus \mathbb{W}' \) pass, according
with this, \( x \) or \( \lambda_1 \) blocks from \( \mathbb{W}' \). Thus, on two points of \( \mathcal{J} \) lie \( \lambda_1 = \lambda - x \)
or \( \lambda_2 = \lambda - \lambda_1 \) common blocks. By [4] (2.1), we get this for the intersec-
tions of the blocks in \( \mathcal{J} \). Thus, \( \mathcal{J} \) is a PSD for

\[
(v - c, k - x, \lambda_1 = \lambda - x, \lambda_2 = \lambda - \lambda_1) =
\]

\[
= (q^{2d+1} + \ldots + q^{d+1}, q^{2d} + \ldots + q^d, q^{2d-1} + \ldots + q^d, q^{2d-1} + \ldots + q^{d-1}) .
\]

Any resolution-class in \( \mathcal{J} \) is formed from all blocks in \( \mathcal{J} \) passing
through any block in \( 2-(c, x, \lambda_1) \) design \( \mathbb{W}' \). One has exactly

\[
\frac{k - c}{x} = \frac{q^{d+1} x}{x} = q^{d+1} \]

blocks in each resolution-class and exactly

\[
\frac{v - c}{q^{d+1}} = \frac{q^{2d+1} + \ldots + q + 1 - (q^{d} + \ldots + q + 1)}{q^{d+1}} = \ldots = c
\]
(disjoint!) classes. There are exactly \((\lambda - x)/x = \ldots = q^d\) blocks from each resolution-class passing through any point in \(\mathcal{F} = \mathcal{O}\setminus \mathcal{O}'\).

Finally, our resolution is strong with inner and outer constant \(\lambda - x\) and \(\lambda - \lambda_1\) respectively. ■

AN ILLUSTRATION. \(\mathcal{O} = PG_4(5, 2)\) (with a tight subdesign for \((7, 7, 7)\)). The initial block in \(\mathcal{O}\) (in the form of a difference set) is

\[1_0 = 0, 1, 2, 3, 4, 6, 7, 8, 9, 12, 13, 14, 16, 18, 19, 24, 26, 27, 28, 32, 33, 35, 36, 38, 41, 45, 48, 49, 52, 54, 56.\]

(All blocks \(1_i\) are formed by \((\text{mod. } 63)\) addition of the \(1, 2, \ldots, 62\) respectively.) The points in \(\mathcal{O}'\) are \(0, 9, 18, 27, 36, 45, 54\).

The resolution-classes are:

\((0) \wedge (9) = (0) \wedge (45) = (9) \wedge (45) =\)

\[= \{1_7, 1_{31}, 1_{37}, 1_{39}, 1_{44}, 1_{56}, 1_{59}, 1_{60}\},\]

\((0) \wedge (18) = (0) \wedge (27) = (18) \wedge (27) =\)

\[= \{1_{11}, 1_{14}, 1_{15}, 1_{25}, 1_{49}, 1_{55}, 1_{57}, 1_{62}\},\]

\((0) \wedge (36) = (0) \wedge (54) = (36) \wedge (54) =\)

\[= \{1_{22}, 1_{28}, 1_{30}, 1_{35}, 1_{47}, 1_{50}, 1_{51}, 1_{61}\},\]

\((9) \wedge (18) = (9) \wedge (54) = (18) \wedge (54) =\)

\[= \{1_2, 1_5, 1_6, 1_{16}, 1_{40}, 1_{46}, 1_{48}, 1_{53}\},\]

\((9) \wedge (27) = (9) \wedge (36) = (27) \wedge (36) =\)

\[= \{1_1, 1_{3}, 1_8, 1_{20}, 1_{23}, 1_{24}, 1_{34}, 1_{53}\},\]

\((18) \wedge (36) = (18) \wedge (45) = (36) \wedge (45) =\)

\[= \{1_4, 1_{10}, 1_{12}, 1_{17}, 1_{29}, 1_{32}, 1_{33}, 1_{43}\},\]

\((27) \wedge (45) = (27) \wedge (54) = (45) \wedge (54) =\)

\[= \{1_{13}, 1_{19}, 1_{21}, 1_{26}, 1_{38}, 1_{41}, 1_{42}, 1_{52}\},\]

This is the 4-resolution with the constants 12 and 14.
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