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Pencils of binary quartics

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Pencils of Binary Quartics.

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1. - Introduction.

The objective of this article is to discuss the classification and invariant theory of pencils of binary quartics. The main motivation for this is the intrinsic interest of the question. Indeed, it is now known (see [1]) that for very few types of quantics (a single homogeneous equation) is the ring of invariants a complete intersection (for degree d in r variables we require either $d \leqslant 2$, $d = 3$, $r \leqslant 4$ or $d \leqslant 6$, $r = 2$), so it is of interest to document such further natural situations where—as here—the ring of invariants is a complete intersection.

There are, however, two additional reasons for writing this paper at this time. One is that the results (or at least part of them) are used in recent work of the author [6] giving an algorithm to list possible configurations of singularities on quartic surfaces in $P^3(\mathbb{C})$, given that the singularities are isolated and not all simple.

The other is that whereas at the time of writing [5] I considered these questions but my notes were incomplete due to the difficulty of calculations of invariants, these can now be performed relatively easily using computer algebra (my preference is for MAPLE).

The paper opens with a section discussing pencils of binary quantics in general, and establishing basic ideas and notation, in particular a symbol descibing the ramification associated with each case. The various types of pencil of binary cubics and quartics are then listed and named, thus defining a stratification *ad hoc*. The map from strata to possible symbols is not surjective; in these degrees it is injective.

The next section discusses the application of geometric invariant theory, and determines the structure of the ring of invariants. Most of the invariants were already given by Salmon in [4], but his list is incomplete.

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The case of pencils of binary cubics was analysed by Newstead in [3], a source I have found very useful.

Then we discuss the notion of symmetry of forms and of pencils. We discuss non-stable strata in a separate section; in the remainder of the paper we describe the stratification of the moduli space.

The discussion of symmetry leads to a more systematic way to view the strata. We re-present the list, and show how to characterise the cases on it.

Finally we calculate the invariants on the examples from our lists of strata, and list the symmetry types that occur in special cases.

1. – General notions.

Consider the space $V = V_n$ of homogeneous polynomials of degree n in the variables x and y. A *pencil of binary* n^{ics} is a 2-dimensional subspace of V. We can define it by picking any two linearly independent members $f = f(x, y)$ and $g = g(x, y)$, and write $\langle f, g \rangle$. Thus a typical member of the pencil is a linear combination $pf + qg$. We regard two pencils as equal if we have the same subspace of V and as isomorphic if they are equivalent under some automorphism in $GL(V)$.

We can distinguish two main subcases, according as f and g do or do not have a common factor (of degree at least 1). If there is no common factor, then for any linear expression $y - tx$ (or x) there is a member of the pencil, unique up to a scalar factor, which it divides: it is determined by $pf(1, t) + qg(1, t) = 0$ (note that by hypothesis, $f(1, t)$ and $g(1, t)$ do not both vanish). Equivalently, the point of projective space $P^1(C)$ determined by the ratio $(x: y)$ determines a unique ratio $(p: q)$, and hence a point of projective space $P^1(C)$. We can thus identify such pencils with rational maps (of degree n) from $P^1(C)$ to $P^1(C)$. Note that we regard these as *different* projective spaces, so will not attempt to iterate such maps. Two ways of obtaining a single polynomial from a pencil are immediate. The *Jacobian* J is defined, as usual, as $\partial f/\partial x \cdot \partial g/\partial y - \partial f/\partial y \cdot \partial g/\partial x$. It thus is homogeneous of degree $2n - 2$. Up to a constant factor, it depends only on the pencil. The *discriminant* Δ is the usual discriminant of $pf + qg$ considered as polynomial in x and y. This is homogeneous of degree $2n - 2$ in p and q.

In the case of a pencil without common factor, J represents the set of branch points of the corresponding rational map, and Δ the set of images of these points.

This statement is made more precise in the following lemma. Here we write $\nu_x(f)$ to denote the multiplicity of x as factor of f: this is equal to r if f is divisible by x^r but not by x^{r+1}.

LEMMA 1.1. (i) *If the pencil has no common factor, then* $\nu_x(J) = r - 1$ *if the member* $pf + qg = h$, *say, of the pencil divisible by* x *has* $\nu_x(h) = r$, *and* $\nu_q(\Delta) = r$ *iff* f *has* $n - r$ *distinct linear factors.*

(ii) *If the pencil* $\langle f, g\rangle$ *has Jacobian* J *and discriminant* Δ *while for* $f' = xf$, $g' = xg$ *we have* J' *and* Δ', *then* $J' = x^2 J$ *(up to a multiplicative constant). If* x *divides both* f *and* g *then* $\Delta' = 0$; *otherwise* x *divides* $p_0 f + q_0 g$ *for a unique* $(p_0 : q_0)$ *and* $\Delta' = (pq_0 - qp_0)^2 \Delta$ *(up to a constant).*

PROOF. The assertions for J follow by elementary calculations. If $\nu_x(f) = r$ and $\nu_x(g) = s > r$ then J is clearly divisible by x^{r+s-1} and we may check that the coefficient of $x^{r+s-1}y^{2n-r-s-1}$ is non-zero.

Now suppose that f has $n - r < n$ distinct linear factors; write f_1 for their product, and $f = f_1 f_2$, so that both $\partial f/\partial x$ and $\partial f/\partial y$ are divisible by f_2. Now Δ is the determinant of the matrix of coefficients of the $x^a y^b$ $(a + b = 2n - 2)$ in $\partial(pf + qg)/\partial x$ and $\partial(pf + qg)/\partial y$. This is a sum of 2^{2n-2} similar determinants each obtained by omitting either p or q in each row, hence each a multiple of some $p^{2n-2-t}q^t$.

If $t < r$ we have at least $2n - 1 - r$ rows formed from f. The corresponding polynomials, of degree $2n - 3$, are all multiples of f_2, of degree r, so form a vector space of dimension at most $2n - 2 - r$. The corresponding determinant thus vanishes. Thus Δ is divisible by q^r

First suppose the pencil has no common factor. Since a general member does not then have a repeated root (by Bertini's theorem), Δ does not vanish identically. We have proved that it is divisible by a product P, say, of terms corresponding to members of the pencil with repeated factors. We will now show that P has the same degree $2n - 2$ as Δ: the desired result will follow.

As described above, the pencil defines a rational map from P^1 to P^1. Let us calculate Euler characteristics *à la* Riemann-Hurwitz. Since $\chi(P^1) = 2$ and the degree is n, the characteristic of the source P^1 is equal to $2n$ minus corrections due to the branching behaviour. A member of the pencil with just $n - r$ distinct factors corresponds to a point with just $n - r$ distinct pre-images, and hence requires a correction of r. Since the characteristic of the source is 2, the sum of these correction terms must equal $2n - 2$. This proves our claim.

Now consider $f' = xf$, $g' = xg$; we may suppose that f is divisible by x, so $q_0 = 0$. Write

$$R = \partial/\partial x(pf + qg), \qquad S = \partial/\partial y(pf + qg).$$

Then Δ is the determinant of $\{x^a y^b R, x^a y^b S \mid a + b = n - 2\}$ with respect to $\{x^c y^d \mid c + d = 2n - 3\}$ and Δ' is the determinant of

$\{x^a y^b R', x^a y^b S' \mid a+b=n-1\}$ with respect to $\{x^c y^d \mid c+d=2n-1\}$, where

$$R' = \partial/\partial x(x(pf+qg)), \qquad S' = \partial/\partial y(x(pf+qg)).$$

It follows from homogeneity that

$$R' = (n+1)xR + nyS, \qquad S' = xS.$$

Thus elementary operations transform the above list to

$$\{x^{a+2} y^b R, x^{a+2} y^b S \mid a+b=n-2\},$$

$$xy^{n-1}S, \ (n+1)\, xy^{n-1}R + ny^n S.$$

The first part of this list has determinant Δ with respect to $\{x^{c+2} y^d \mid c+d=2n-3\}$, so we must extract the coefficients of xy^{2n-2}, y^{2n-1} in the last two terms. We thus see that (up to a non-zero scalar) Δ is multiplied by the square of the coefficient of y^{n-1} in S, or equivalently, that of y^n in g, which is zero if x also divides g, and is a non-zero multiple of q otherwise. ■

We use the above to define a symbol for any pencil to describe its branching behaviour. First suppose there is no common factor. For each root of Δ (of multiplicity r, say) we list the multiplicities of the factors (there are $n-r$ of them) of the corresponding member $pf+qg$ of the pencil, separated by commas, and enclose this set in parentheses. Do this for each root of Δ, and enclose the whole in (square) brackets.

In the case of a pencil with common factor h, say $\langle fh, gh \rangle$, first we write the symbol of the pencil $\langle f, g \rangle$, Now for each linear factor of h, of multiplicity r, if this has already occurred as a factor of one of the members of the pencil with a repeated root, we underline the corresponding number in the symbol r times. If not, we attach a new blank parenthesis to the symbol and underline that instead. The definition is amply illustrated in the lists to follow.

For a pencil with no common factor, we can interpret the symbol topologically. We have a branched n-fold covering of a 2-sphere; at each branch point, the symbol provides a partition of n which describes the nature of the branching. The monodromy given by encircling the point once is a permutation of the n branches whose cycle type is given by this partition.

Globally, if we puncture the sphere at the k branch points we obtain an unramified covering. The fundamental group of the punctured sphere is generated by the classes $\{\alpha_i\}$ of loops circling these points, which are subject to the sole relation $\alpha_1 \alpha_2 \dots \alpha_k = 1$ (so generate a free group of rank $k-1$). This acts transitively by permutations on a set of cardinality n; the fundamental group of the cover is a stabiliser subgroup.

As we have noted in the above proof, the numerical conditions ensure that when we compactify this covering by filling in the punctures we obtain a surface of genus 0.

2. – Enumeration.

We begin by enumerating types of pencil of degree $n \leqslant 3$. Note that pencils with a common factor are enumerated by removing the factor, and then considering the different places in which it can be inserted in the symbol. The enumeration with $n=3$ is given in [3]; see also [5], where a notation is introduced and normal forms discussed more fully (though the emphasis there is on the real case).

The first column in Table 1 gives the names assigned in [5] to the various cases with $n=3$. The fourth column is the dimension of the stratum (as a subset of the Grassmannian of 2-dimensional subspaces of the $n+1$-dimensional vector space V). The column headed J gives the multiplicities of the linear factors of J. We observe that in each case the number of such factors equals the dimension. The corresponding column for Δ would be the same except for case (e), where $\Delta \equiv 0$.

TABLE 1.

	Symbol	*Normal form*		*J*
		$\underline{n=1}$		
	[ ]	$\langle x, y \rangle$	0	( )
		$\underline{n=2}$		
	$[(\underline{\cdot})]$	$\langle x^2, xy \rangle$	1	(2)
	[(2)(2)]	$\langle x^2, y^2 \rangle$	2	(1,1)
		$\underline{n=3}$		
(e)	$[(\underline{\underline{\cdot}})]$	$\langle x^3, x^2 y \rangle$	1	(4)
(c)	$[(\underline{\cdot})(\underline{\cdot})]$	$\langle x^2 y, xy^2 \rangle$	2	(2,2)
(d)	$[(\underline{2})(2)]$	$\langle x^3, xy^2 \rangle$	2	(3,1)
(b)	$[(2)(2)(\underline{\cdot})]$	$\langle x^2(x+y), y^2(x+y) \rangle$	3	(2,1,1)
(C)	[(3)(3)]	$\langle x^3, y^3 \rangle$	2	(2,2)
(B)	[(3)(2, 1)(2, 1)]	$\langle x^3, x^2 y + y^3 \rangle$	3	(2,1,1)
(A)	[(2, 1)(2, 1)(2, 1)(2, 1)]	$\langle x^3 + \varrho x y^2, \varrho x^2 y + y^3 \rangle$	4	(1,1,1,1)

TABLE 2.

	Symbol	Normal form		J
(a)	$[(\underline{\underline{\underline{\cdot}}})]$	$\langle x^4,\, x^3y\rangle$	1	(6)
(b)	$[(\underline{\underline{\cdot}})(\underline{\cdot})]$	$\langle x^3y,\, x^2y^2\rangle$	2	(4,2)
(c)	$[(\underline{\cdot})(\underline{\cdot})(\underline{\cdot})]$	$\langle x^2y(x+y),\, xy^2(x+y)\rangle$	3	(2,2,2)
(d)	$[(\underline{\underline{2}})(2)]$	$\langle x^4,\, x^2y^2\rangle$	2	(5,1)
(e)	$[(\underline{2})(\underline{2})]$	$\langle x^3y,\, xy^3\rangle$	2	(3,3)
(f)	$[(\underline{2})(2)(\underline{\cdot})]$	$\langle x^3(x+y),\, xy^2(x+y)\rangle$	3	(3,2,1)
(g)	$[(2)(2)(\underline{\cdot})]$	$\langle x^2(x+y)^2,\, y^2(x+y)^2\rangle$	3	(4,1,1)
(h)	$[(2)(2)(\underline{\cdot})(\underline{\cdot})]$	$\langle x^2(x^2+2axy+y^2),$ $y^2(x^2+2axy+y^2)\rangle$	4	(2,2,1,1)
(i)	$[(\underline{3})(3)]$	$\langle x^4,\, xy^3\rangle$	2	(4,2)
(j)	$[(3)(3)(\underline{\cdot})]$	$\langle x^3(x+y),\, y^3(x+y)\rangle$	3	(2,2,2)
(k)	$[(\underline{3})(2,\,1)(2,\,1)]$	$\langle x^4,\, x^3y+xy^3\rangle$	3	(4,1,1)
(l)	$[(3)(\underline{2},\,1)(2,\,1)]$	$\langle x^3y,\, xy^3+y^4\rangle$	3	(3,2,1)
(m)	$[(3)(2,\,\underline{1})(2,\,1)]$	$\langle x^3(x+y), (x+y)^2y^2\rangle$	3	(2,2,1,1)
(n)	$[(3)(2,\,1)(2,\,1)(\underline{\cdot})]$	$\langle x^3(x+ky),\, y^2(x+y)(x+ky)\rangle$	4	(2,2,1,1)
(o)	$[(\underline{2},\,1)(2,\,1)(2,\,1)(2,\,1)]$	$\langle x^3y,\, x(x^3+3bxy^2+y^3)\rangle$	4	(3,1,1,1)
(p)	$[(2,\,\underline{1})(2,\,1)(2,\,1)(2,\,1)]$	$\langle x^2y^2,\, y(x^3+3bxy^2+y^3)\rangle$	4	(2,1,1,1,1
(q)	$[(2,\,1)(2,\,1)(2,\,1)(2,\,1)(\underline{\cdot})]$	$\langle x^4+ax^3y+bx^2y^2,\, x^2y^2+xy^3\rangle$	5	(2,1,1,1,1

We now begin the same for pencils of quartics. First we consider cases with a common factor, which we list in Table 2: as before, we accomplish this by inserting a common factor in the cases in Table 1.

The precise conditions on the parameters defining these strata will be discussed below. An exception occurs in case (h), where if $a = 0$ the pencil contains $(x^2+y^2)^2$, and Δ has type $(2, 2, 2)$. We will distinguish this case and denote it by (h_0).

We next enumerate the cases with no common factor. The list is summarized in Table 3. In order to display all the data, each case occupies two lines.

To obtain the list in Table 3, we must argue directly. First suppose the pencil contains a fourth power, say $f = x^4$. We may take the coefficient of y^4 in g as 1; perform a substitution $y' = y + kx$ to remove the term xy^3 and add a multiple of f to g to reduce g to the form $ax^3y + bx^2y^2 + y^4$. (In future we will omit essentially trivial arguments of this type.) We now find (it is easy using MAPLE) that $\Delta = q^3D$, where D is a

TABLE 3.

Name / *Dimension*	*Symbol* / *Normal form*	J / Δ
(A)	$[(4)(4)]$	$(3, 3)$
2	$\langle x^4, y^4\rangle$	$(3, 3)$
(B)	$[(4)(2, 2)(2, 1, 1)]$	$(3, 1, 1, 1)$
3	$\langle x^4, x^2y^2 + y^4\rangle$	$(3, 2, 1)$
(C)	$[(4)(3, 1)(2, 1, 1)]$	$(3, 2, 1)$
3	$\langle x^4, y^3(x+y)\rangle$	$(3, 2, 1)$
(D)	$[(4)(2, 1, 1)(2, 1, 1)(2, 1, 1)]$	$(3, 1, 1, 1)$
4	$\langle x^4, x^3y + 3bx^2y^2 + y^4\rangle$	$(3, 1, 1, 1)$
(E)	$[(2, 2)(2, 2)(2, 2)]$	$(1,1,1,1,1,1)$
3	$\langle x^2y^2, (x^2+y^2)^2\rangle$	$(2, 2, 2)$
(F)	$[(2, 2)(2, 2)(2, 1, 1)(2, 1, 1)]$	$(1,1,1,1,1,1)$
4	$\langle x^2y^2, (x^2 + 2axy + y^2)^2\rangle$	$(2, 2, 1, 1)$
(G)	$[(2, 2)(3, 1)(3, 1)]$	$(2, 2, 1, 1)$
3	$\langle x^3(x-2y), y^3(4x+y)\rangle$	$(2, 2, 2)$
(H)	$[(2, 2)(3, 1)(2, 1, 1)(2, 1, 1)]$	$(2, 1, 1, 1, 1)$
4	$\langle x^3y, (x^2 + 2axy + y^2)^2\rangle$	$(2, 2, 1, 1)$
(I)	$[(3, 1)(3, 1)(3, 1)]$	$(2, 2, 2)$
3	$\langle x^3(x+2y), y^3(2x+y)\rangle$	$(2, 2, 2)$
(J)	$[(3, 1)(3, 1)(2, 1, 1)(2, 1, 1)]$	$(2, 2, 1, 1)$
4	$\langle x^3(x+4ky), y^3(x+y)\rangle$	$(2, 2, 1, 1)$
(K)	$[(2, 2)(2, 1, 1)(2, 1, 1)(2, 1, 1)(2, 1, 1)]$	$(1,1,1,1,1,1)$
5	$\langle x^2y^2, x^4 + bx^3y + dxy^3 + y^4\rangle$	$(2, 1, 1, 1, 1)$
(L)	$[(3, 1)(2, 1, 1)(2, 1, 1)(2, 1, 1)(2, 1, 1)]$	$(2, 1, 1, 1, 1)$
5	$\langle x^3y, x^4 + cx^2y^2 + dxy^3 + y^4\rangle$	$(2, 1, 1, 1, 1)$
(M)	$[(2,1,1)(2,1,1)(2,1,1)(2,1,1)(2,1,1)(2,1,1)]$	$(1,1,1,1,1,1)$
6	$\langle x^4 - x^2y^2, ax^4 + bx^3y + dxy^3 + y^4\rangle$	$(1,1,1,1,1,1)$

cubic (not divisible by q) with discriminant $a^2(27a^2 + 8b^3)^3$. We thus have cases (A), where $a = b = 0$, (B) with $a = 0$, $b \neq 0$, (C) with $a = 1$, $b = -3/2$, and (D) where none of these occur. A more convenient normal form for (C) is given in Table 3.

Next we enumerate cases where the pencil contains no fourth power but has two members each with only 2 distinct prime factors. First suppose both of these have multiplicities $(2, 2)$. Then we can take $f = x^2 y^2$, $g = (x^2 + 2axy + y^2)^2$. Here we have $\Delta = p^2 q^2 D$ where q divides D only if $a^2 = 1$ (which is excluded since the pencil contains no fourth power) and D has a repeated root if and only if $a = 0$. We thus have cases (E) with $a = 0$ and (F) with $a \neq 0$.

Next suppose we have multiplicities $(3, 1)$ and $(2, 2)$, so we can take $f = x^3 y$ and $g = (x^2 + 2axy + y^2)^2$. Then again $\Delta = p^2 q^2 D$ with neither p nor q dividing D; the discriminant of D is $(a^2 + 3)^3$ (up to a constant). We thus have cases (G) with $a^2 = -3$ and (H) otherwise. If both f and g have multiplicities $(3, 1)$ we can take $f = x^3(ax + by)$ and $g = y^3(cx + dy)$, where as there is no common factor, $ad(ad - bc) \neq 0$ and as there is no fourth power, $bc \neq 0$. We again have $\Delta = p^2 q^2 D$ with neither p nor q dividing D; the discriminant of D is $(ad - bc)(8ad + bc)(4ad - bc)$. The case $8ad + bc = 0$ again yields the stratum (G); otherwise we have case (I) with $4ad = bc$ and case (J) where the above discriminant does not vanish.

There remain three cases: (K) where f has factor multiplicities $(2, 2)$ but each other member of the pencil has at least 3 distinct linear factors; (L) similarly with $(3, 1)$ and the generic case (M).

In all cases in Table 2 and Table 3, the dimension of the stratum is equal to the number of distinct linear factors of Δ, except in cases $(a), (b), (d), (g)$ where there is a common repeated factor and $\Delta \equiv 0$; in these cases, we may count the factors of J instead. Apart from these, the factors of Δ correspond to those of J except whenever the symbol contains $(2, 2)$ (in Table 2, only cases (m) and (p)).

A more interesting remark is that the symbol $[(2, 2)(2, 2)(3, 1)]$ does *not* appear. This could be predicted from the topological interpretation. For if such a cover existed, we could find permutations of a set of 4 objects, of respective cycle types $(2, 2)$ $(2, 2)$ and $(3, 1)$, whose product is the identity. But this is manifestly impossible.

3. – Invariant theory.

To apply invariant theory in its simplest form, we consider generic quartics (or indeed n^{ics}) f, g, and the action of $GL_2 \times GL_2$ on the polynomial ring in the coefficients of f and g, where one copy of GL_2 acts by linear substitutions in x and y and the other by linear substitutions in f and g.

There are several ways to construct invariants of a pencil of binary quartics. One may consider $pf + qg$ as a quartic in x and y and take its invariants: its self-transvectant $I(p, q)$ and its catalecticant $J(p, q)$. We then have the discriminant DI of $I(p, q)$, which has weight 2 (in the coefficients of f and also in those of g); the resultant DR of $I(p, q)$ and $J(p, q)$, which has weight 6; and the discriminant DJ of $J(p, q)$, which also has weight 6. We may also form the Hessian determinant $H(p, q)$ of $J(p, q)$ and the transvectant DH of $H(p, q)$ and $I(p, q)$, which has weight 4.

Alternatively we may start with the Jacobian J and construct its invariants as a binary sextic (our version of these - including the algorithm to compute them - is taken from [2], p.156). We denote these by JI_2, JI_4, JI_6, JI_{10} and JI_{15}, where the suffix gives the weight in each case. We recall that the first four of these are independent, while the square of JI_{15} may be expressed as a polynomial in them. More precisely, we call an invariant Φ *even* if $\Phi(-f) = \Phi(f)$, and *odd* if the sign is changed. Then the even invariants form a polynomial ring in JI_2, JI_4, JI_6 and JI_{10}; each odd invariant is divisible by JI_{15}, with quotient an even invariant.

(We could also consider the invariants for Δ but here the weights would be tripled, and in any case as $\Delta = 4I(p, q)^3 + 27J(p, q)^2$ many of these can be recovered from the above covariants of $I(p, q)$ and $J(p, q)$).

We also have the resultant $Res(f, g)$ of f and g.

We choose the following as our basic invariants:

$$I_2 = DI\,, \qquad J_2 = (JI_2 - 3DI)/4\,, \qquad I_4 = 64\,Res(f, g)\,, \qquad I_6 = DR\,.$$

These are essentially the same as the invariants given by Salmon [4], Arts 213-217.

There are various conventions as to the precise definition of invariants, so that each one has a somewhat variable scalar factor. Our convention is, in effect, defined by the computer programme we have used. It can be inferred from the following examples which give the invariants for normal forms for the top dimensional strata $(K), (L), (M)$. Most other cases may be obtained by substituting in these. In subsequent calculations we write conventionally C_* to denote an unimportant scalar factor (usually involving a large power of 2).

Write F for the generic quartic $F = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$. Then the invariants for the pencil $\langle F, x^4\rangle$ (generically in case (D)) are

$$(48e^2, 8e^2, 64e^4, 0);$$

for the pencil $\langle F, x^3y\rangle$ (if we set $a = e = 1$ this is the normal form for case (L)) we have

$$(3d^2, 4ce - d^2, 64ae^3, 0);$$

and for the pencil $\langle F, x^2y^2\rangle$ (here we may set $a = e = 1$ to give the normal form for case (K)) we have

$$(4bd - 16ae,\ 8ae,\ 64a^2e^2, (16ae - bd)^3 + 27(ad^2 - b^2e)^2)\,.$$

The two latter cases are, of course, independent of b and c respectively, since these do not affect the pencil. Similarly in the invariants for the generic case $\langle F,\ x^4 - x^2y^2\rangle$, the coefficients a and c only appear in the combination $a + c$. Taking $c = 0$ (for the normal form for (M) we also have $e = 1$), we have

$$\begin{aligned}(4bd - 16ae + 48e^2,\ &-4d^2 + 8ae + 8e^2,\ 64e^2(a - b - d + e)(a + b + d + e),\\ 27a^2d^4 &- bd^3(b + 9d)^2 + 6abd^2e(6d - b) - 384a^2de^2(2b + 3d) +\\ &+ 27be^2(b^3 + 20b^2d + 116bd^2 + 128d^3) - 128ae^3(-32a^2 + 108bd + 27b^2) +\\ &+ 24^3b^2e^4)\,.\end{aligned}$$

As a final example, for the given normal form for case (q), the invariants are

$$(3a^2 - 4ab,\ 4b - a^2,\ 0,\ b^3(a^3 - 9a^2 + 27b))\,.$$

In our enumeration of strata, we defined 3 codimension 1 conditions:

there is a common factor,

the pencil contains a perfect square,

the pencil contains a quartic divisible by a cube.

The invariant I_6 is defined as the resultant DR of $I_{p,q}$ and $J_{p,q}$. It thus vanishes if and only if these have a common factor. This means that for some element $pf + qg$ in the pencil, both its invariants I and J vanish, so it has a factor of multiplicity at least 3. Hence the third condition is invariantly characterised by the vanishing of I_6.

The invariant I_4 is defined as the resultant of f and g. It thus vanishes if and only if f and g have a common factor: the first condition above.

These conclusions were obtained *à priori*. By direct calculation we also find that the second condition holds if and only if $J_2^2 = I_4$. These are the considerations which guided our choice of the invariants to take as basic. The conditions $I_6 = 0$, $I_4 = 0$ and $I_4 = J_2^2$ are illustrated on the above examples.

PROPOSITION 3.1. *The even invariants of J are expressed in terms* *of the above by the following formulae:*

$$JI_2 = C_*(3J_2 - I_2);$$

$$JI_4 = C_*(7I_2^2 - 22I_2J_2 + 63J_2^2 - 135I_4);$$

$$JI_6 = C_*(-37I_2^3 + 678I_2^2J_2 - 2709I_2J_2^2 + 324J_2^3 - \\ -810I_4(I_2 - 3J_2) - 1600I_6);$$

$$JI_{10} = -\frac{157}{45}I_2^5 + 99I_2^4J_2 - 969I_2^3J_2^2 + 3567I_2^2J_2^3 - \\ -2538I_2J_2^4 - \frac{486}{5}J_2^5 + \frac{45}{4}I_4(-7I_2^3 + 129I_2^2J_2 - 657I_2J_2^2 - 81J_2^3) - \\ -\frac{320}{9}I_6(7I_2^2 - 72I_2J_2 + 63J_2^2) + \frac{3645}{4}I_4^2(I_2 - 3J_2) + 2800I_4I_6.$$

The odd invariant JI_{15} *is a product* $C_*\,FI_9\,FI_6$ *of irreducible factors, where*

$$FI_6 = 243I_4(I_2 - 9J_2) + 256I_6,$$

and FI_9^2 *is obtained by substituting* $A = I_2$, $B = 3J_2$, $C = 81I_4$ *and* $D = = 64I_6$ *in*

$$A(A-3B)^2(A+B)^6 - 3AC(A+B)^3(A-3B)(A^2-4AB+B^2) + \\ +\frac{2}{3}AC^2(5A^4 - 48A^3B + 114A^2B^2 - 40AB^3 + 9B^4) + \\ +\frac{4}{3}AC^3(A^2 - 4AB + B^2) + \frac{1}{9}AC^4 + \\ +D(3A^6 + 3A^4B^2 + 40A^3B^3 + 57A^2B^4 + 24AB^5 + B^6) + \\ +CD(A^4 - 40A^3B + 38A^2B^2 - 16AB^3 + B^4) + \\ +\frac{1}{3}C^2D(19A^2 - 24AB + B^2) + \frac{1}{27}C^3D + \\ +D^2(3A^3 + 15AB^2 + 2B^3) + CD^2(5A - 2B) + D^3.$$

PROOF. The result was established by direct calculation using MAPLE. The calculations were performed as follows. It is enough to consider the generic case $\langle F, x^4 - x^2y^2\rangle$ considered above, with $c = 0$. Note that I_4 has a factor e. Substituting $e = 0$ yields

$$I_2 = 4bd, \quad J_2 = -4d^2, \quad I_6 = 27a^2d^4 - bd^3(b + 9d)^2.$$

We may solve these equations as

$$d = \frac{1}{2}\sqrt{-J_2}, \qquad b = \frac{I_2}{4d}, \qquad a = \frac{\sqrt{3I_6 + bd^3(b+9d)^2}}{9d^2}$$

and hence define substitutions (with new variables i_2, j_2, i_6)

$$d = \frac{1}{2}\sqrt{-j_2}, \qquad b = \frac{i_2}{4d}, \qquad a = \frac{\sqrt{3i_6 + bd^3(b+9d)^2}}{9d^2}.$$

Now for any invariant Φ of J (or indeed of Δ) we proceed as follows. Write X for Φ computed on the above example. Substitute $e = 0$ to obtain A_1. Now make the above substitutions for a, b, d in A_1 and simplify: this yields a polynomial B_1 in i_2, j_2 and i_6. Substituting the computed values of I_2, J_2, I_6 for i_2, j_2, i_6 in B_1 gives an expression C_1. Now divide $A_1 - C_1$ by the computed value of I_4 and simplify to obtain an expression A_2 (the fact that this division is possible is a crucial check on the calculation). Repeat the entire procedure, obtaining in turn B_2, C_2, A_3 and so on. Since the degree is lowered at each step, the procedure terminates.

The desired formula is now given by the final value D_n where D_r is inductively defined by $D_0 = 0$, $D_r = D_{r-1} + i_4^{r-1}B_r$, with I_2 for i_2, etc. This procedure is easily programmed, and running the programme gives the results above. ■

We can also run the programme on other invariants—e.g. those of the binary sextic Δ (after the first, we just obtain complicated expressions) and, more interestingly, the discriminants, giving

$$\operatorname{discrim} J = C_* I_4 I_6,$$

$$\operatorname{discrim} \Delta = C_* I_4 (I_4 - J_2^2)^2 I_6^3,$$

conforming to the considerations of Section 1.

THEOREM 3.1. *The ring of invariants of a pencil of binary quartics is generated by I_2, J_2, I_4, I_6 and FI_9. The only syzygy is that expressing the square of FI_9 in terms of the rest.*

The invariant FI_9 was overlooked by Salmon. We shall show below how to interpret it.

PROOF. We will model our proof on the corresponding argument given by Newstead [3] for pencils of cubics. We begin with the vector space V of quartics, containing f and g. A 2-dimensional subspace of V is determined by its 10 Plücker coordinates, or equivalently by the exterior product $f \wedge g \in \wedge^2(V)$.

As representation of GL_2 (acting on x and y), $\wedge^2(V)$ splits as direct sum of representations of degrees 7 and 3. The Jacobian J is obtained by projecting $f \wedge g$ onto the 7-dimensional component. As projective variety, the Grassmannian has dimension 6 and we are projecting onto a 6-dimensional linear space. This projection is generically smooth, as we see by a simple calculation in local coordinates. Since the Grassmannian variety has degree 5, this is also the degree of the projection. It follows that the field of invariants of J is an extension of degree 5 of the field of invariants of the pencil.

The field of invariants of J is an extension of degree 2 of the field generated by the independent invariants JI_2, JI_4, JI_6 and JI_{10}. We have expressed these as polynomials in our basic invariants I_2, J_2, I_4, I_6. Comparing the degrees of these invariants we see that we have a field extension of degree $(2 \times 4 \times 6 \times 10)/(2 \times 2 \times 4 \times 6) = 5$. Moreover the quadratic extension given by JI_{15} corresponds to the quadratic extension given by FI_9. Since our invariants generate a field over which the invariants of J generate an extension of degree 5, it follows that we do indeed have the field of all invariants.

It remains to show that this field contains no polynomials which are not expressible as polynomials in the given invariants. Suppose it does, and that A, B are polynomials, with no non-trivial common factor, such that

$$\frac{A(I_2, J_2, I_4, I_6)}{B(I_2, J_2, I_4, I_6)},$$

when evaluated for a generic pair $\langle f, g \rangle$ of binary quartics, reduces to a polynomial in the coefficients. We may suppose without loss of generality that A and B are homogeneous, and that we have a counter-example of minimal degree.

We recall that substituting $f = F$, $g = x^3 y$ gives

$$(I_2, J_2, I_4, I_6) = (3d^2, 4ce - c^2, 64ae^3, 0).$$

Since the first three of these are algebraically independent, substituting in B will yield zero identically only if I_6 divides B. But, in the case when f and g are generic, I_6 is irreducible (it is so already in Case (K) for example), so since A/B becomes polynomial in this case, I_6 must also divide A: a contradiction.

Thus $A(3d^2, 4ce - d^2, 64ae^3, 0)/B(3d^2, 4ce - d^2, 64ae^3, 0)$ has non-vanishing denominator, so is a homogeneous polynomial in a, c, d and e: say $P(a, c, d, e)$. It follows that

$$\frac{A(I_2, J_2, I_4, I_6)}{B(I_2, J_2, I_4, I_6)} = P\left(\frac{I_4}{64}, \frac{I_2 + 3J_2}{4}, \sqrt{\frac{I_2}{3}}, 1\right),$$

since this holds on substituting $I_2 = 3d^2$, $J_2 = 4c - d^2$, $I_4 = a$. Since the left hand side is a rational function, $\sqrt{I_2}$ can only appear on the right hand side *via* its square. Hence the right hand side is a polynomial $C(I_2, J_2, I_4)$.

Now set

$$A'(I_2, J_2, I_4, I_6) = A(I_2, J_2, I_4, I_6) - B(I_2, J_2, I_4, I_6)\, C(I_2, J_2, I_4).$$

Then A'/B has the same properties as A/B, and vanishes for the above example. But this example is the generic element of its stratum. Thus A'/B vanishes whenever $I_6 = 0$. As above, it follows that this, and hence A', is formally divisible by I_6: $A' = I_6 A''$. But now A''/B has the same properties and lower degree, contradicting the inductive hypothesis. ■

PROOF. Since we have established our invariants by comparison with the ring of invariants of the jacobian J, it is of some interest to consider the induced map of moduli spaces in more detail. We confine ourselves to finding its branch locus. This is given by the vanishing of

$$\frac{\partial(JI_2, JI_4, JI_6, JI_{10})}{\partial(I_2, J_2, I_4, I_6)} = C_*(I_2 + 3J_2)\, FI_6$$

(after a brief calculation in MAPLE). The vanishing locus of this function meets our strata in a rather complicated manner: we will not investigate this further.

4. – Pencils with symmetry.

As a preliminary to discussing symmetry of pencils, it is convenient to note the situation for binary sextics, or more generally, n^{ics}. We define the symmetry group of a form F to be that subgroup of GL_2 which transforms F to a multiple of itself, or (better) its image in PGL_2.

For the n^{th} power x^n of a single linear form, we obtain the full group of upper triangular matrices. For a monomial $x^r y^{n-r}$ we have the group of diagonal matrices, augmented, if $r = n - r$, by the transposition of the two factors.

In all other cases, the subgroup of PGL_2 preserves a finite set of at least 3 points on the projective line, hence is itself finite. It may thus be taken to be contained in a maximal compact subgroup $PU_2 \cong SO_3$ of $PGL_2(\mathbb{C})$. This group is thus cyclic, dihedral, or one of the 3 polyhedral groups. Let us write C_n for a cyclic group of order n; D_{2n} for a dihedral group of order $2n$; and $Tet, Oct, Icos$ for the respective polyhedral groups.

If there is a symmetry of order k, we may conjugate it to diagonal form. This transforms f to the product of a monomial by a polynomial in x^k and y^k; this still leaves the freedom to multiply x and y by constants. Excluding the case when f is a monomial, it follows that $k \leqslant n$.

Now we fix $n = 6$. For f not a monomial, we have the following cases:

$$\begin{aligned}
k &= 5 \quad f_5 = x(x^5 - y^5),\\
k &= 3 \quad f_3 = ax^6 + bx^3y^3 + cy^6,\\
k &= 3 \quad g_3 = x^5y + x^2y^4,\\
k &= 2 \quad f_2 = ax^6 + bx^4y^2 + cx^2y^4 + dy^6,\\
k &= 2 \quad g_2 = ax^5y + bx^3y^3 + cxy^5.
\end{aligned}$$

The case $k = 6$ is given by taking $b = 0$ in f_3; the case $k = 4$ by $b = d = 0$ in f_2 or by $b = 0$ in g_2. For f_3 and g_2, if $ac \neq 0$ we can reduce to $a = c = 1$ and have a further symmetry interchanging x and y, giving a dihedral group. In fact all cases admitting the four group D_4 can be reduced to g_2.

Case f_2 is characterised by the vanishing of the invariant I_{15}. The only case of a group neither cyclic nor dihedral is given by $xy(x^4 - y^4)$, representing the vertices of a regular octahedron and thus admitting the octahedral group.

Now we turn to the case of pencils. Here we consider the product $GL_2 \times GL_2$ of the groups of linear substitutions on the variables (x, y) and on the forms (p, q), and consider the action of this group on pairs $\langle p, q\rangle$. Since, by hypothesis, p and q are linearly independent, the sta-

TABLE 4.

Name	f	g	*Symmetry group*
s_4	x^4+y^4	x^3y	C_4
t_4	x^4+y^4	x^2y^2	*Oct*
s_3	x^4+xy^3	ax^3y+by^4	D_6
t_3	x^4+xy^3	x^2y^2	C_3
s_2	$ax^4+bx^2y^2+cy^4$	dx^3y+exy^3	C_2
t_2	$ax^4+bx^2y^2+cy^4$	$a'x^4+b'x^2y^2+c'y^4$	D_4

biliser intersects $1\times GL_2$ trivially, so projects isomorphically into the first factor. Since the subgroup $(x, y, p, q)\to(ax, ay, a^{-4}p, a^{-4}q)$ always acts trivially, we may project further to PGL_2. The resulting group is, of course, a subgroup of the automorphism group of the Jacobian of p and q, which we have just described.

In particular, if J has just two linear factors, we may take both p and q to be monomials: for pencils of quartics, this arises in cases $(a), (b), (d), (e), (i), (A)$. Here the group is infinite: in case (a) we have the triangular matrices, in the rest the diagonal ones, with interchange possible for (e) and (A).

Otherwise we have a finite group, which may be regarded as a subgroup of SO_3. If a cyclic group of order m acts by symmetries, we may diagonalise the group to multiply x by m^{th} roots of unity, say, leaving y invariant, and look for this to multiply each of p and q by a constant. This implies that each of p and q is equal to a monomial multiplied by a polynomial in x^m and y^m.

For pencils of quartics, with the case when both p and q are monomials excluded, this implies $m\leq 4$, and leads to the list in Table 4. Here the cases $(s_4), (t_4)=(E), (t_3)$ are unique up to isomorphism; the 1-parameter families (s_3) and (t_2) (which is essentially the same as (F)) contain some special cases, listed in Section 6 below. The 2-parameter family (s_2) has a number of interesting specialisations, which are also discussed in detail in Section 6.

5. – Non-stable pencils.

In the next section we will use our invariants to describe strata in the moduli space. As this ignores non-stable cases, we consider these more fully now.

Applying geometric invariant theory (in its naïve form) to our situation, we find 3 cases for $\langle f, g\rangle$ not semistable:

(i) $g \equiv 0$;

(ii) x divides f and x^4 divides g;

(iii) x^2 divides f and x^3 divides g;

and 3 cases for $\langle f, g\rangle$ not stable:

(iv) x^4 divides g;

(v) x divides f and x^3 divides g;

(vi) x^2 divides f and x^2 divides g.

However, (i) does not correspond to a pencil; and if (vi) holds, then some other member of the pencil is divisible by x^3, so we reduce to case (iii). The correct list of possibilities is thus the following.

A non-semistable pencil is either of type (ii), which occurs if and only if the pencil contains a 4^{th} power and has a common factor: this gives cases $(a), (d), (i), (k)$; or of type (iii), which occurs if and only if there is a repeated common factor, viz, for cases $(a), (b), (g)$.

A semi-stable pencil which is not stable is either of type (iv), which occurs when there is a fourth power, hence for cases $(A), (B), (C), (D)$; or of type (v), which implies, but is not equivalent to, having a common factor: it comprises cases $(e), (f), (l)$ and (o)

All invariants vanish for the non-stable pencils; for each of the semi-stable groups (iv), (v), the invariants take a unique value (up to scaling by constants) for the whole group. Such a group is typified by the unique case representing a closed orbit: this is (A) for type (iv) and (e) for type (v). In the next section we will refer to the corresponding points in moduli space as $(A)^+$ and $(e)^+$.

We now go through these cases considering first, the precise conditions on the normal form in Tables 2 and 3, and second, what is the symmetry group in each case.

The first question is quickly answered in most cases, since only for cases (D) and (o) does the proposed normal form contain a parameter. For each of these, our reduction to normal form fixed x and y up to scalar factors: x as the repeated factor; y since in the (o) case, the pencil contains x^3y, in the (D) case we eliminated the term in xy^3. Now the only scalars which leave the form invariant involve multiplying x (or y) by a cube root of unity: this multiplies b by such a root. Thus in both cases b^3 is the parameter determining the pencil up to isomorphism.

It follows that the pencil $\langle x^4, y(y^3 + 3dx^2y + ex^3)\rangle$ has type (A) if $d = e = 0$, type (B) if $e = 0$, $d \neq 0$, type (C) if $e \neq 0$, $8d^3 + e^2 = 0$ and otherwise type (D), with parameter $d^3/e^2 \neq -1/8$. The pencil $\langle x^3y, x(y^3 +$

$+ 3dxy^2 + ex^3)\rangle$ has type (e) if $d = e = 0$, type (f) if $e = 0$, $d \neq 0$, type (l) if $e \neq 0$, $d^3 = e$ and otherwise type (o), with parameter $d^3/e \neq 1$.

The same argument as gave the normal form shows also that for each of (D) and (o) a generic pencil has no non-trivial automorphism; the case $b = 0$ admits a group of order 3.

We have already discussed the automorphism group in the cases when each of f and g is a monomial. Here the group is infinite, so the case is necessarily non-stable. These are the cases $(a), (b), (d), (e), (i), (A)$

For the remaining cases we recall the list of multiplicities of linear factors of J:

(f)	(g)	(k)	(l)	(B)	(C)
$(3, 2, 1)$	$(4, 1, 1)$	$(4, 1, 1)$	$(3, 2, 1)$	$(3, 1, 1, 1)$	$(3, 2, 1)$

In cases $(f), (l), (C)$, J has no non-trivial automorphism, hence nor has the pencil. In cases (g) and (k) we obtain a group of order 2 (interchange x and y, resp. change a sign). In case (B), the same argument as for (D) shows that x and y are determined up to scalars; it is now clear that the automorphism group has order 2.

6. – Stratification of the moduli space.

If f and g are subjected to a linear transformation with determinant δ, an invariant of weight r is multiplied by δ^r. When we calculate the invariants (for a case which is at least semi-stable) we shall regard them as defining the point in weighted projective space $P(1, 1, 2, 3)$ with coordinates (I_2, J_2, I_4, I_6): the results below are given with this understanding. Note that if δ is taken as -1, these four invariants are unchanged but FI_9 changes sign. Since FI_9 is determined up to sign by the other invariants, its value does not yield additional information, and we omit it from the tabulations to follow.

We next tabulate invariants for all cases giving a unique point in moduli space. As well as the 1-point strata of Tables 2 and 3, we list other points picked out by symmetry considerations. The prefix S in the last two cases denotes an element of a stratum with extra symmetry.

We observe that (s_4) and (t_3) are the two points at which the moduli space $P(1, 1, 2, 3)$ is not smooth.

It will clarify our discussion of 1-dimensional strata if we first discuss cases with symmetry. The generic symmetric case (s_2) is characterised by the equation $FI_9 = 0$, for a direct calculation shows that FI_9 vanishes on it, and FI_9 is irreducible, so its zero locus also is. Special cases of this may be obtained by substituting in FI_9^2 and factorising. This leads to 6

TABLE 5.

Case	I_2	J_2	I_4	I_6
$(e)^+$	3	-1	0	0
$(A)^+$	6	1	1	0
(c)	-1	-1	0	1
(h_0)	-4	0	0	1
(j)	9	1	0	0
(m)	1	0	0	0
(E)	-2	1	1	8
(G)	-18	1	1	0
(I)	0	-3	-3	0
(s_4)	0	0	1	0
(t_3)	0	0	0	1
(SH)	0	1	1	0
(Sn)	0	1	0	0

cases, as follows.

Name	*Condition*	*Equation*
$(S1)$	$I_6 = 0$	$I_2 = 0$
$(S2)$	$I_6 = 0$	$3(27I_4 + I_2^2 - 12I_2J_2 + 9J_2^2)^2 = 4I_2(I_2 - 6J_2)^3$
$(S3)$	$I_4 = 0$	$64I_6 + I_2(I_2 - 9J_2)^2 = 0$
$(S4)$	$I_4 = 0$	$64I_6 + (I_2 + 3J_2)^3 = 0$
$(S5)$	$I_4 = J_2^2$	$64I_6 + I_2(I_2 + 18J_2)^2 = 0$
$(S6)$	$I_4 = J_2^2$	$64I_6 + (I_2 - 6J_2)^3 = 0$

The factors defining $(S4)$ and $(S6)$ each occur squared when we substitute in FI_9^2. We can at once identify three of these cases: we have $(S2) = (J)$, $(S4) = (h)$, $(S6) = (F)$. For if we take new normal forms for

TABLE 6.

	$I_6 = 0 = FI_9$	$I_6 \neq 0 = FI_9$	$I_6 = 0 \neq FI_9$	$I_6 \neq 0 \neq FI_9$
$I_4 = 0,\ I_4 = J_2^2$	(*unstable*)	(h_0)	(m)	$(p), (t_3)$
$I_4 = 0,\ I_4 \neq J_2^2$	$(e)^+, (j), (Sn)$	$(h), (S3), (c)$	(n)	(q)
$I_4 \neq 0,\ I_4 = J_2^2$	$(A)^+, (G), (SH)$	$(F), (S5), (E)$	(H)	(K)
$I_4 \neq 0,\ I_4 \neq J_2^2$	$(J), (S1), (I), (s_4)$	(s_2)	(L)	$(M), (s_3)$

TABLE 7.

Name			*Normal form*		
Zero			*Invariants*		
Parameter			*Special parameter values*		
			Nature of special case		
$(S1)$			$\langle x^3y,\ x^4+2ax^2y^2+y^4\rangle$		
$I_6,\ FI_9$			$(0,\ 1,\ a^{-2},\ 0)$		
a^2	0	1	3	∞	
	(s_4)	(SH)	(I)	(Sn)	
$(S2)=(J)$			$\langle x^3(x+4ky),\ y^3(x+y)\rangle$		
$I_6,\ FI_9$			$(6(k-1)^2,\ 1-2k-2k^2,\ 1-4k,\ 0)$		
k	0	1	4	-8	∞
	$(A)^+$	(j)	(I)	(G)	$(e)^+$
$(S3)$			$\langle x^4+x^2y^2,\ dx^3y+xy^3\rangle$		
$I_4,\ FI_9$			$(4d,\ 4,\ 0,\ -d(d-9)^2)$		
d	0	1	9	∞	
	(Sn)	(c)	(j)	(h_0)	
$(S4)=(h)$			$\langle x^2(x^2+2axy+y^2),\ y^2(x^2+2axy+y^2)\rangle$		
$I_4,\ FI_9$			$(3a^2-4,\ -a^2,\ 0,\ 1)$		
a^2	0	1	∞		
	(h_0)	(c)	$(e)^+$		
$(S5)$			$\langle x^2y^2,\ x^4+2cx^3y-2cxy^3-y^4\rangle$		
$I_4-J_2^2,\ FI_9$			$(4(c^2-1),\ 2,\ 4,(1-c^2)(c^2+8)^2)$		
c^2	0	1	-8	∞	
	(E)	(SH)	(G)	(h_0)	
$(S6)=(F)$			$\langle x^2y^2,(x^2+2axy+y^2)^2\rangle$		
$I_4-J_2^2,\ FI_9$			$(2(4a^2-1),\ 1,\ 1,\ -8(a^2-1)^3)$		
a^2	0	1	∞		
	(E)	$(A)^+$	(h_0)		
(n)			$\langle x^3y,\ y^2(x^2+2axy+y^2)\rangle$		
$I_4,\ I_6$			$(3a^2,\ 1-a^2,\ 0,\ 0)$		
a^2	0	3/4	1	∞	
	(Sn)	(j)	(m)	$(e)^+$	
(p)			$\langle x^2y^2,\ y(x^3+3bxy^2+y^3)\rangle$		
$I_4,\ I_4-J_2^2$			$(-4,\ 0,\ 0,\ 1-b^{-3})$		
b^3	0	1	∞		
	(t_3)	(m)	(h_0)		
(H)			$\langle x^3y,(x^2+2axy+y^2)^2\rangle$		
$I_6,\ I_4-J_2^2$			$(6a^2,\ 1,\ 1,\ 0)$		
a^2	0	1	-3	∞	
	(SH)	$(A)^+$	(G)	(m)	
(s_3)			$\langle x^4+xy^3,\ ax^3y+y^4\rangle$		
FI_9		$(3(a-4)^2,\ 8+20a-a^2,\ -64(a-1)^3,\ 729a^2(a-4)^3)$			
a	0	1	4	-8	∞
	$(A)^+$	(c)	(L)	(M)	$(e)^+$

these latter as

$$(J) \quad \langle x^3(x+ky),\, y^3(y+kx)\rangle,$$

$$(h) \quad \langle xy(x^2+kxy),\, xy(y^2+kxy)\rangle,$$

$$(F) \quad \langle x^2(x+ky)^2,\, y^2(y+kx)^2\rangle,$$

we see that in each case there is a symmetry interchanging x and y. Thus for (J) it interchanges the two terms divisible by a cube; for (h) it swaps the two common factors and for (F) we interchange the two terms which are perfect squares.

The remaining cases are more easily identified as special cases of the given normal form $\langle ax^4+bx^2y^2+cy^4,\, dx^3y+exy^3\rangle$ for (s_2): viz. $(S1)$ as the case $e=0$ (or $d=0$), $(S3)$ as the case $c=0$ and $(S5)$ as the case $b^2=4ac$. These identifications are all confirmed by checking the invariants.

We may thus organise our strata according to which of I_6, I_4, $I_4-J_2^2$, FI_9 vanish on them. This gives Table 6, which also shows how all strata are defined by equations, and hence characterises the cases in Tables 2 and 3 in terms of invariants. Where there are several entries in one square in the table, either we have two of the (Sr) strata, with equations given above, or we have a 0-dimensional stratum, with invariants given in Table 5.

In Table 7 we list properties of the 1-dimensional strata. Each is regarded as a rational curve in moduli space. We identify a parameter for this curve and list those values of the parameter at which some special behaviour is encountered, thus describing the closure of the stratum in moduli space. In virtually all cases this is given by the vanishing of one of I_6, I_4, $I_4-J_2^2$, FI_9 which does not vanish on the whole stratum. Outside the cases listed, these strata are disjoint.

REFERENCES

[1] N. D. BEKLEMISHEV, *Algebras of invariants of forms that are complete intersections*, Math. USSR. Izv., **47vi** (1983), pp. 423-429.

[2] J. H. GRACE - A. YOUNG, *The Algebra of Invariants*, Cambridge University Press (1903).

[3] P. E. NEWSTEAD, *Invariants of pencils of binary cubics*, Math. Proc. Cambridge Philos. Soc., **89** (1981), pp. 201-209.

[4] G. SALMON, *Lessons Introductory to the Modern Higher Algebra* (2nd edition), Hodges, Smith & Co., Dublin (1866).

[5] C. T. C. WALL, *Pencils of real binary cubics*, Math. Proc. Cambridge Philos. Soc., **93** (1983), pp. 477-484.

[6] C. T. C. WALL, *Sextic curves and quartic surfaces with higher singularities*, preprint, available via http://www.mi.aau.dk/ esn.

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