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Valuations and Group Algebras.

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1. - Introduction.

In [5], Dubrovin constructed a chain domain which has a prime ideal which is not completely prime. This ring was obtained by considering a right ordered group Γ such that the group algebra $K[\Gamma]$ can be embedded in a skew field D. Thus a partial answer was given to the Malcev-Problem [7]: *Let F be a field and G be a left orderable group. Can the group ring $F[G]$ be enclosed in a skew field?* This question has a stronger version which can be found for instance in Passmann's book [10]: *Determine the right ordered groups Γ with positive cone Π for which the skew-group-algebra $R[\Gamma, \sigma]$ is an order in a division algebra D whenever R is an Ore domain.* In the following, the pair (Γ, Π) denotes a right ordered group Γ with its positive cone Π, while $\sigma: \Gamma \to \mathrm{Aut}(R)$ is a group homomorphism. A perhaps more natural reformulation of Passmann's problem is to ask for which Γ the group algebra $R[\Gamma, \sigma]$ is a right Ore domain whenever R is one. This question has been discussed in detail in [1], and most constructions of chain orders in skew fields are based on the results of this paper, as one can see for instance in [2].

The primary goal of this paper is to investigate the structure of the group algebras and chain rings obtained via the localization techniques which were discussed in [1]. Our discussion focuses on a pair Γ_1 and Γ_2 of right ordered groups with positive cones Π_1 and Π_2 respectively. We

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consider a subsemigroup Δ of Γ_1 containing Π_1 and a cone preserving semigroup-morphism $\phi: \Delta \to \Gamma_2$, and define the right and left ϕ-values of non-zero elements of $R[\Delta, \sigma]$. Theorem 2.2 shows that these ϕ-values give rise to a pair of generalized, conjugated valuations in the sense of [1]. The same result also shows that $R[\Delta, \sigma]$ has a zero Jacobson-radical and that its group of units $U(R[\Gamma_1, \sigma])$ is $U(R)U(\Delta)$. In particular, Theorem 2.2 permits to solve the isomorphism problem for right orderable groups: Two right orderable groups Γ_1 and Γ_2 are isomorphic if and only if $R[\Gamma_1] \cong R[\Gamma_2]$ for all rings R (Corollary 2.3). This extends the well-known result that torsion-free abelian groups are isomorphic if and only if the corresponding group algebras are isomorphic. Since the ring structure of $R[\Gamma_1, \sigma]$ is independent of the chosen right order on Γ_1, no statement can be made about Γ_1 as a right ordered group. The remaining part of Section 2 investigates the valuation ring S^ϕ associated with the ϕ-values.

Section 3 considers the chain rings S_T^ϕ arising as localizations of S^ϕ inside the classical right ring of quotients of $R[\Gamma_1, \sigma]$ in the case that $R[\Gamma_1, \sigma]$ is a right Ore ring. Theorem 3.1 determines the Jacobson radical $J(S^\phi)$ of this ring and shows that $S^\phi/J(S^\phi)$ is the classical right ring of quotients of $R[H, \sigma|H]$ where $H = \ker\phi$. Furthermore, the pair of generalized, conjugated valuations on $R[\Gamma_1, \sigma]$ induces a left valuation $|\ |_l$ on S_T^ϕ such that $|a|_l \leq |b|$ if and only if $bS_T^\phi \subseteq aS_T^\phi$ for all a, $b \in S_T^\phi$.

In the following, all rings have a multiplicative identity. The symbols $J(R)$ and $U(R)$ denote the Jacobson-radical and the group of units of R respectively. All groups are written multiplicatively.

2. – Group rings and cones.

Let Γ be a group. A subsemigroup $\Pi \subseteq \Gamma_1$ is called a *cone* if $\Pi \cap \Pi^{-1} = \{\varepsilon\}$ and $\Pi \cup \Pi^{-1} = \Gamma$ hold. Note that in the case where Π is invariant, Γ is an ordered group. In general, setting $\alpha \leqslant_r \beta$ iff $\beta\alpha^{-1} \in \Pi$ resp. $\alpha \leqslant_l \beta$ iff $\alpha^{-1}\beta \in \Pi$ allows to view the group Γ as a *right-ordered* resp. *left-ordered* group.

We consider the pair (Γ_1, Π_1) where Γ_1 is a group and Π_1 a cone. Further let R be a domain, i.e. a ring without zero divisors. For a group homomorphism $\sigma: \Gamma_1 \to \mathrm{Aut}(R)$, we define a ring multiplication on the free left R-module with basis Γ_1 by $\alpha r = r^{\sigma(\alpha)}\alpha$ for all $\alpha \in \Gamma_1$ and $r \in R$. The resulting ring is denoted by $R[\Gamma_1, \sigma]$. Write a non-zero x in $R[\Gamma_1, \sigma]$

as $x = \Sigma_{\alpha \in \Gamma_1} r_\alpha \alpha$, and let $\mathrm{supp}(x) = \{\alpha \mid r_\alpha \neq 0\}$ denote the *support of* x.

If $T = \{x \in R[\Gamma_1, \sigma] \mid \varepsilon_1 \in \mathrm{supp}(x) \subseteq \Pi_1\}$, then every $x \in R[\Gamma_1, \sigma]$ has unique factorizations $x = u\alpha = \beta v$ with $\alpha, \beta \in \Gamma_1$ and $u, v \in T$. We refer to this decomposition as the *$T - \Gamma_1$-factorization of* x and say that $|x|_l = \beta$ resp. $|x|_r = \alpha$ are *the left resp. right Γ_1-values of* x. In a previous paper we had introduced the concept of generalized valuations, however this terminology can be omitted in this context. Naturally, the question arises at this point if the conditions on Π_1 can be weakened by assuming that the subsemigroup Π_1 is a *right cone* only, i.e.

(a) Π_1 generates Γ_1.

(b) If $a, b \in \Pi_1$ with $a^{-1} b \notin \Pi_1$, then $b^{-1} a \in \Pi_1$.

This question is answered in a negative way by

PROPOSITION 2.1. *A right cone Π_1 of Γ_1 is a cone if and only if $x \in R[\Gamma_1, \sigma]$ has a unique $T - \Gamma_1$-decomposition.*

PROOF. The uniqueness property guarantees immediately that Π_1 cannot contain any units but ε_1. If $\alpha \in \Gamma_1$, then we can find $\delta \in \Gamma_1$ and $\pi_1, \pi_2 \in \Pi_1$ with $\varepsilon_1 + \alpha = \delta(\pi_1 + \pi_2)$. Without loss of generality, $\varepsilon_1 = \delta\pi_1$ and $\alpha = \delta\pi_2$. Since Π_1 is a right cone, we may assume that $\pi_1^{-1}\pi_2 \in \Pi_1$, say $\pi_2 = \pi_1\pi$ for some $\pi \in \Pi_1$. Then, $\alpha = \delta\pi_2 = \delta\pi_1\pi = \pi \in$ $\in \Pi_1$. In the same way, $\pi_2^{-1}\pi_1 \in \Pi_1$ yields $\alpha^{-1} \in \Pi_1$. ■

Let (Γ_2, Π_2) be a further right-ordered group. We consider a subsemigroup Δ of Γ_1 containing Π_1. A semigroup map $\phi\colon \Delta \to \Gamma_2$ is called a *cone preserving homomorphism*, provided ϕ maps Π_1 into Π_2. It is natural to define the ϕ-values of an element $x \in R[\Delta, \sigma]$ to be the values under ϕ of $|x|_r$ resp. $|x|_l$. To be more precise, we set $|x|_r^\phi = \phi|x|_r$ resp. $|x|_l^\phi = \phi|x|_l$. We call $(\Gamma_1, \Gamma_2, \Delta)$ *a cone-valuated triple with associated map* ϕ if Δ contains α^{-1} for every $\alpha \in \ker\phi$. In the case that $\Delta = \Gamma_1$, we omit any reference of Δ and speak of *a cone-valuated pair* instead.

THEOREM 2.2. *Let $(\Gamma_1, \Gamma_2, \Delta)$ be a cone-valuated triple with associated map ϕ. The following hold for any domain R and any group-homomorphism $\sigma\colon \Gamma_1 \to \mathrm{Aut}(R)$:*

(a) *For all non-zero $x, y, z \in R[\Delta, \sigma]$, the following conditions are satisfied:*

(i) $|x|_r^\phi \in \Pi_2$ *if and only if* $|x|_l^\phi \in \Pi_2$.

(ii) $|x|_r^\phi = \varepsilon_2$ *if and only if* $|x|_l^\phi = \varepsilon_2$.

(iii) *If* $x + y \neq 0$, *then* $|x+y|_l^\phi \geq \min_l \{|x|_l^\phi, |y|_l^\phi\}$ *and* $|x+y|_r^\phi \geqslant_r \min_r \{|x|_r^\phi, |y|_r^\phi\}$.

(iv) *If* $|x|_r^\phi \geqslant_r |y|_r^\phi$, *then* $|xz|_r^\phi \geqslant_r |yz|_r^\phi$, *while* $|x|_l^\phi \geqslant_l |y|_l^\phi$ *yields* $|zx|_l^\phi \geqslant_l |zy|_l^\phi$.

(*b*) $U(R[\Delta, \sigma]) = U(R)U(\Delta) = \{u\alpha \,|\, u \in U(R)$ *and* $\alpha \in U(\Delta)\}$.

(*c*) $J(R[\Delta, \sigma]) = 0$ *if* $|\Gamma_1| > 1$.

Proof. (*a*) Since $|x|_r^\phi = \min_r \{\phi(\alpha) \,|\, \alpha \in \operatorname{supp}(x)\} \in \Pi_2$, we have $\phi(\alpha) \in \Pi_2$ for all $\alpha \in \operatorname{supp}(x)$. But then, $|x|_l^\phi = \min_l \{\phi(\alpha) \,|\, \alpha \in \operatorname{supp}(x)\}$ has to be in Π_2 too. Thus, (i) holds by symmetry. Furthermore, if $|x|_r^\phi = \varepsilon_2$, then $\phi(\alpha) \in \Pi_2$ for all $\alpha \in \operatorname{supp}(x)$ by what has been shown so far. If $\alpha_0 = \min_r \operatorname{supp}(x)$, then $\phi(\alpha_0) = \varepsilon_2$, and $\varepsilon_2 \leqslant_l |x|_l^\phi \leqslant_l \phi(\alpha_0) = \varepsilon_2$ from which (ii) follows by symmetry.

To show (iii), let $\alpha_0 = \min_r \operatorname{supp}(x+y)$. Then $\alpha_0 \in \operatorname{supp}(x)$ or $\alpha_0 \in \operatorname{supp}(y)$. In the first case, $|x+y|_r^\phi = \phi(\alpha_0) \geqslant_r |x|_r^\phi \geqslant_r \min_r \{|x|_r^\phi, |y|_r^\phi\}$ as desired. The second case is treated similarly. By symmetry, (iii) is satisfied. For (iv), we suppose $|x|_r^\phi \leqslant_r |y|_r^\phi$. Choose $u_1, u_2, u_3 \in T$ and $\alpha_1, \alpha_2, \alpha_3 \in \Delta$ such that $x = u_1\alpha_1$, $y = u_2\alpha_2$ and $z = u_3\alpha_3$. For $j = 1, 2$ write $\alpha_j u_3 = v_j\beta_j$ with $v_j \in T$ and $\beta_j \in \Delta$. We obtain $xz = u_1 v_1 \beta_1 \alpha_3$ and $yz = u_2 v_2 \beta_2 \alpha_3$. But this gives $|xz|_r^\phi = \phi(\beta_1\alpha_3)$ and $|yz|_r^\phi = \phi(\beta_2\alpha_3)$. Since $\phi(\alpha_1) \leqslant_r \phi(\alpha_2)$, we have $\phi(\beta_1) = \min_r \phi(\operatorname{supp}(\alpha_1 u_3)) \leqslant_r \min_r \phi(\operatorname{supp}(\alpha_2 u_3)) = \phi(\beta_2)$, from which the first part of (*c*) follows. In view of the symmetry of the problem, *a*) has been shown.

To prove (*b*) consider an element $x \in R[\Delta, \sigma]$ which has an inverse $y \in R[\Delta, \sigma]$. We write $x = v\beta$ with $v \in T$ and $\beta \in \Delta$. If $|\operatorname{supp}(x)| > 1$, then $\operatorname{supp}(v)$ contains an element of $\Pi_1 \setminus \{\varepsilon_1\}$. We write $y = u\alpha$ with $u \in T$ and $\alpha \in \Delta$ and select $w \in T$ and $\gamma \in \Delta$ with $\beta u = w\gamma$. Since $\varepsilon_1 = vw\gamma\alpha$, we have $\gamma\alpha = \varepsilon_1$ and $vw = \varepsilon_1$ by the uniqueness of $T - \Gamma_1$-factorizations. In $\operatorname{supp}(v)$, choose an element α which is maximal in the *left* order induced by Π_1, while in $\operatorname{supp}(w)$ choose β maximal in the *right* order. Since $|\operatorname{supp}(v)| \geqslant 2$, we have $\alpha >_r \varepsilon_1$, from which we obtain $\alpha\beta >_r > \beta \geqslant_r \varepsilon_1$. Because R is a domain, $\alpha\beta$ has a non-zero coefficient in the product vw, but is not an element of $\operatorname{supp}(vw)$ since $vw = \varepsilon_1$. Hence, there are $\alpha' \in \operatorname{supp}(v)$ and $\beta' \in \operatorname{supp}(w)$ with $\alpha\beta = \alpha'\beta'$ and $\alpha \neq \alpha'$ or $\beta \neq \beta'$. A straightforward calculation shows that $\alpha \neq \alpha'$ and $\beta \neq \beta'$. Since $\beta >_r \beta'$

by the choice of β, we can find $\pi \in \Pi_1$ with $\beta = \pi\beta'$. Then $\alpha'\beta' = \alpha\beta = \alpha\pi\beta'$ yields $\alpha' = \alpha\pi$ from which $\alpha' \geqslant_l \alpha$ follows. However, we have $\alpha' <_l \alpha$ by the choice of α. The resulting contradiction shows that x cannot have more than one element in its support, i.e. $v \in R$. Then, $|\operatorname{supp}(y)| = |\operatorname{supp}(xy)| = 1$, and $u \in R$. In particular, $\varepsilon_1 = xy = vu^{\sigma(\beta)}\beta\alpha = yx = uv^{\sigma(\alpha)}\alpha\beta$ yields that α is a unit of Δ. Moreover, v is a unit of R whose inverse is $u^{\sigma(\beta)}$. The converse is obvious. For the proof of (c), let x be a non-zero element of $J(R[\Delta, \sigma])$ and write $x = u\alpha$ where $u \in T$ and $\alpha \in \Delta$. If $\alpha \leqslant_r \varepsilon_1$, then $\alpha^{-1} \geqslant_r \varepsilon_1$. We choose any $\beta \in \Pi_1 \setminus \{\varepsilon_1\}$, and observe that $x\alpha^{-2}\beta$ is a non-zero element of $J(R[\Delta, \sigma])$ with $x\alpha^{-2}\beta = u\alpha^{-1}\beta$ and $\alpha^{-1}\beta >_r \varepsilon_1$. Hence, no generality is lost, if we assume that $\alpha >_r \varepsilon_1$. Since $\varepsilon_1 \notin \operatorname{supp}(x)$ in this case, we have $\operatorname{supp}(\varepsilon_1 - x) = \operatorname{supp}(x) \cup \{\varepsilon_1\}$. Since $\operatorname{supp}(x)$ is not empty, $\varepsilon_1 - x$ cannot have a right inverse in $R[\Delta, \sigma]$ by what has been shown previously. On the other hand, $J(R[\Delta, \sigma])$ is a quasi-regular ideal, which results in a contradiction. ■

Theorem 2.2 shows in particular that the maps $|\ |_r$ and $|\ |_l$ defined in part (a) form a pair of generalized, conjugated valuations in the sense of [1].

COROLLARY 2.2. *Let Γ_1 and Γ_2 be right orderable groups. Then, $\Gamma_1 \cong \Gamma_2$ as groups if and only if $R[\Gamma_1] \cong R[\Gamma_2]$ for all rings R ($\mathbb{Q}[\Gamma_1] \cong \mathbb{Q}[\Gamma_2]$).*

PROOF. By Theorem 2.2 (b), we known that $U(\mathbb{Q}[\Gamma_i]) = U(\mathbb{Q})\Gamma_i$. Observe that $N_i = U(\mathbb{Q})\varepsilon_i$ is a normal subgroup of $U(\mathbb{Q}[\Gamma_i])$ since it is contained in the center of $\mathbb{Q}[\Gamma_i]$. Every ring isomorphism $\sigma\colon \mathbb{Q}[\Gamma_1] \to \mathbb{Q}[\Gamma_2]$ induces a group isomorphism $\tau\colon U(\mathbb{Q})\Gamma_1 \to U(\mathbb{Q})\Gamma_2$. Since τ was induced by the ring-map σ, we have $\tau(r\varepsilon_1) = r\varepsilon_2$ for all $r \in U(\mathbb{Q})$. Thus, $\tau | N_1$ maps N_1 onto N_2. Since $U(\mathbb{Q}[\Gamma_i])$ is the direct product of N_i and Γ_i, we obtain that Γ_1 and Γ_2 are isomorphic as groups. ■

We are particularly interested in the ring

$$S^\phi = \{x \in R[\Delta, \sigma] \mid |x|_r^\phi \geqslant \varepsilon_2\} = \{x \in R[\Delta, \sigma] \mid \operatorname{supp}(x) \subseteq \Pi_1 \cup \ker\phi\}.$$

Observe that $0 \in S^\phi$ since $|0|_l = |0|_r = \infty > \varepsilon_2$ by convention.

PROPOSITION 2.4. *Consider a cone-valuated triple $(\Gamma_1, \Gamma_2, \Delta)$ with associated map ϕ whose kernel is denoted by H, a domain R, and a group homomorphism $\sigma\colon \Gamma_1 \to \operatorname{Aut}(R)$. Then $I^\phi = \{x \in$*

$\in S^\phi \mid |x|_r^\phi > \varepsilon_2\}$ is a two-sided ideal of S^ϕ such that $S^\phi/I^\phi \cong R[H, \tau]$ where $\tau = \sigma|H$.

PROOF. If x is a non-zero element of I^ϕ, then $|x|_l^\phi >_l \varepsilon_2$. Observe that, for every non-zero $y \in S^\phi$, the inequality $|y|_l^\phi \geqslant_l \varepsilon_2 = |\varepsilon_1|_l^\phi$ yields $|xy|_l^\phi \geqslant_l |x|_l^\phi > \varepsilon_2$ from which $|xy|_r^\phi >_r \varepsilon_2$ follows. On the other hand, $|y|_r^\phi \geqslant_r \varepsilon_2 = |\varepsilon_1|_r^\phi$ implies $|yx|_r^\phi \geqslant_r |\varepsilon_1 x|_r^\phi = |x|_r^\phi >_r \varepsilon_2$. Thus, $xy, yx \in I^\phi$. Moreover, if $a \neq b$ are in I^ϕ, then $|a-b|_r^\phi \geqslant_r \min_r \{|a|_r^\phi, |b|_r^\phi\} >_r \varepsilon_2$, and $a - b \in I^\phi$, and I^ϕ is a two-sided ideal of S^ϕ. Let $x = \Sigma_{\alpha \in \mathrm{supp}(x)} r_\alpha \alpha$ be a non-zero element of S^ϕ, and write $x' = \Sigma_{\alpha \in \mathrm{supp}(x) \cap H} r_\alpha \alpha$. We define a map $\lambda\colon S^\phi/I^\phi \to R[H, \tau]$ by $\lambda(x) = x'$. If $x - y \in I^\phi$, then $0 = (x-y)' = x' - y'$, and λ is well-defined. Moreover, if $x, y \in S^\phi$, then $x - x', y - y' \in I^\phi$, and $xy - x'y' = (x - x')y + x'(y - y') \in I^\phi$ since I^ϕ is a two-sided ideal. Then, $\lambda(xy) = \lambda(x'y') = x'y' = \lambda(x)\lambda(y)$. It is now routine to show that λ is an isomorphism. ■

COROLLARY 2.5. *Consider a cone-valuated triple $(\Gamma_1, \Gamma_2, \Delta)$ with associated map ϕ, a domain R, and a group homomorphism $\sigma\colon \Gamma_1 \to \mathrm{Aut}(R)$.*

(a) I^ϕ is maximal as a right ideal of S^ϕ if and only if ϕ is one-to-one and R is a division algebra.

(b) S^ϕ has the property that $|x|_r^\phi = \varepsilon_2$ yields that x is a unit of S^ϕ if and only if $|\Gamma_1| = 1$ and R is a division algebra.

PROOF. (*a*) Suppose that I^ϕ is a maximal right ideal of S^ϕ. Using the notation of Proposition 2.4, $R[H, \tau]$ is a division algebra. But this is only possible if $H = \{\varepsilon_1\}$ by part (*b*) of Theorem 2.2. Consequently, $R \cong S^\phi/I^\phi$ is a division algebra. The converse is obvious.

(*b*) Suppose that S^ϕ is regular. If Γ_1 contains two elements, we can find $\alpha \in \Gamma_1$ such that $\alpha >_r \varepsilon_1$. Then, $\varepsilon_1 + \alpha$ is an element of S^ϕ with $|\varepsilon_1 + \alpha|_r^\phi = \varepsilon_2$. But then, $\varepsilon_1 + \alpha$ is a unit of S^ϕ which is impossible by Theorem 2.2. The rest of the proof is obvious. ■

3. - Localizations.

In this section, R always is a right Ore-ring and we assume that (Γ_1, Π_1) has the property that $R[\Gamma_1, \sigma]$ is a right Ore ring. Turning to the valuation rings which we considered in Proposition 2.4 and Corollary

2.5, we consider the set $X = \{x \in S^{\phi} \mid |x|_r^{\phi} = \varepsilon_2\}$. It easy to see that the elements of X are precisely the elements $x \in R[\Gamma_1, \sigma]$ of the form $x = u\alpha$ for some $u \in T$ and $\alpha \in \ker\phi$. Since the elements of $\ker\phi$ are units in S^{ϕ}, it follows that X is an Ore sets, and that $S_X^{\phi} \cong S_T^{\phi}$ is a chain-order in D in the sense of Dubrovin (for details see [1, Theorem 4.2]). In particular, D is the classical ring of quotients of S_{ϕ}.

THEOREM 3.1. *Consider a right Ore-domain R, a cone-valuated triple $(\Gamma_1, \Gamma_2, \Delta)$ with associated map ϕ whose kernel is denoted by H, and a group-homomorphism σ: $\Gamma_1 \to \mathrm{Aut}(R)$. If $R[\Gamma_1, \sigma]$ is a right Ore-domain, then*

(*a*) *S_T^{ϕ} is a chain-domain with maximal ideal I_T^{ϕ}.*

(*b*) *$R[H, \tau]$ is a right Ore-domain whose classical ring of quotients is S_T^{ϕ}/I_T^{ϕ}.*

(*c*) *$|\ |_l$ induces a generalized left valuation $|\ |_l$ on S_T^{ϕ}.*

(*d*) *For all a, $b \in S_T^{\phi}$ we have $|a|_l \leqslant |b|_l$ if and only if $aS_T^{\phi} \supseteq bS_T^{\phi}$.*

PROOF. (*a*) To see that S_T^{ϕ} is a right and left chain domain, we consider non-zero elements a_1 and a_2 of S_T^{ϕ}, and write $a_i = \alpha_i u_i t_i^{-1}$ with $\alpha_i \in \Gamma_1$ and u_i, $t_i \in T$ for $i = 1, 2$. Then, $a_i S_T^{\phi} = \alpha_i S_T^{\phi}$. If $\alpha_2 \geqslant_l \alpha_1$, then $\alpha_2 = \alpha_1 \pi$ for some $\pi \in \Pi_1 \subseteq S^{\phi}$. Therefore, $a_2 S_T^{\phi} \subseteq a_1 S_T^{\phi}$. Because of the symmetry of the problem, S_T^{ϕ} is a right chain domain.

Observe that I_T^{ϕ} is a right ideal of S_T^{ϕ} whose elements are of the form at^{-1} with $a \in I^{\phi}$ and $t \in T$. We show that I_T^{ϕ} consists of the non-units of S_T^{ϕ}. To see this, suppose that $1 \in I_T^{\phi}$. It implies $1 = at^{-1}$ for some $a \in I^{\phi}$ and $t \in T$. We obtain $a = t$ and $\varepsilon_2 <_r |a|_r = |t|_r = \varepsilon_2$, a contradiction. Thus, I_T^{ϕ} consists only of non-units. On the other hand, if $at^{-1} \in S_T^{\phi}$ is not a unit, then $|a|_r > \varepsilon_2$ since otherwise $a \in X$, and at^{-1} is a unit of S_T^{ϕ}. But $|a|_r > \varepsilon_2$ implies $a \in I^{\phi}$. Therefore, I_T^{ϕ} indeed is the collection of non-units of S_T^{ϕ}. This shows that I_T^{ϕ} is the Jacobson-radical of the chain-ring S_T^{ϕ}. In particular, $D' = S_T^{\phi}/I_T^{\phi}$ is a division algebra.

(*b*) In the first step, we compute $S^{\phi} \cap I_T^{\phi}$. Choose $a \in I^{\phi}$ and $t \in T$ such that $at^{-1} \in S^{\phi}$, say $at^{-1} = b$. If $b \notin I^{\phi}$, then $|b|_r = \varepsilon_2$, and $b \in X$. Consequently, I^{ϕ} contains an element of X, and $I_T^{\phi} = S_T^{\phi}$ which is not possible. Therefore, $S^{\phi} \cap I_T^{\phi} = I^{\phi}$. In the division algebra D', we consider the ring $L = (S^{\phi} + I_T^{\phi})/I_T^{\phi}$ and show that it is essential in D' as a L-submodule: If $at^{-1} \in S_T^{\phi}$ with $a \in S^{\phi}$ and $t \in T$, but $at^{-1} \notin I_T^{\phi}$, then

$|a|_r \geq_r \varepsilon_2$, and $a + I_T^\phi = (at^{-1} + I_T^\phi)(t + I_T^\phi)$ is a non-zero element of $(at^{-1} + I_T^\phi) L \cap L$. Therefore, D' is the maximal right ring of quotients of L by [6, Propositions A. 2.11 and Corollary C. 2.31]. Moreover, $L \cong S^\phi / S^\phi \cap I_T^\phi = S^\phi / I^\phi \cong R[H, \tau]$ yields that $R[H, \tau]$ is a right Ore-ring whose classical right ring of quotients is isomorphic to D' as desired.

To show (c) and (d), consider $a, b \in S_T^\phi$. We can find $\alpha, \beta \in \Gamma_1$ and $u, v, x, y \in T$ such that $a = \alpha u x^{-1}$ and $b = \beta v y^{-1}$, where the $T - \Gamma_1$-factorizations have the property $\alpha u, \beta v \in S^\phi$. Then $|\alpha u|_l = \phi(\alpha)$, and $|\beta v|_l = \phi(\beta)$. We obtain that ux^{-1} and vy^{-1} are units in S_T^ϕ. Therefore, $aS_T^\phi = \alpha S_T^\phi$ and $bS^\phi = \beta S_T^\phi$.

Suppose $\alpha S_T^\phi \supseteq \beta S_T^\phi$. We can find $s \in S^\phi$ and $z \in T$ with $\beta = \alpha s z^{-1}$, and write $s = \sum_{i=1}^n r_i \sigma_i$ such that $\phi(\sigma_i) \geq_l e_2$ for all i and $z = \sum_{j=1}^m t_j \delta_j$ where $\phi(\delta_j) \geq_l e_2$ and $\phi(\delta_1) = e_2$. We obtain $\beta z = \sum_{j=1}^m t_j^{\sigma(\beta)} \beta \delta_j$ and $\alpha s = \sum_{i=1}^n r_i^{\sigma(\alpha)} \alpha \sigma_i$. There exists $i_0 \in \{1, \ldots, n\}$ with $\beta \delta_1 = \alpha \sigma_{i_0}$. This shows $\phi(\beta) = \phi(\alpha \delta_1) = \phi(\alpha)\, \phi(\sigma_{i_0}) \geq_l \phi(\alpha)$ since $\phi(\sigma_{i_0}) \geq_l e_2$.

Define $|a|_l = \phi(\alpha)$ where $a = \alpha u x^{-1}$ is a factorization of a as before. To show that this map is well-defined, we consider a second factorization $a = \tilde{\alpha} \tilde{u} \tilde{x}^{-1}$. Since $\alpha S_T^\phi = \tilde{\alpha} S_T^\phi$, the results verified up to this point yield $\phi(\alpha) \leq_l \phi(\tilde{\alpha}) \leq_l \phi(\alpha)$. This shows that the map $|\ |_l$ is indeed well-defined. Moreover, every $a \in S^\phi$ can be written as $a = \alpha u 1^{-1}$ with $u \in T$. Then, $|a|_l = \phi(\alpha) = |a|_l^\phi$. Thus, $|\ |_l$ extends $|\ |_l^\phi$ as desired.

We consider $a, b \in S_T^\phi$ with $|a|_l \geq_l |b|_l$, and choose a decomposition of a and b as before. Observe $\phi(\alpha) \geq_l \phi(\beta)$. If $\alpha \geq_l \beta$ in Γ_1, then $\beta^{-1} \alpha \in \Pi_1 \subseteq S^\phi$, and there is $s \in S^\phi$ with $\alpha = \beta s$. Thus, $aS_T^\phi = \alpha S_T^\phi \subseteq \beta S_T^\phi = bS_T^\phi$. On the other hand, if $\alpha \leq_l \beta$, then $\phi(\alpha) \leq_l \phi(\beta)$; and hence $\phi(\alpha) = \phi(\beta)$. Since $\alpha^{-1} \beta \in \Pi_1 \subseteq \Delta$, we have $\phi(\beta) = \phi(\alpha)\phi(\alpha^{-1}\beta) = \phi(\beta)\phi(\alpha^{-1}\beta)$. Hence, $\alpha^{-1}\beta \in \ker \phi$, and hence $\beta^{-1}\alpha \in \Delta$. But then, $\beta^{-1}\alpha \in \ker(\phi) \subseteq X$ yields that $\beta^{-1}\alpha$ is a unit of S_T^ϕ. In this case $\alpha S_T^\phi = \beta S_T^\phi$. In either case, we have shown $|a|_l \geq_l |b|_l$ if and only if $aS_T^\phi \subseteq bS_T^\phi$.

Using the last result and the fact that S_T^ϕ is a chain ring, it is now possible to show that the map $|\ |_l$ defines a generalized left valuation on S_T^ϕ using standard arguments. ■

In the last result, we assumed that $R[\Gamma_1, \sigma]$ is a right Ore ring in order to embed S^ϕ as an essential submodule into a ring Q in which the elements of T are units. We now show that the Ore condition on $R[\Gamma_1, \sigma]$ is necessary and sufficient for the existence of such a ring Q.

COROLLARY 3.2. *Consider a right Ore-domain R, a cone-valuated pair (Γ_1, Γ_2) with associated map ϕ, and a group-homomorphism $\sigma: \Gamma_1 \to \mathrm{Aut}(R)$. Then, the following conditions are equivalent*:

(*a*) *$R[\Gamma_1, \sigma]$ is a right Ore ring.*

(*b*) *S^ϕ can be embedded as an essential S^ϕ-submodule of a ring Q in which the elements of T are units.*

PROOF. $(a) \Rightarrow (b)$ is an immediate consequence of the Theorem 3.1.

$(b) \Rightarrow (a)$ Let $t \in T$ and $s \in S^\phi$. No generality is lost if we assume that $s \neq 0$. Because of (*b*), we can find elements $t_1, t_2 \in S^\phi$ with $t^{-1} s t_1 = t_2$. Choose $u_1, u_2 \in T$ and $\alpha_1, \alpha_2 \in \Gamma_1$ with $t_i = u_i \alpha_i$ for $i = 1, 2$. Write $s u_1 \alpha_1 = t u_2 \alpha_2$ to obtain $s u_1 = t u_2 (\alpha_2 \alpha_1^{-1})$ in $R[\Gamma_1, \sigma]$. Then, $s u_1 = \sum_{i=1}^{n} a_i \delta_i$ where $\mathrm{supp}(s u_1) = \{\delta_1, \ldots, \delta_n\} \subseteq \Gamma_1$ and $\phi(\delta_i) \geqslant_r \varepsilon_2$. Similarly, $t u_2 = \sum_{j=1}^{m} b_j \varrho_j$ where $\mathrm{supp}(t u_2) = \{\varrho_1, \ldots, \varrho_m\}$ with $\phi(\varrho_j) \geqslant_r \varepsilon_2$. Then $\sum_{i=1}^{n} a_i \delta_i = \left(\sum_{j=1}^{m} b_j \varrho_j \right) \alpha_2 \alpha_1^{-1} = \sum_{j=1}^{m} b_j (\varrho_j \alpha_2 \alpha^{-1})$. This shows

$$\varepsilon_2 \leqslant_r |s u_1|_r = \min_{i=1}^{n} \phi(\delta_i) = \min_{j=1}^{m} \phi(\varrho_j \alpha_2 \alpha_1^{-1}) =$$

$$= \left(\min_{j=1}^{m} \phi(\varrho_j) \right) \phi(\alpha_2 \alpha_1^{-1}) = \varepsilon_2 \phi(\alpha_2 \alpha_1^{-1}) = \phi(\alpha_2 \alpha_1^{-1}).$$

Moreover $t u_2 \in T$, and hence $\alpha_2 \alpha_1^{-1} \in S^\phi$. Furthermore, $u_2 \in T$ yields $u_2 \alpha_2 \alpha^{-1} \in S^\phi$. Therefore, $s u_1 = t(u_2 \alpha_2 \alpha_1^{-1})$ with $u_1 \in T$ and $u_2 \alpha_2 \alpha_1^{-1} \in S^\phi$; and T is right Ore in S^ϕ.

To show (a), let r_1 and r_2 be two non-zero elements of $R^\sigma[\Gamma_1]$. Choose $u_1, u_2 \in T$ and $\alpha_1, \alpha_2 \in \Gamma_1$ with $r_i = u_i \alpha_i$ for $i = 1, 2$. By (b) there are $v_1, v_2 \in T$ with $u_2 v_2 = u_1 v_1$. Then $r_2 \alpha_2^{-1} v_2 = u_2 v_2 = u_1 v_1 = r_1 r_1^{-1} v_1$ is non-zero, and $\alpha_i^{-1} v_i \in R[\Gamma_1, \sigma]$. Hence $R[\Gamma_1, \sigma]$ is a right Ore ring. ■

The last result of this section shows that, in the setting of Mathiak's work ([8]), the pair of conjugated generalized valuations can be extended to the localization S_T^ϕ just like standard valuations.

COROLLARY 3.3. *Let R be a right and left, Ore-domain, $(\Gamma_1, \Gamma_2, \Delta)$ a cone-valuated triple with associated map ϕ, and $\sigma: \Gamma_1 \to \mathrm{Aut}(R)$ be a group homomorphism such that $R[\Gamma_1, \sigma]$ is a right and left Ore-domain.*

(a) The pair $(|\ |_l, |\ |_r)$ *of generalized, conjugated valuations on* S^{Φ} *extends to a pair* $(|\ |_l, |\ |_r)$ *of conjugated, generalized valuations on* S_T^{Φ}.

(b) The induced valuations in a) are order-anti-isomorphic to the generalized valuations on S_T^{Φ} *which are induced by the linear ordering of the one-sided ideals of* S_T^{Φ}.

PROFF. By Theorem 3.1, there exist extensions $|\ |_l$ and $|\ |_r$ of $|\ |_l$ and $|\ |_r$ which are one-sided generalized valuations and satisfy condition (*b*). Since $|a|_l \geqslant_l \varepsilon_2$ and $|a|_r \geqslant_r \varepsilon_2$ for all $a \in S_T^{\Phi}$, it remains to show $|a|_l = \varepsilon_2$ if and only if $|a|_r = \varepsilon_2$. If $|a|_l = \varepsilon_2$ then $aS^{\Phi} T = S_T^{\Phi}$ and a is a unit of S_T^{Φ}. In this case, $S_T^{\Phi} a = S_T^{\Phi}$ and $|a|_r = \varepsilon_2$. The converse is verified in exactly the same way. ■

4. – Examples.

Dubrovin showed in [5] that the property that $R[\Gamma_1, \sigma]$ is a right Ore-domain whenever R is a right Ore-domain, is inherited by subgroups. Using this, we can easily establish the following

PROPOSITION 4.1 (*a*) *The following conditions are equivalent for a group* Γ, *a right Ore-domain* R, *and a group-homomorphism* $\sigma\colon \Gamma \to \mathrm{Aut}(R)$:

(i) $R[\Gamma, \sigma]$ *is a right Ore-domain.*

(ii) *For every finitely generated subgroup* U *of* Γ, *the ring* $R[U, \sigma|U]$ *is a right Ore-domain.*

(iii) Γ *is the union of a smooth ascending chain* $\{\Gamma_\nu \mid \nu < \kappa\}$ *of subgroups* Γ_ν *such that* $R[\Gamma_\nu, \sigma_{|\Gamma_\nu}]$ *is a right Ore-domain.*

(*b*) *The following conditions are equivalent for a group* Γ *which is the semi-direct product of a normal subgroup* N *by a subgroup* U:

(i) $R[\Gamma, \sigma]$ *is a right Ore domain for all right Ore domains* R *and all* $\sigma\colon \Gamma \to \mathrm{Aut}(R)$.

(ii) α) $R[N, \sigma]$ *is a right Ore domain for all right Ore domains* R *and all* $\sigma\colon N \to \mathrm{Aut}(R)$.

β) $R[U, \tau]$ *is a right Ore domain for all right Ore domains* R *and all* $\tau\colon U \to \mathrm{Aut}(R)$.

PROOF. (*a*) The implication (i)$\Rightarrow$(iii) is obvious. (iii)$\Rightarrow$(ii): If U is a finitely generated subgroup of Γ, then $U \subseteq \Gamma_\nu$ for some $\nu < \kappa$. Since $(\sigma|_{\Gamma_\nu})|_U = \sigma|_U$, we have that $R[U, \sigma|_U]$ is a right Ore domain by Dubrovin's result. (ii)$\Rightarrow$(i): Whenever a and b are non-zero elements of $R[\Gamma, \sigma]$, then there is a finitely generated subgroup U of Γ with $a, b \in$ $\in R[U, \sigma|_U]$. But the latter ring is a right Ore-domain. (*b*) (i)$\Rightarrow$(ii): Condition (i) holds by (*a*). To show the second condition, we observe that every homomorphism $\sigma: U \to \mathrm{Aut}(R)$ can be extended to a homomorphism $\tilde{\sigma}: \Gamma \to \mathrm{Aut}(R)$ by setting $\tilde{\sigma}(nu) = \sigma(u)$ for all $n \in N$ and $u \in U$. Now, we apply Dubrovin's result again.

(ii)$\Rightarrow$(i): By condition (ii), $R[N, \sigma|_N]$ is a right Ore domain, and $(R[N, \sigma_{|N}])[U, \tau]$ is a right Ore domain for all homomorphisms $\tau: U \to$ $\to \mathrm{Aut}(R[N, \sigma|_N])$. In particular consider the map τ_U which is defined by $[\tau_U(u)]\left(\sum_{a \in N} r_a a\right) = \sum_{a \in N} r^{\sigma(a)} a$ for all $u \in U$. By [1, Lemma 3.1], $R[\Gamma, \sigma]$ is isomorphic to the right Ore domain $(R[N, \sigma|_N])[U, \tau_U]$. ■

EXAMPLE 4.2. *Let G and H be infinite groups such that $R[G]$ and $R[H]$ are right Ore domains for all right Ore domains R. Then, $\Gamma =$ $= G \wr H$ has the property that $R[\Gamma]$* is a right Ore domain for all right Ore domains R, but Γ has a trivial center. Here, $\wr$ denotes the restricted wreath-product of G by H.

PROOF. By Proposition 4.1, it is enough to show that $\bigoplus_I G$ is Ore for all index sets I. Because of Proposition 4.1, it suffices to consider the case that I is finite. However, a finite direct sum of Ore groups is an Ore group by Proposition 4.1. ■

For instance, consider the following family $\{G_n \,|\, n < \omega\}$ of groups: Set $G_0 = \mathbb{Z}$ and $G_{n+1} = G_n \wr \mathbb{Z}$. We observe that each G_n is a solvable finitely generated Ore group with trivial center whose $(n-1)^{st}$ commutator subgroup is non-trivial. By Proposition 4, the group $\Gamma = \bigoplus_\omega G_n$ is an Ore group which is not solvable.

We conclude with some examples relating our results to previous work by Brungs' and Törner.

EXAMPLE 4.3. *Since every skewpolynomial ring $R[x, \sigma]$ can be viewed as a subring of $R[\mathbb{Z}, \sigma]$ Theorem 2.2 shows that the chain rings constructed in this paper include those from* [2].

In [1], we investigate groups Γ which are the union of a smooth ascending chain $\{\Gamma_\nu\}_{\nu<\kappa}$ of normal subgroups such that $\Gamma_{\nu+1}/\Gamma_\nu$ is torsion-free abelian.

THEOREM 4.4. *Let R be a right Ore-ring and Γ a group which is the union of a smooth chain $\{\Gamma_\nu\}_{\nu<\kappa}$ of normal subgroups of Γ with $\Gamma_0 = \{\varepsilon\}$. Then, Γ can be right ordered in such a way that for all $\alpha<\kappa$*

a) Γ/Γ_α carries a natural right order induced by the Γ_ν's such that the canonical projection $\pi_\alpha\colon \Gamma\to\Gamma/\Gamma_\alpha$ is an order preserving map.

b) $S_T^{\pi_\alpha}/J(S_T^{\pi_\alpha})$ is the classical right ring of quotients of $R[\Gamma_\alpha]$.

PROOF. Using [4, Lemma 3.7], we can right order Γ in such a way that an element $x\in\Gamma_{\nu+1}\setminus\Gamma_\nu$ is positive in Γ exactly if $x\Gamma_\nu$ is positive in $\Gamma_{\nu+1}/\Gamma_\nu$. We fix $\alpha<\kappa$, and observe that the group $[\Gamma_{\sigma+1}/\Gamma_\alpha]/[\Gamma_\sigma/\Gamma_\nu]\cong\Gamma_{\sigma+1}/\Gamma_\sigma$ is torsion-free abelian. We right order the group on the left in such a way that the natural isomorphism becomes order-preserving. Once Γ and Γ/Γ_α are right ordered as has been detailed in the first paragraph, the canonical projection $\pi_\alpha\colon\Gamma\to\Gamma/\Gamma_\alpha$ is order preserving. To see this let $x\in\Gamma$ be positive and choose $\sigma<\kappa$ minimal with $x\in\Gamma_\sigma$. Then, $\sigma=\nu+1$, and $x\Gamma_\nu$ is positive in Γ_σ/Γ_ν. Only the case $\sigma>\alpha$ needs further consideration. In this case, $x\Gamma_\alpha\in[\Gamma_\sigma/\Gamma_\alpha]\setminus[\Gamma_\nu/\Gamma_\alpha]$, and hence $x\Gamma_\alpha$ is positive since the isomorphism $[\Gamma_{\sigma+1}/\Gamma_\alpha]/[\Gamma_\sigma/\Gamma_\nu]\cong\Gamma_{\sigma+1}/\Gamma_\sigma$ is order preserving and $x\Gamma_\nu$ is positive in Γ_σ/Γ_ν. The theorem is now an immediate consequence of the results of Section 3. ∎

EXAMPLE 4.5. *Suppose that Γ is a right Ore-group which contains a normal subgroup N such that N and Γ/N are both right ordered groups, e.g. Γ is the semi-direct product of N and a suitable subgroup U. Then, Γ can be right ordered in such a way that the projection-map $\phi\colon\Gamma\to\Gamma/N$ is order preserving. We obtain that $S_T^\phi/J(S_T^\phi)$ is the classical ring of quotients of the group algebra $R[N]$, while the chain-ring $S_T^{1_\Gamma}$ which is obtained by using $1_\Gamma\colon\Gamma\to\Gamma$ to define the generalized valuations satisfies $S_T^{1_\Gamma}/J(S_T^{1_\Gamma})\cong Q(R)$ where $Q(R)$ is the classical ring of quotients of R. In the case of [1, Example 2], the first ring is the classical ring of quotients of $R[\mathbb{Z}]$ and is not associated with the cone Π_1.*

The last example also applies in the following case. Let $\Gamma = \mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$ in which $\Pi_{\mathbb{Z} \wr \mathbb{Z}} \mathbb{Z}$ is the kernel of the induced map $\phi \colon \Gamma \to \mathbb{Z} \wr \mathbb{Z}$, and consider the group-algebra $K[\Gamma]$ over a field K. Since this kernel is an abelian group, the induced valuation ring S^{ϕ} has the property that $S^{\phi}_T / J(S^{\phi}_T)$ is a commutative ring not isomorphic to K although $K[\Gamma]$ is non-commutative.

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