Ulrich Albrecht
Günter Törner

Valuations and group algebras

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1. – Introduction.

In [5], Dubrovin constructed a chain domain which has a prime ideal which is not completely prime. This ring was obtained by considering a right ordered group \( \Gamma \) such that the group algebra \( K[\Gamma] \) can be embedded in a skew field \( D \). Thus a partial answer was given to the Malcev-Problem [7]: Let \( F \) be a field and \( G \) be a left orderable group. Can the group ring \( F[G] \) be enclosed in a skew field? This question has a stronger version which can be found for instance in Passmann's book [10]: Determine the right ordered groups \( \Gamma \) with positive cone \( \Pi \) for which the skew-group-algebra \( R[\Gamma, \sigma] \) is an order in a division algebra \( D \) whenever \( R \) is an Ore domain. In the following, the pair \( (\Gamma, \Pi) \) denotes a right ordered group \( \Gamma \) with its positive cone \( \Pi \), while \( \sigma: \Gamma \to \text{Aut}(R) \) is a group homomorphism. A perhaps more natural reformulation of Passmann's problem is to ask for which \( \Gamma \) the group algebra \( R[\Gamma, \sigma] \) is a right Ore domain whenever \( R \) is one. This question has been discussed in detail in [1], and most constructions of chain orders in skew fields are based on the results of this paper, as one can see for instance in [2].

The primary goal of this paper is to investigate the structure of the group algebras and chain rings obtained via the localization techniques which were discussed in [1]. Our discussion focuses on a pair \( \Gamma_1 \) and \( \Gamma_2 \) of right ordered groups with positive cones \( \Pi_1 \) and \( \Pi_2 \) respectively. We
consider a subsemigroup $\Delta$ of $\Gamma_1$ containing $\Pi_1$ and a cone preserving semigroup-morphism $\phi: \Delta \to \Gamma_2$, and define the right and left $\phi$-values of non-zero elements of $R[\Delta, \sigma]$. Theorem 2.2 shows that these $\phi$-values give rise to a pair of generalized, conjugated valuations in the sense of [1]. The same result also shows that $R[\Delta, \sigma]$ has a zero Jacobson-radical and that its group of units $U(R[\Gamma_1, \sigma]) = U(R) U(\Delta)$. In particular, Theorem 2.2 permits to solve the isomorphism problem for right orderable groups: Two right orderable groups $\Gamma_1$ and $\Gamma_2$ are isomorphic if and only if $R[\Gamma_1] \cong R[\Gamma_2]$ for all rings $R$ (Corollary 2.3). This extends the well-known result that torsion-free abelian groups are isomorphic if and only if the corresponding group algebras are isomorphic. Since the ring structure of $R[\Gamma_1, \sigma]$ is independent of the chosen right order on $\Gamma_1$, no statement can be made about $\Gamma_1$ as a right ordered group. The remaining part of Section 2 investigates the valuation ring $S^\phi$ associated with the $\phi$-values.

Section 3 considers the chain rings $S^\phi_T$ arising as localizations of $S^\phi$ inside the classical right ring of quotients of $R[\Gamma_1, \sigma]$ in the case that $R[\Gamma_1, \sigma]$ is a right Ore ring. Theorem 3.1 determines the Jacobson radical $J(S^\phi)$ of this ring and shows that $S^\phi / J(S^\phi)$ is the classical right ring of quotients of $R[H, \sigma | H]$ where $H = \ker \phi$. Furthermore, the pair of generalized, conjugated valuations on $R[\Gamma_1, \sigma]$ induces a left valuation $| \cdot |_l$ on $S^\phi_T$ such that $|a|_l \leq |b|$ if and only if $bS^\phi_T \subseteq aS^\phi_T$ for all $a$, $b \in S^\phi_T$.

In the following, all rings have a multiplicative identity. The symbols $J(R)$ and $U(R)$ denote the Jacobson-radical and the group of units of $R$ respectively. All groups are written multiplicatively.

2. - Group rings and cones.

Let $\Gamma$ be a group. A subsemigroup $\Pi \subseteq \Gamma_1$ is called a cone if $\Pi \cap \Pi^{-1} = \{\varepsilon\}$ and $\Pi \cup \Pi^{-1} = \Gamma$ hold. Note that in the case where $\Pi$ is invariant, $\Gamma$ is an ordered group. In general, setting $\alpha \leq_r \beta$ iff $\beta \alpha^{-1} \in \Pi$ resp. $\alpha \leq_l \beta$ iff $\alpha^{-1} \beta \in \Pi$ allows to view the group $\Gamma$ as a right-ordered resp. left-ordered group.

We consider the pair $(\Gamma_1, \Pi_1)$ where $\Gamma_1$ is a group and $\Pi_1$ a cone. Further let $R$ be a domain, i.e. a ring without zero divisors. For a group homomorphism $\sigma: \Gamma_1 \to \Aut(R)$, we define a ring multiplication on the free left $R$-module with basis $\Gamma_1$ by $ar = r^{\sigma(a)} \alpha$ for all $a \in \Gamma_1$ and $r \in R$. The resulting ring is denoted by $R[\Gamma_1, \sigma]$. Write a non-zero $x$ in $R[\Gamma_1, \sigma]$
as \( x = \sum_{a \in \Pi_1} r_a \alpha \), and let \( \text{supp}(x) = \{ \alpha \mid r_a \neq 0 \} \) denote the support of \( x \).

If \( T = \{ x \in R[\Gamma_1, \sigma] \mid \epsilon_1 \in \text{supp}(x) \subseteq \Pi_1 \} \), then every \( x \in R[\Gamma_1, \sigma] \) has unique factorizations \( x = ua = \beta v \) with \( \alpha, \beta \in \Gamma_1 \) and \( u, v \in T \). We refer to this decomposition as the \( T-\Gamma_1 \)-factorization of \( x \) and say that \(|x|_1 = \beta \) resp. \(|x|_r = \alpha \) are the left resp. right \( \Gamma_1 \)-values of \( x \). In a previous paper we had introduced the concept of generalized valuations, however this terminology can be omitted in this context. Naturally, the question arises at this point if the conditions on \( \Pi_1 \) can be weakened by assuming that the subsemigroup \( \Pi_1 \) is a right cone only, i.e.

(a) \( \Pi_1 \) generates \( \Gamma_1 \).

(b) If \( a, b \in \Pi_1 \) with \( a^{-1} b \notin \Pi_1 \), then \( b^{-1} a \in \Pi_1 \).

This question is answered in a negative way by

**Proposition 2.1.** A right cone \( \Pi_1 \) of \( \Gamma_1 \) is a cone if and only if \( x \in R[\Gamma_1, \sigma] \) has a unique \( T-\Gamma_1 \)-decomposition.

**Proof.** The uniqueness property guarantees immediately that \( \Pi_1 \) cannot contain any units but \( \epsilon_1 \). If \( a \in \Gamma_1 \), then we can find \( \delta \in \Gamma_1 \) and \( \pi_1, \pi_2 \in \Pi_1 \) with \( \epsilon_1 + a = \delta (\pi_1 + \pi_2) \). Without loss of generality, \( \epsilon_1 = \delta \pi_1 \) and \( a = \delta \pi_2 \). Since \( \Pi_1 \) is a right cone, we may assume that \( \pi_1^{-1} \pi_2 \in \Pi_1 \), say \( \pi_2 = \pi_1 \pi \) for some \( \pi \in \Pi_1 \). Then, \( a = \delta \pi_2 = \delta \pi_1 \pi = \pi \in \Pi_1 \). In the same way, \( \pi_2^{-1} \pi_1 \in \Pi_1 \) yields \( a^{-1} \in \Pi_1 \).

Let \( (\Gamma_2, \Pi_2) \) be a further right-ordered group. We consider a subsemigroup \( \Delta \) of \( \Gamma_1 \) containing \( \Pi_1 \). A semigroup map \( \phi : \Delta \to \Gamma_2 \) is called a cone preserving homomorphism, provided \( \phi \) maps \( \Pi_1 \) into \( \Pi_2 \). It is natural to define the \( \phi \)-values of an element \( x \in R[\Delta, \sigma] \) to be the values under \( \phi \) of \(|x|_r \) resp. \(|x|_l \). To be more precise, we set \(|x|_r^\phi = \phi |x|_r \) resp. \(|x|_l^\phi = \phi |x|_l \). We call \( (\Gamma_1, \Gamma_2, \Delta) \) a cone-valuated triple with associated map \( \phi \) if \( \Delta \) contains \( a^{-1} \) for every \( a \in \ker \phi \). In the case that \( \Delta = \Gamma_1 \), we omit any reference of \( \Delta \) and speak of a cone-valuated pair instead.

**Theorem 2.2.** Let \( (\Gamma_1, \Gamma_2, \Delta) \) be a cone-valuated triple with associated map \( \phi \). The following hold for any domain \( R \) and any group-homomorphism \( \sigma : \Gamma_1 \to \text{Aut}(R) \):

(a) For all non-zero \( x, y, z \in R[\Delta, \sigma] \), the following conditions are satisfied:

(i) \(|x|_r^\phi \in \Pi_2 \) if and only if \(|x|_l^\phi \in \Pi_2 \).
PROOF. (a) Since we have \( o(a) \in 172 \) for all \( a \in \text{supp}(x) \). But then, \( \frac{a}{a} \) has to be in \( 172 \) too. Thus, (i) holds by symmetry. Furthermore, if \( a \in 2 \), then \( q5(a) \in 112 \) for all \( a \in \text{supp}(x) \) by what has been shown so far. If \( a_0 = \min \text{supp}(x) \), and from \( r \), which (ii) follows by symmetry.

To show (iii), let Then \( a_0 \in \text{supp}(x) \) or \( a_0 \in \text{supp}(y) \). In the first case, \( \frac{a}{a} \) is desired. The second case is treated similarly. By symmetry, (iii) is satisfied. For (iv), we suppose \( \phi(\alpha) \in 1 \). We obtain and \( f_3 \in 1 \). But this gives

\[
\text{Since } o(a_1) \text{ and } o(a_2), \text{ we have from which the first part of (c) follows.}
\]

In view of the symmetry of the problem, (a) has been shown.

To prove (b) consider an element \( x \in R[\Delta, \sigma] \) which has an inverse \( y \in R[\Delta, \sigma] \). We write \( x = v\beta \) with \( v \in T \) and \( \beta \in \Delta \). If \( |\text{supp}(x)| > 1 \), then \( \text{supp}(v) \) contains an element of \( \Pi_1 \backslash \{\varepsilon_1\} \). We write \( y = ua \) with \( u \in T \) and \( a \in \Delta \) and select \( w \in T \) and \( \gamma \in \Delta \) with \( \beta u = w \gamma \). Since \( \varepsilon_1 = w \gamma \alpha \), we have \( \gamma \alpha = \varepsilon_1 \) and \( vw = \varepsilon_1 \) by the uniqueness of \( T - \Gamma_1 \)-factorizations. In \( \text{supp}(v) \), choose an element \( a \) which is maximal in the left order induced by \( \Pi_1 \), while in \( \text{supp}(w) \) choose \( \beta \) maximal in the right order. Since \( |\text{supp}(v)| \geq 2 \), we have \( \alpha \geq \varepsilon_1 \), from which we obtain \( a\beta \geq \varepsilon_1 \). Because \( R \) is a domain, \( a\beta \) has a non-zero coefficient in the product \( vw \), but is not an element of \( \text{supp}(vw) \) since \( vw = \varepsilon_1 \). Hence, there are \( a' \in \text{supp}(v) \) and \( \beta' \in \text{supp}(w) \) with \( a\beta = a'\beta' \) and \( a \neq a' \) or \( \beta \neq \beta' \). A straightforward calculation shows that \( a \neq a' \) and \( \beta \neq \beta' \). Since \( \beta \geq a' \beta' \)
by the choice of \( \beta \), we can find \( \pi \in \Pi_1 \) with \( \beta = \pi \beta' \). Then \( \alpha' \beta' = \alpha \beta = \pi \alpha \beta' \) yields \( \alpha' = \pi \alpha \) from which \( \alpha' \geq \alpha \) follows. However, we have \( \alpha' < \alpha \) by the choice of \( \alpha \). The resulting contradiction shows that \( x \) cannot have more than one element in its support, i.e. \( v \in R \). Then, \( |\text{supp}(y)| = |\text{supp}(xy)| = 1 \), and \( u \in R \). In particular, \( \varepsilon_1 = xy = vu^{\sigma(\beta)} \beta \alpha = yx = uv^{\sigma(\alpha)} \alpha \beta \) yields that \( \alpha \) is a unit of \( \Delta \). Moreover, \( v \) is a unit of \( R \) whose inverse is \( u^{\sigma(\beta)} \). The converse is obvious. For the proof of (c), let \( x \) be a non-zero element of \( J(R[\Delta, \sigma]) \) and write \( x = u \alpha \) where \( u \in T \) and \( \alpha \in \Delta \). If \( \alpha \leq_r \varepsilon_1 \), then \( \alpha^{-1} \geq_r \varepsilon_1 \). We choose any \( \beta \in \Pi_1 \setminus \{\varepsilon_1\} \), and observe that \( x \alpha^{-2} \beta \) is a non-zero element of \( J(R[\Delta, \sigma]) \) with \( x \alpha^{-2} \beta = u \alpha \alpha^{-1} \beta \) and \( \alpha^{-1} \beta \geq_r \varepsilon_1 \). Hence, no generality is lost, if we assume that \( \alpha >_r \varepsilon_1 \). Since \( \varepsilon_1 \notin \text{supp}(x) \) in this case, we have \( \text{supp}(x - \varepsilon_1) = \text{supp}(x) \cup \{\varepsilon_1\} \). Since \( \text{supp}(x) \) is not empty, \( \varepsilon_1 - x \) cannot have a right inverse in \( R[\Delta, \sigma] \) what has been shown previously. On the other hand, \( J(R[\Delta, \sigma]) \) is a quasi-regular ideal, which results in a contradiction.  

Theorem 2.2 shows in particular that the maps \( | \) \( |_r \) and \( |_l \) defined in part (a) form a pair of generalized, conjugated valuations in the sense of [1].

**Corollary 2.2.** Let \( \Gamma_1 \) and \( \Gamma_2 \) be right orderable groups. Then, \( \Gamma_1 \equiv \Gamma_2 \) as groups if and only if \( R[\Gamma_1] \equiv R[\Gamma_2] \) for all rings \( R \).

**Proof.** By Theorem 2.2 (b), we known that \( U(Q[\Gamma_1]) = U(Q) \Gamma_i \). Observe that \( N_i = U(Q) \varepsilon_i \) is a normal subgroup of \( U(Q[\Gamma_i]) \) since it is contained in the center of \( Q[\Gamma_i] \). Every ring isomorphism \( \sigma: Q[\Gamma_1] \rightarrow Q[\Gamma_2] \) induces a group isomorphism \( \tau: U(Q) \Gamma_i \rightarrow U(Q) \Gamma_2 \). Since \( \tau \) was induced by the ring-map \( \sigma \), we have \( \tau(r \varepsilon_1) = r \varepsilon_2 \) for all \( r \in U(Q) \). Thus, \( \tau | N_i \) maps \( N_1 \) onto \( N_2 \). Since \( U(Q[\Gamma_1]) \) is the direct product of \( N_i \) and \( \Gamma_i \), we obtain that \( \Gamma_1 \) and \( \Gamma_2 \) are isomorphic as groups.

We are particularly interested in the ring

\[
S^\phi = \{ x \in R[\Delta, \sigma] \mid |x|_r^\phi \geq \varepsilon_2 \} = \{ x \in R[\Delta, \sigma] \mid \text{supp}(x) \subseteq \Pi_1 \cup \ker \phi \}.
\]

Observe that \( 0 \in S^\phi \) since \( |0|_l = |0|_r = \infty > \varepsilon_2 \) by convention.

**Proposition 2.4.** Consider a cone-valuated triple \((\Gamma_1, \Gamma_2, \Delta)\) with associated map \( \phi \) whose kernel is denoted by \( H \), a domain \( R \), and a group homomorphism \( \sigma: \Gamma_1 \rightarrow \text{Aut}(R) \). Then \( I^\phi = \{ x \in \)
is a two-sided ideal of \( S^\phi \) such that \( S^\phi /I^\phi \equiv R[H, \tau] \) where \( \tau = \sigma|H \).

**Proof.** If \( x \) is a non-zero element of \( I^\phi \), then \( |x|^\phi > \varepsilon_2 \). Observe that, for every non-zero \( y \in S^\phi \), the inequality \( |y|^\phi \geq \varepsilon_1 \varepsilon_2 = |x_1|^\phi \) yields \( |xy|^\phi \geq |x|^\phi > \varepsilon_2 \) from which \( |xy|^\phi > \varepsilon_2 \) follows. On the other hand, \( |y|^\phi \geq \varepsilon_1 \varepsilon_2 \) implies \( |yx|^\phi \geq \varepsilon_1 |x|^\phi = |x|^\phi > \varepsilon_2 \). Thus, \( xy, yx \in I^\phi \). Moreover, if \( a, b \) are in \( I^\phi \), then \( |a - b|^\phi \geq \min \{ |a|^\phi, |b|^\phi \} > \varepsilon_2 \), and \( a - b \in I^\phi \), and \( I^\phi \) is a two-sided ideal of \( S^\phi \). Let \( x = \Sigma_{\alpha \in \supp(x)} r_\alpha \alpha \) be a non-zero element of \( S^\phi \), and write \( x' = \Sigma_{\alpha \in \supp(x) \cap H} r_\alpha \alpha \). We define a map \( \lambda: S^\phi /I^\phi \rightarrow R[H, \tau] \) by \( \lambda(x) = x' \). If \( x - y \in I^\phi \), then \( 0 = (x - y)' = x' - y' \), and \( \lambda \) is well-defined. Moreover, if \( x, y \in S^\phi \), then \( x - x', y - y' \in I^\phi \), and \( xy - x'y' = (x - x')y + x'(y - y') \in I^\phi \) since \( I^\phi \) is a two-sided ideal. Then, \( \lambda(xy) = \lambda(x'y') = x'y' = \lambda(x)\lambda(y) \). It is now routine to show that \( \lambda \) is an isomorphism. \( \blacksquare \)

**Corollary 2.5.** Consider a cone-valuated triple \((\Gamma_1, \Gamma_2, \Lambda)\) with associated map \( \phi \), a domain \( R \), and a group homomorphism \( \sigma: \Gamma_1 \rightarrow \text{Aut}(R) \).

(a) \( I^\phi \) is maximal as a right ideal of \( S^\phi \) if and only if \( \phi \) is one-to-one and \( R \) is a division algebra.

(b) \( S^\phi \) has the property that \( |x|^\phi = \varepsilon_2 \) yields that \( x \) is a unit of \( S^\phi \) if and only if \( |\Gamma_1| = 1 \) and \( R \) is a division algebra.

**Proof.** (a) Suppose that \( I^\phi \) is a maximal right ideal of \( S^\phi \). Using the notation of Proposition 2.4, \( R[H, \tau] \) is a division algebra. But this is only possible if \( H = \{\varepsilon_1\} \) by part (b) of Theorem 2.2. Consequently, \( R \equiv S^\phi /I^\phi \) is a division algebra. The converse is obvious.

(b) Suppose that \( S^\phi \) is regular. If \( \Gamma_1 \) contains two elements, we can find \( \alpha \in \Gamma_1 \) such that \( \alpha \geq \varepsilon_1 \). Then, \( \varepsilon_1 + \alpha \) is an element of \( S^\phi \) with \( |\varepsilon_1 + \alpha|^\phi = \varepsilon_2 \). But then, \( \varepsilon_1 + \alpha \) is a unit of \( S^\phi \) which is impossible by Theorem 2.2. The rest of the proof is obvious. \( \blacksquare \)

3. – Localizations.

In this section, \( R \) always is a right Ore-ring and we assume that \((\Gamma_1, \Pi_1)\) has the property that \( R[\Gamma_1, \sigma] \) is a right Ore ring. Turning to the valuation rings which we considered in Proposition 2.4 and Corollary
2.5, we consider the set \( X = \{ x \in S^\phi \mid |x|^\phi = \varepsilon_2 \} \). It easy to see that the elements of \( X \) are precisely the elements \( x \in R[\Gamma_1, \sigma] \) of the form \( x = ua \) for some \( u \in T \) and \( \alpha \in \ker \phi \). Since the elements of \( \ker \phi \) are units in \( S^\phi \), it follows that \( X \) is an Ore set, and that \( S^\phi_X \equiv S^\phi_T \) is a chain-order in \( D \) in the sense of Dubrovin (for details see [1, Theorem 4.2]). In particular, \( D \) is the classical ring of quotients of \( S^\phi \).

**THEOREM 3.1.** Consider a right Ore-domain \( R \), a cone-valuated triple \((\Gamma_1, \Gamma_2, \Delta)\) with associated map \( \phi \) whose kernel is denoted by \( H \), and a group-homomorphism \( \sigma : \Gamma_1 \to \text{Aut}(R) \). If \( R[\Gamma_1, \sigma] \) is a right Ore-domain, then

(a) \( S^\phi_T \) is a chain-domain with maximal ideal \( I^\phi_T \).

(b) \( R[H, \tau] \) is a right Ore-domain whose classical ring of quotients is \( S^\phi_T/I^\phi_T \).

(c) \( | \cdot | \) induces a generalized left valuation \( | \cdot |_l \) on \( S^\phi_T \).

(d) For all \( a, b \in S^\phi_T \) we have \( |a|_l \leq |b|_l \) if and only if \( aS^\phi_T \supseteq bS^\phi_T \).

**Proof.** (a) To see that \( S^\phi_T \) is a right and left chain domain, we consider non-zero elements \( a_1 \) and \( a_2 \) of \( S^\phi_T \), and write \( a_i = a_i u_i t_i^{-1} \) with \( a_i \in \Gamma_1 \) and \( u_i, t_i \in T \) for \( i = 1, 2 \). Then, \( a_1 S^\phi_T = a_1 S^\phi_T \). If \( a_2 \geq a_1 \), then \( a_2 = \alpha_1 \tau \) for some \( \alpha \in \Pi_1 \subseteq S^\phi \). Therefore, \( a_2 S^\phi_T \subseteq a_1 S^\phi_T \). Because of the symmetry of the problem, \( S^\phi_T \) is a right chain domain.

Observe that \( I^\phi_T \) is a right ideal of \( S^\phi_T \) whose elements are of the form \( at^{-1} \) with \( a \in \phi \) and \( t \in T \). We show that \( I^\phi_T \) consists of the non-units of \( S^\phi_T \). To see this, suppose that \( 1 \in I^\phi_T \). It implies \( 1 = at^{-1} \) for some \( a \in \phi \) and \( t \in T \). We obtain \( a = t \) and \( \varepsilon_2 < |a|_\rho = |t|_\rho = \varepsilon_2 \), a contradiction. Thus, \( I^\phi_T \) consists only of non-units. On the other hand, if \( at^{-1} \in S^\phi \) is not a unit, then \( |a|_\rho > \varepsilon_2 \) since otherwise \( a \in X \), and \( at^{-1} \) is a unit of \( S^\phi_T \). But \( |a|_\rho > \varepsilon_2 \) implies \( a \in \phi \). Therefore, \( I^\phi_T \) indeed is the collection of non-units of \( S^\phi_T \). This shows that \( I^\phi_T \) is the Jacobson-radical of the chain-ring \( S^\phi_T \). In particular, \( D' = S^\phi_T/I^\phi_T \) is a division algebra.

(b) In the first step, we compute \( S^\phi \cap I^\phi_T \). Choose \( a \in \phi \) and \( t \in T \) such that \( at^{-1} \in S^\phi \), say \( at^{-1} = b \). If \( b \notin \phi \), then \( |b|_\rho = \varepsilon_2 \), and \( b \in X \). Consequently, \( \phi \) contains an element of \( X \), and \( I^\phi_T = S^\phi_T \) which is not possible. Therefore, \( S^\phi \cap I^\phi_T = \phi \). In the division algebra \( D' \), we consider the ring \( L = (S^\phi + I^\phi_T)/I^\phi_T \) and show that it is essential in \( D' \) as a \( L \)-submodule: If \( at^{-1} \in S^\phi_T \) with \( a \in S^\phi \) and \( t \in T \), but \( at^{-1} \notin I^\phi_T \), then
is a non-zero element of 
( at^{-1} + I_{r}^{\phi} ) L \cap L. Therefore, D' is the maximal right ring of quotients of L by [6, Propositions A. 2.11 and Corollary C. 2.31]. Moreover, $L = S^\phi / S^\phi \cap I^\phi_r = S^\phi / I^\phi \cong R[H, \tau]$ yields that $R[H, \tau]$ is a right Ore-ring whose classical right ring of quotients is isomorphic to $D'$ as desired.

To show (c) and (d), consider $a, b \in S^\phi_T$. We can find $\alpha, \beta \in \Gamma_1$ and $u, v, x, y \in T$ such that $a = \alpha ux^{-1}$ and $b = \beta vy^{-1}$, where the $T - \Gamma_1$-factorizations have the property $\alpha u, \beta v \in S^\phi$. Then $|\alpha u|_i = \phi(\alpha)$, and $|\beta v|_i = \phi(\beta)$. We obtain that $ux^{-1}$ and $vy^{-1}$ are units in $S^\phi_T$. Therefore, $aS^\phi_T = aS^\phi_T$ and $bS^\phi_T = \beta S^\phi_T$.

Suppose $aS^\phi_T \supseteq \beta S^\phi_T$. We can find $s \in S^\phi$ and $z \in T$ with $\beta = \alpha z^{-1}$, and write $s = \sum_{i=1}^{n} r_i \sigma_i$ such that $\phi(\sigma_i) \geq_1 e_2$ for all $i$ and $z = \sum_{j=1}^{m} t_j \delta_j$ where $\phi(\delta_j) \geq_1 e_2$ and $\phi(\delta_1) = e_2$. We obtain $\beta z = \sum_{j=1}^{m} t_j^{(\beta)} \beta \delta_j$ and $as = \sum_{i=1}^{n} r_i^{(\alpha)} a \alpha \sigma_i$. There exists $i_0 \in \{1, \ldots, n\}$ with $\beta \delta_1 = a \sigma_{i_0}$. This shows $\phi(\beta) = \phi(a \delta_1) = \phi(\alpha) \phi(\sigma_{i_0}) \geq_1 \phi(\alpha)$ since $\phi(\sigma_{i_0}) \geq_1 e_2$.

Define $|a|_i = \phi(\alpha)$ where $a = \alpha ux^{-1}$ is a factorization of $a$ as before. To show that this map is well-defined, we consider a second factorization $a = \tilde{a} \tilde{u} \tilde{x}^{-1}$. Since $aS^\phi_T = \tilde{a}S^\phi_T$, the results verified up to this point yield $\phi(\alpha) \leq \phi(\tilde{\alpha}) \leq \phi(\alpha)$. This shows that the map $|\cdot|_i$ is indeed well-defined. Moreover, every $a \in S^\phi$ can be written as $a = \alpha u x^{-1}$ with $u \in T$. Then, $|a|_i = \phi(\alpha) = |a|_i^\phi$. Thus, $|\cdot|_i$ extends $|\cdot|_i^\phi$ as desired.

We consider $a, b \in S^\phi_T$ with $|a|_i \geq_1 |b|_i$, and choose a decomposition of $a$ and $b$ as before. Observe $\phi(\alpha) \geq_1 \phi(\beta)$. If $\alpha \geq_1 \beta$ in $\Gamma_1$, then $\beta^{-1} \alpha \in \Pi_1 \subset \zeta S^\phi$, and there is $s \in S^\phi_T$ with $\alpha = \beta s$. Thus, $aS^\phi_T = \alpha S^\phi_T \subset \beta S^\phi_T = bS^\phi_T$. On the other hand, if $\alpha \leq_1 \beta$, then $\phi(\alpha) \leq_1 \phi(\beta)$; and hence $\phi(\alpha) = \phi(\beta)$. Since $\alpha^{-1} \beta \in \Pi_1 \subset \Delta$, we have $\phi(\beta) = \phi(\alpha) \phi(\alpha^{-1} \beta) = \phi(\beta) \phi(\alpha^{-1} \beta)$. Hence, $\alpha^{-1} \beta \in \ker \phi$, and hence $\beta^{-1} \alpha \in \Delta$. But then, $\beta^{-1} \alpha \in \ker(\phi) \subset X$ yields that $\beta^{-1} \alpha$ is a unit of $S^\phi_T$. In this case $aS^\phi_T = \beta S^\phi_T$. In either case, we have shown $|a|_i \geq_1 |b|_i$ if and only if $aS^\phi_T \subseteq bS^\phi_T$.

Using the last result and the fact that $S^\phi_T$ is a chain ring, it is now possible to show that the map $|\cdot|_i$ defines a generalized left valuation on $S^\phi_T$ using standard arguments.

In the last result, we assumed that $R[\Gamma_1, \sigma]$ is a right Ore ring in order to embed $S^\phi$ as an essential submodule into a ring $Q$ in which the elements of $T$ are units. We now show that the Ore condition on $R[\Gamma_1, \sigma]$ is necessary and sufficient for the existence of such a ring $Q$. 

Corollary 3.2. Consider a right Ore-domain $R$, a cone-valuated pair $(\Gamma_1, \Gamma_2)$ with associated map $\phi$, and a group-homomorphism $\sigma: \Gamma_1 \rightarrow \text{Aut}(R)$. Then, the following conditions are equivalent:

(a) $R[\Gamma_1, \sigma]$ is a right Ore ring.

(b) $S^\phi$ can be embedded as an essential $S^\phi$-submodule of a ring $Q$ in which the elements of $T$ are units.

Proof. (a)$\Rightarrow$(b) is an immediate consequence of the Theorem 3.1.

(b)$\Rightarrow$(a) Let $t \in T$ and $s \in S^\phi$. No generality is lost if we assume that $s \neq 0$. Because of (b), we can find elements $t_1, t_2 \in S^\phi$ with $t^{-1}st_1 = t_2$. Choose $u_1, u_2 \in T$ and $\alpha_1, \alpha_2 \in \Gamma_1$ with $t_i = u_i\alpha_i$ for $i = 1, 2$. Write $su_1\alpha_1 = tu_2\alpha_2$ to obtain $su_1 = tu_2(\alpha_2\alpha_1^{-1})$ in $R[\Gamma_1, \sigma]$. Then, $su_1 = \sum_{i=1}^n a_i \delta_i$ where $\text{supp}(su_1) = \{\delta_1, \ldots, \delta_n\} \subseteq \Gamma_1$ and $\phi(\delta_i) \geq \varepsilon_2$. Similarly, $tu_2 = \sum_{i=1}^m b_j \dot{q}_j$ where $\text{supp}(tu_2) = \{q_1, \ldots, q_m\}$ with $\phi(q_j) \geq \varepsilon_2$.

Then $\sum_{i=1}^n a_i \delta_i = \left( \sum_{j=1}^m b_j \dot{q}_j \right) \alpha_2\alpha_1^{-1} = \sum_{j=1}^m b_j(q_j\alpha_2\alpha_1^{-1}).$ This shows $\varepsilon_2 \leq_s |su_1|_r = \min_{i=1}^n \phi(\delta_i) = \min_{j=1}^m \phi(q_j\alpha_2\alpha_1^{-1}) = \left( \min_{j=1}^m \phi(q_j) \right) \phi(\alpha_2\alpha_1^{-1}) = \varepsilon_2 \phi(\alpha_2\alpha_1^{-1}) = \phi(\alpha_2\alpha_1^{-1}).$

Moreover $tu_2 \in T$, and hence $\alpha_2\alpha_1^{-1} \in S^\phi$. Furthermore, $u_2 \in T$ yields $u_2\alpha_2\alpha_1^{-1} \in S^\phi$. Therefore, $su_1 = t(u_2\alpha_2\alpha_1^{-1})$ with $u_1 \in T$ and $u_2\alpha_2\alpha_1^{-1} \in S^\phi$; and $T$ is right Ore in $S^\phi$.

To show (a), let $r_1$ and $r_2$ be two non-zero elements of $R^\sigma[\Gamma_1]$. Choose $u_1, u_2 \in T$ and $\alpha_1, \alpha_2 \in \Gamma_1$ with $r_i = u_i\alpha_i$ for $i = 1, 2$. By (b) there are $v_1, v_2 \in T$ with $u_2v_2 = u_1v_1$. Then $r_2\alpha_2^{-1}v_2 = u_2v_2 = u_1v_1 = r_1r_1^{-1}v_1$ is non-zero, and $\alpha_i^{-1}v_1 \in R[\Gamma_1, \sigma]$. Hence $R[\Gamma_1, \sigma]$ is a right Ore ring.

The last result of this section shows that, in the setting of Mathiak’s work ([8]), the pair of conjugated generalized valuations can be extended to the localization $S^\phi_T$ just like standard valuations.

Corollary 3.3. Let $R$ be a right and left Ore-domain, $(\Gamma_1, \Gamma_2, \Delta)$ a cone-valuated triple with associated map $\phi$, and $\sigma: \Gamma_1 \rightarrow \text{Aut}(R)$ be a group homomorphism such that $R[\Gamma_1, \sigma]$ is a right and left Ore-domain.
(a) The pair $(\|_l, \|_r)$ of generalized, conjugated valuations on $S^\phi$ extends to a pair $(\|_l, \|_r)$ of conjugated, generalized valuations on $S^\phi_T$.

(b) The induced valuations in a) are order-anti-isomorphic to the generalized valuations on $S^\phi_T$ which are induced by the linear ordering of the one-sided ideals of $S^\phi_T$.

PROOF. By Theorem 3.1, there exist extensions $\|_l$ and $\|_r$ of $\|_l$ and $\|_r$ which are one-sided generalized valuations and satisfy condition (b). Since $|a|_l \geq_l \epsilon_2$ and $|a|_r \geq_r \epsilon_2$ for all $a \in S^\phi_T$, it remains to show $|a|_l = \epsilon_2$ if and only if $|a|_r = \epsilon_2$. If $|a|_l = \epsilon_2$ then $aS^\phi T = S^\phi_T$ and $a$ is a unit of $S^\phi_T$. In this case, $S^\phi_T a = S^\phi_T$ and $|a|_r = \epsilon_2$. The converse is verified in exactly the same way. ■

4. – Examples.

Dubrovin showed in [5] that the property that $R[\Gamma, \sigma]$ is a right Ore-domain whenever $R$ is a right Ore-domain, is inherited by subgroups. Using this, we can easily establish the following

Proposition 4.1 (a) The following conditions are equivalent for a group $\Gamma$, a right Ore-domain $R$, and a group-homomorphism $\sigma: \Gamma \to \Aut(R)$:

(i) $R[\Gamma, \sigma]$ is a right Ore-domain.

(ii) For every finitely generated subgroup $U$ of $\Gamma$, the ring $R[U, \sigma|_U]$ is a right Ore-domain.

(iii) $\Gamma$ is the union of a smooth ascending chain $\{\Gamma_\nu | \nu < \kappa\}$ of subgroups $\Gamma_\nu$ such that $R[\Gamma_\nu, \sigma|_{\Gamma_\nu}]$ is a right Ore-domain.

(b) The following conditions are equivalent for a group $\Gamma$ which is the semi-direct product of a normal subgroup $N$ by a subgroup $U$:

(i) $R[\Gamma, \sigma]$ is a right Ore domain for all right Ore domains $R$ and all $\sigma: \Gamma \to \Aut(R)$.

(ii) $\alpha)$ $R[N, \sigma]$ is a right Ore domain for all right Ore domains $R$ and all $\sigma: N \to \Aut(R)$.

$\beta)$ $R[U, \tau]$ is a right Ore domain for all right Ore domains $R$ and all $\tau: U \to \Aut(R)$. 
PROOF. (a) The implication (i) $\Rightarrow$ (iii) is obvious. (iii) $\Rightarrow$ (ii): If $U$ is a finitely generated subgroup of $\Gamma$, then $U \subseteq \Gamma_v$ for some $v < \kappa$. Since $(\sigma|_{\Gamma_v})|_U = \sigma|_U$, we have that $R[U, \sigma|_U]$ is a right Ore domain by Dubrovin’s result. (ii) $\Rightarrow$ (i): Whenever $a$ and $b$ are non-zero elements of $R[\Gamma, \sigma]$, then there is a finitely generated subgroup $U$ of $\Gamma$ with $a, b \in R[U, \sigma|_U]$. But the latter ring is a right Ore-domain. (b) (i) $\Rightarrow$ (ii): Condition (i) holds by (a). To show the second condition, we observe that every homomorphism $\sigma: U \rightarrow \text{Aut}(R)$ can be extended to a homomorphism $\tilde{\sigma}: \Gamma \rightarrow \text{Aut}(R)$ by setting $\tilde{\sigma}(nu) = \sigma(u)$ for all $n \in N$ and $u \in U$. Now, we apply Dubrovin’s result again.

(ii) $\Rightarrow$ (i): By condition (ii), $R[N, \sigma|_N]$ is a right Ore domain, and $(R[N, \sigma|_N])[U, \tau]$ is a right Ore domain for all homomorphisms $\tau: U \rightarrow \text{Aut}(R[N, \sigma|_N])$. In particular consider the map $\tau_U$ which is defined by $[\tau_U(u)](\sum_{a \in N} r_a \alpha) = \sum_{a \in N} r_{\sigma(a)} \alpha$ for all $u \in U$. By [1, Lemma 3.1], $R[\Gamma, \sigma]$ is isomorphic to the right Ore domain $(R[N, \sigma|_N])[U, \tau_U]$. 

EXAMPLE 4.2. Let $G$ and $H$ be infinite groups such that $R[G]$ and $R[H]$ are right Ore domains for all right Ore domains $R$. Then, $\Gamma = \Gamma = G \Join H$ has the property that $R[\Gamma]$ is a right Ore domain for all right Ore domains $R$, but $\Gamma$ has a trivial center. Here, $\Join$ denotes the restricted wreath-product of $G$ by $H$.

PROOF. By Proposition 4.1, it is enough to show that $\bigoplus_I G$ is Ore for all index sets $I$. Because of Proposition 4.1, it suffices to consider the case that $I$ is finite. However, a finite direct sum of Ore groups is an Ore group by Proposition 4.1.

For instance, consider the following family $\{G_n \mid n < \omega\}$ of groups: Set $G_0 = \mathbb{Z}$ and $G_{n+1} = G_n \Join \mathbb{Z}$. We observe that each $G_n$ is a solvable finitely generated Ore group with trivial center whose $(n-1)^{st}$ commutator subgroup is non-trivial. By Proposition 4, the group $\Gamma = \bigoplus G_n$ is an Ore group which is not solvable.

We conclude with some examples relating our results to previous work by Brungs’ and Törner.

EXAMPLE 4.3. Since every skewpolynomial ring $R[x, \sigma]$ can be viewed as a subring of $R[Z, \sigma]$ Theorem 2.2 shows that the chain rings constructed in this paper include those from [2].
In [1], we investigate groups $\Gamma$ which are the union of a smooth ascending chain $\{\Gamma_v\}_{v<\kappa}$ of normal subgroups such that $\Gamma_{v+1}/\Gamma_v$ is torsion-free abelian.

**Theorem 4.4.** Let $R$ be a right Ore-ring and $\Gamma$ a group which is the union of a smooth chain $\{\Gamma_v\}_{v<\kappa}$ of normal subgroups of $\Gamma$ with $\Gamma_0 = \{1\}$. Then, $\Gamma$ can be right ordered in such a way that for all $\alpha < \kappa$

a) $\Gamma/\Gamma_\alpha$ carries a natural right order induced by the $\Gamma_v$'s such that the canonical projection $\pi_\alpha: \Gamma \to \Gamma/\Gamma_\alpha$ is an order preserving map.

b) $S^*_\Gamma/J(S^*_\Gamma)$ is the classical right ring of quotients of $R[\Gamma_\alpha]$.

**Proof.** Using [4, Lemma 3.7], we can right order $\Gamma$ in such a way that an element $x \in \Gamma_{v+1} \setminus \Gamma_v$ is positive in $\Gamma$ exactly if $x\Gamma_v$ is positive in $\Gamma_{v+1}/\Gamma_v$. We fix $\alpha < \kappa$, and observe that the group $[\Gamma_{\alpha+1}/\Gamma_\alpha]/[\Gamma_\alpha/\Gamma_v] \cong \Gamma_{\alpha+1}/\Gamma_\alpha$ is torsion-free abelian. We right order the group on the left in such a way that the natural isomorphism becomes order-preserving. Once $\Gamma$ and $\Gamma/\Gamma_\alpha$ are right ordered as has been detailed in the first paragraph, the canonical projection $\pi_\alpha: \Gamma \to \Gamma/\Gamma_\alpha$ is order preserving. To see this let $x \in \Gamma$ be positive and choose $\sigma < \kappa$ minimal with $x \in \Gamma_\sigma$. Then, $\sigma = v + 1$, and $x\Gamma_v$ is positive in $\Gamma_\sigma/\Gamma_v$. Only the case $\sigma > \alpha$ needs further consideration. In this case, $x\Gamma_\alpha \in \Gamma_\sigma/\Gamma_\alpha \setminus [\Gamma_\sigma/\Gamma_\alpha]$, and hence $x\Gamma_\alpha$ is positive since the isomorphism $[\Gamma_{\alpha+1}/\Gamma_\alpha]/[\Gamma_\alpha/\Gamma_v] \cong \Gamma_{\alpha+1}/\Gamma_\alpha$ is order preserving and $x\Gamma_v$ is positive in $\Gamma_\alpha/\Gamma_v$. The theorem is now an immediate consequence of the results of Section 3.

**Example 4.5.** Suppose that $\Gamma$ is a right Ore-group which contains a normal subgroup $N$ such that $N$ and $\Gamma/N$ are both right ordered groups, e.g. $\Gamma$ is the semi-direct product of $N$ and a suitable subgroup $U$. Then, $\Gamma$ can be right ordered in such a way that the projection-map $\phi: \Gamma \to \Gamma/N$ is order preserving. We obtain that $S^*_\Gamma/J(S^*_\Gamma)$ is the classical ring of quotients of the group algebra $R[N]$, while the chain-ring $S_{\ell}^1$, which is obtained by using $1_\Gamma: \Gamma \to \Gamma$ to define the generalized valuations satisfies $S_{\ell}^1/J(S_{\ell}^1) \cong Q(R)$ where $Q(R)$ is the classical ring of quotients of $R$. In the case of [1, Example 2], the first ring is the classical ring of quotients of $R[\mathbb{Z}]$ and is not associated with the cone $\Pi_1$. 
The last example also applies in the following case. Let $\Gamma = \mathbb{Z} \cdot (\mathbb{Z} \cdot \mathbb{Z})$ in which $\prod_{\mathbb{Z} \cdot \mathbb{Z}} \mathbb{Z}$ is the kernel of the induced map $\phi: \Gamma \rightarrow \mathbb{Z} \cdot \mathbb{Z}$, and consider the group-algebra $K[\Gamma]$ over a field $K$. Since this kernel is an abelian group, the induced valuation ring $S^\phi$ has the property that $S^\phi_T / J(S^\phi_T)$ is a commutative ring not isomorphic to $K$ although $K[\Gamma]$ is non-commutative.

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