# Rendiconti del Seminario Matematico della Università di Padova



Francesco Barioli


**Decreasing diagonal elements in completely positive matrices**


*Rendiconti del Seminario Matematico della Università di Padova*, tome 100 (1998), p. 13-25

<http://www.numdam.org/item?id=RSMUP_1998__100__13_0>

Rend. Sem. Mat. Univ. Padova, Vol. 100 (1998)

# Decreasing Diagonal Elements in Completely Positive Matrices.

Francesco Barioli (*)

All matrices considered in this paper are real matrices (1).

It is well known that a diagonally dominant symmetric matrix with nonnegative diagonal elements is positive semidefinite; actually, this fact is an immediate consequence of the Gerschgorin circles theorem.

An analogous result, not equally well known, holds for completely positive matrices, i.e. for those matrices $A$ that can be factorized in the form $A = VV^T$, where $V$ is a nonnegative matrix. In fact Kaykobad [4] proved that a nonnegative symmetric diagonally dominant matrix is completely positive.

Hence, if the diagonal elements of a symmetric matrix $A$ have sufficiently large positive values, then $A$ is positive semidefinite and, if $A$ is nonnegative, it is completely positive.

In this paper we consider the question arising by assuming the opposite point of view. Let $A$ be a positive semidefinite or a completely positive matrix; then we will consider the following question: how much a diagonal element of $A$ can be decreased while preserving the semidefinite positivity or, respectively, the complete positivity of $A$?

We will answer this question concerning positive semidefinite matrices in Section 2; the answer is simple and follows from an inductive test

(*) Indirizzo dell'A.: Dipartimento di Matematica Pura e Applicata, Università di Padova, Via Belzoni 7, 35131 Padova, Italy.

(1) This paper is part of the doctoral dissertation under the supervision of Prof. Luigi Salce.

for semidefinite positivity (see [7], Cor. 1.3]), which involves the Moore-Penrose pseudo-inverse matrix of a maximal principal submatrix.

As to completely positive matrices the answer arises from the connection of the above question with the property PLSS (positivity of least square solutions) introduced in [7]. More precisely, we will consider in Section 3 *i-minimizable matrices* ($1 \leqslant i \leqslant n$, where $n$ is the order of the considered matrix), which are those completely positive matrices such that the minimal value of the $i^{th}$ diagonal element making the matrix positive semidefinite makes the matrix completely positive too. We will show that 1-minimizability is equivalent to property PLSS, that singular completely positive matrices are $i$-minimizable for some $i$, and that completely positive matrices with completely positive associated graph (see [5]) are $i$-minimizable for all $i$.

Recent results (see [1]) show that completely positive matrices with cyclic graph of odd lenght are not $i$-minimizable for any $i$. Conversely, it follows from results by Berman and Grone [2] that completely positive matrices with cyclic graph of even lenght are $i$-minimizable for all $i$. Thus the following problem arises: to find examples of completely positive matrices with more complicate behaviour with respct to minimizability.

In Section 4 we will investigate *excellent* completely positive matrices, already introduced in [1], and defined in terms of their graphs, which are «almost» completely positive. We will give a characterization of excellent 1-minimizable completely positive matrices, which enables us to produce several examples of excellent matrices with different behaviour with the respect to minimizability.

## 2. – Minimized positive semidefinite matrices.

We recall the following definitions. A symmetric matrix $A$ of order $n$ is *positive semidefinite* if $\underline{x}^T A \underline{x} \geqslant 0$ for all vectors $\underline{x} \in \mathbb{R}^n$. The matrix $A$ is *doubly nonnegative* if it is both semidefinite and (entrywise) nonnegative; it is *completely positive* if there exist a nonnegative (not necessarily square) matrix $V$ such that $A = VV^T$.

It is well known that a completely positive matrix is doubly nonnegative and that the converse is not generally true for matrices of order larger than four.

We will denote by $R(A)$ the column space (range) of the matrix $A$, by $A^+$ its Moore-Penrose pseudo-inverse and by $\mathrm{rk}(A)$ its rank. For unexplained notation we refer to [6].

Let $A$ be a symmetric semidefinite matrix of order $n > 1$ in bordered form

(1) $$A = \begin{pmatrix} a & \underline{b}^T \\ \underline{b} & A_1 \end{pmatrix}.$$

In [7] the following result, which provides an inductive test for semidefinite positivity, was proved:

LEMMA 2.1. *The symmetric matrix $A$ in* (1) *is positive semidefinite if and only if $A_1$ is positive semidefinite, $\underline{b} \in R(A_1)$ and $a \geqslant \underline{b}^T A_1^+ \underline{b}$.*

It follows from Lemma 2.1 that the minimal value of $t \in \mathbb{R}$ such that the matrix

(2) $$A(t) = \begin{pmatrix} t & \underline{b}^T \\ \underline{b} & A_1 \end{pmatrix}$$

is positive semidefinite is $\underline{b}^T A_1^+ \underline{b}$; we will denote it by $a_\mu$. The matrix

(3) $$A_\mu = \begin{pmatrix} a_\mu & \underline{b}^T \\ \underline{b} & A_1 \end{pmatrix}$$

is called the 1-*minimized* of $A$. We say that $A$ itself is 1-*minimized* if $A = A_\mu$. If $1 \leqslant i \leqslant n$, the *i-minimized* of $A$ is the matrix $(P_i^T A P_i)_\mu$, where $P_i$ is the permutation matrix obtained from the identity matrix by transposing the first and the $i^{\text{th}}$ row. The matrix $A$ is said to be *i-minimized* if $P_i^T A P_i = (P_i^T A P_i)_\mu$.

A 1-minimized (or, more generally, an *i-minimized*) matrix is singular. This fact is trivial if $A_1$ is singular, otherwise it follows from the equality $\operatorname{Det}(A) = \operatorname{Det}(A_1)(a - \underline{b}^T A_1^{-1} \underline{b})$ applied to $A = A_\mu$.

The converse is not generally true, as the trivial example

(4) $$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

shows. However, we have the following result:

THEOREM 2.2. *A singular positive semidefinite matrix $n$-by-$n$ is $i$-minimized for some $i \leqslant n$.*

In order to prove the preceding theorem, we need two lemmas, whose straightforward proofs are given for sake of completeness.

LEMMA 2.3. *Let $A$ be a positive semidefinite $n$-by-$n$ matrix. Then its rank coincides with the maximum order of a non-singular principal submatrix.*

PROOF. Let $\mathrm{rk}(A) = k$. Since $A$ is positive semidefinite, there exists a real $n$-by-$k$ matrix $W$ such that $A = WW^T$. Since $\mathrm{rk}(W) = k$ too, we can find a non-singular $k$-by-$k$ submatrix of $W$. Let it $W_1$. Finally $A' = W_1 W_1^T$ is a non-singular principal submatrix of $A$ of order $k$. ∎

The second result that we need is the following:

LEMMA 2.4. *Let $A$ be a positive semidefinite matrix in bordered form* (1), *with* $\mathrm{rk}(A_1) = k$, *and let $M$ a non-singular principal submatrix of $A_1$ of order $k$. Then $a_\mu = \underline{c}^T M^{-1} \underline{c}$, where $\underline{c}$ is the vector obtained by taking the coordinates of $\underline{b}$ corresponding to the rows of $M$.*

PROOF. The existence of $M$ is ensured by Lemma 2.3. There is no loss of generality if we consider $A$ in the form

$$\begin{pmatrix} a & \underline{c}^T & \underline{d}^T \\ \underline{c} & M & D^T \\ \underline{d} & D & N \end{pmatrix}. \tag{5}$$

There exists a matrix $W_1$ of rank $k$ such that $A_1 = W_1 W_1^T$ and clearly $A_\mu = WW^T$ for $W^T = (W_1^+ \underline{b} \ \ W_1^T)$, because

$$a_\mu = \underline{b}^T A_1^+ \underline{b} = \underline{b}^T (W_1 W_1^T)^+ \underline{b} = \underline{b}^T (W_1^T)^+ W_1^+ \underline{b} = (W_1^+ \underline{b})^T W_1^+ \underline{b}.$$

Thus $\mathrm{rk}(A_\mu) = k$ and the principal submatrix

$$\begin{pmatrix} a_\mu & \underline{c}^T \\ \underline{c} & M \end{pmatrix} \tag{6}$$

is singular and positive semidefinite, thus $a_\mu = \underline{c}^T M^{-1} \underline{c}$, by Lemma 2.1. ∎

We are now able to give the

PROOF OF THEOREM 2.2 Let $B$ the singular positive semidefinite matrix, $k = \mathrm{rk}(B)$. There exists a permutation matrix of the form $P_i$ for some $i \leq n$, such that $A = P_i^T B P_i$, written in bordered form (1), has $\mathrm{rk}(A_1) = k$.

By Lemma 2.4, there exists a non-singular principal submatrix $M$ of order $k$ of $A_1$, so that $A$ is cogredient to the matrix

$$A^* = \begin{pmatrix} a & \underline{c}^T & \underline{d}^T \\ \underline{c} & M & D^T \\ \underline{d} & D & N \end{pmatrix}. \tag{7}$$

We will show that this matrix is 1-minimized, hence the matrix $B$ will be $i$-minimized. The principal submatrix

$$\begin{pmatrix} a & \underline{c}^T \\ \underline{c} & M \end{pmatrix} \tag{8}$$

is singular, hence $a = \underline{c}^T M^{-1} \underline{c}$ and, by Lemma 2.4, this is the minimal value that makes $A^*$ positive semidefinite. ∎

From the preceding results we get the following

COROLLARY 2.5. *Let $A$ be a singular positive semidefinite matrix in bordered form* (1). *Then $A$ is* 1*-minimized if and only if* $\mathrm{rk}(A) = \mathrm{rk}(A_1)$.

PROOF. Assume $A$ 1-minimized. The proof of Lemma 2.4 shows that $W$ has the same number of columns as $W_1$, hence $\mathrm{rk}(A) = \mathrm{rk}(A_1)$. Conversely, if $\mathrm{rk}(A) = \mathrm{rk}(A_1)$, in the proof of Theorem 2.2, $M$ can be choosen as a submatrix of $A_1$ without using the permutation matrix $P_i$, and one can conclude that $A$ itself is 1-minimized. ∎

## 3. – Minimizable completely positive matrices.

Let $A$ be a completely positive matrix of order $n$ in bordered form (1). An immediate consequence of the fact that the class of the completely positive matrices of order $n$ is closed in the class of the symmetric matrices of the same order (see [3]) is that there exists a minimal value $a_e \in \mathbb{R}$

such that the matrix

$$A_e = \begin{pmatrix} a_e & \underline{b}^T \\ \underline{b} & A_1 \end{pmatrix} \tag{9}$$

is completely positive; obviously $a_e \geqslant a_\mu$. The matrix $A_e$ is called the 1-*extremized* of $A$. $A$ itself is called an 1-*extreme* matrix if $A = A_e$. If $i \leqslant n$, the *i-extremized* of $A$ is the 1-extremized of $P_i^T A P_i$ and $A$ itself is an *i-extreme* matrix if $P_i^T A P_i$ is 1-extreme.

DEFINITION. We say that a completely positive matrix $A$ in bordered form (1) is *i-minimizable* if the $i$-minimized matrix of $A$ is also completely positive, i.e. if $(P_i^T A P_i)_\mu = (P_i^T A P_i)_e$.

DEFINITION. We say that a completely positive matrix $A$ in bordered form (1) is *totally minimizable* if it is $i$-minimizable for all $i \leqslant n$; equivalently, if all the matrices cogredient to $A$ are 1-minimizable.

The apparently new notion of 1-minimizable completely positive matrix is equivalent to the notion of «property PLSS» already introduced in [7], and simply called «property (P)» in [6]. Recall that the symmetric non-negative matrix $A$ in bordered form (1) satisfies property PLSS if there exists a nonnegative matrix $V_1$ such that $A_1 = V_1 V_1^T$ and $V_1^+ \underline{b} \geqslant \underline{0}$.

PROPOSITION 3.1. *A completely positive matrix $A$ in bordered form (1) is 1-minimizable if and only if it satisfies the property* PLSS.

PROOF. Let $A$ satisfy the property PLSS. Then $A_\mu$ also satisfies this property; hence, by ([7], Theorem 2.1), $A_\mu$ is completely positive.

Conversely, let $A$ be 1-minimizable. By ([7], Proposition 1.2), there exists a nonnegative matrix $V_1$ and a nonnegative vector $\underline{v}$ such that:

$$A_1 = V_1 V_1^T \qquad \underline{b} = V_1 \underline{v} \qquad \underline{b}^T A_1^+ \underline{b} = \underline{v}^T \underline{v}.$$

From the equalities

$$\|\underline{v}\|_2^2 = \underline{v}^T \underline{v} = \underline{b}^T A_1^+ \underline{b} = \underline{b}^T (V_1 V_1^T)^+ \underline{b} =$$

$$= \underline{b}^T (V_1^T)^+ V_1^+ \underline{b} = \underline{b}^T (V_1^+)^T V_1^+ \underline{b} = \|V_1^+ \underline{b}\|_2^2$$

and from the uniqueness of the least square solution of the linear system

$V_1 \underline{x} = \underline{b}$, there follows that $\underline{v} = V_1^+ \underline{b}$. Thus $V_1^+ \underline{b} \geqslant \underline{0}$ and property PLSS holds. ■

The class of totally minimizable matrices is quite large, as the following result shows. Recall that a graph $\Gamma$ is called *completely positive* if every doubly nonnegative matrix $A$ whose associated graph $\Gamma(A)$ equals $\Gamma$ is completely positive. Kogan and Berman [5] proved that a graph is completely positive if and only if it does not contain a cycle of odd length larger than three.

PROPOSITION 3.2. *The class of totally minimizable matrices contains all the completely positive matrices with completely positive graphs.*

PROOF. Let $A$ be a completely positive matrix such that its graph $\Gamma(A)$ is completely positive. Let $B$ the $i$-minimized matrix of $A$ for some $i \leqslant n$. Obviously $\Gamma(B)$ is completely positive, thus $B$, which is obviously doubly nonnegative, is completely positive. ■

The connection between singularity and minimizability is illustrated in the following

PROPOSITION 3.3. *Let $A$ be a singular completely positive matrix in bordered form* (1). *Then $A$ is $i$-minimizable for some $i \leqslant n$ and is* 1*-minimizable if* $\mathrm{rk}(A) = \mathrm{rk}(A_1)$.

PROOF. By Theorem 2.2, $A$ is $i$-minimized for a certain $i$; let $A^* = P_i^T A P_i$. $A^*$ is completely positive and 1-minimized, so it is obviously 1-minimizable, hence $A$ is $i$-minimizable. Furthermore, if $\mathrm{rk}(A) = \mathrm{rk}(A_1)$, by Corollary 2.5 it follows that $A$ is 1-minimizable. ■

We conclude this section with a result providing the construction of a completely positive singular matrix (hence $i$-minimizable for some $i \leqslant n$) which fails to be 1-minimizable; this matrix is obtained by bordering in a suitable way a non-singular not 1-minimizable completely positive matrix. Such a matrix does exists (see Example 4.6).

PROPOSITION 3.4. *Let $C$ be a non-singular completely positive matrix of order $n > 1$ in bordered form*

$$C = \begin{pmatrix} a & \underline{c}^T \\ \underline{c} & C_1 \end{pmatrix} \tag{10}$$

*which is not 1-minimizable. Then there exists a completely positive matrix $A$ obtained by suitably bordering the matrix $C$*

$$A = \begin{pmatrix} a & \underline{c}^T & d \\ \underline{c} & C_1 & \underline{d} \\ d & \underline{d}^T & e \end{pmatrix} \tag{11}$$

*which is not 1-minimizable, but is $i$-minimizable for some $1 < i \leq n$.*

PROOF. There exists a matrix $V \geq 0$ such that $C = VV^T$. Let $V^T = (\underline{v}_1 \ \ V_1^T)$. Then $\mathrm{rk}(V) = \mathrm{rk}(C) = n$ and $\mathrm{rk}(V_1) = \mathrm{rk}(C_1) = n - 1$. Let $W^T = (\underline{v}_1 \ \ V_1^T \ \ \underline{v})$, where $\underline{v}$ is a nonnegative vector of the column space $R(V_1^T)$. Obviously $\mathrm{rk}(W^T) = n$. Let now $A = WW^T$. Then $A$ is completely positive of order $n + 1$ and rank $n$, and has the form (11) for $d = \underline{v}_1^T \underline{v}$, $\underline{d} = V_1 \underline{v}$ and $e = \underline{v}^T \underline{v}$. By Proposition 3.3, $A$ is $i$-minimizable for some $i$. Assume, by way of contraddiction, that $A$ is 1-minimizable; then the 1-minimized matrix $A_\mu$ of $A$ is completely positive, hence the its submatrix

$$\begin{pmatrix} a_\mu & \underline{c}^T \\ \underline{c} & C_1 \end{pmatrix} \tag{12}$$

is completely positive too. By Lemma 2.4, $a_\mu = \underline{c}^T C_1^{-1} \underline{c}$, hence we reach the contraddiction, since $C$ is not 1-minimizable. ■

## 4. – Minimizability of excellent completely positive matrices.

A cycle of even length is a bipartite graph, hence it is a completely positive graph; therefore, in view of Proposition 3.2, a cyclic completely positive matrix of even order is totally minimizable.

Cyclic doubly nonnegative matrices of odd order which are not completely positive can be easily produced (see [3]). In [1] it is proved that a doubly nonnegative cyclic matrix $A$ of odd order is completely positive if and only if $\mathrm{Det}(A) \geqslant 4h_1 h_2 \dots h_n$, where the $h_i$'s are the non-zero elements over the main diagonal. It follows that such a matrix cannot be singular and consequently, given any $i \leqslant n$, it is not $i$-minimizable.

In this section we study a family of completely positive matrices, containing the cyclic ones, which provides more interesting examples with respect to minimizability.

In [1] doubly nonnegative matrices $A = (a_{ij})$ in bordered form (1) have been considered satisfying the following properties:

1) $a_{2,n} = a_{n,2} = 0$;

2) $\underline{b} = (\beta_2 \; 0 \; 0 \; \dots \; 0 \; \beta_n)^T$ with $\beta_2 \neq 0 \neq \beta_n$;

3) The graph $\Gamma(A_1)$ associated to $A_1$ is completely positive.

The graph $\Gamma(A)$ associated to the matrix $A$ is obtained from the graph $\Gamma(A_1)$ by adding one more vertex, connected with only two vertices of $\Gamma(A_1)$, which are not connected each other. We will say that such a graph is an *excellent graph*, and that a doubly nonnegative matrix $A$ satisfying the preceding properties is an *excellent matrix*.

In the preceding notation, let us denote by $A^-$ the matrix obtained from $A$ by substituting $\beta_n$ by $-\beta_n$, or, equivalently, substituting the vector $\underline{b}$ by the vector $\underline{b}_1 - \underline{b}_2$, where $\underline{b}_1 = (\beta_2 \; 0 \; 0 \; \dots \; 0)^T$ and $\underline{b}_2 = (0 \; 0 \; \dots \; 0 \; \beta_n)^T$. In [1], part of the following result was proved.

THEOREM 4.1. *Let $A$ be an excellent doubly nonnegative $n$-by-$n$ matrix with $n > 2$ in bordered form* (1), *with $\underline{b} = (\beta_2 \; 0 \; 0 \; \dots \; 0 \; \beta_n)^T$. The following facts are equivalent*:

1) *$A$ is completely positive*;

2) *$A^-$ is positive semidefinite*;

3) *The vector $(1 \; 0 \; \dots \; 0 \; 0)^T$ of $\mathbb{R}^{n-1}$ belongs to the column space $R(A_1)$ of $A_1$ and $a \geqslant a_\mu - 4\beta_2\beta_n\alpha$, where $\alpha$ is the element in the first row and last column of $A_1^+$.*

PROOF. For 1) $\Leftrightarrow$ 2) see [1]. For 2) $\Leftrightarrow$ 3), observe that, by Lemma 2.1, $A^-$ is positive semidefinite if and only if $\underline{b}_1 - \underline{b}_2 \in R(A_1)$ and $a \geqslant (\underline{b}_1^T - \underline{b}_2^T) A_1^+ (\underline{b}_1 - \underline{b}_2)$. But since $\underline{b} = \underline{b}_1 + \underline{b}_2 \in R(A_1)$, $\underline{b}_1 - \underline{b}_2 \in R(A_1)$ if and only if $\underline{b}_1 \in R(A_1)$ (or equivalently $\underline{b}_2 \in R(A_1)$), if and only if

$\underline{b} = (1\ 0\ \dots\ 0\ 0)^T \in R(A_1)$. Moreover, we have:

$$(\underline{b}_1^T - \underline{b}_2^T)\, A_1^+ (\underline{b}_1 - \underline{b}_2) = \underline{b}_1^T A_1^+ \underline{b}_1 + \underline{b}_2^T A_1^+ \underline{b}_2 - 2\, \underline{b}_2^T A_1^+ \underline{b}_1 =$$

$$= \underline{b}_1^T A_1^+ \underline{b}_1 + \underline{b}_2^T A_1^+ \underline{b}_2 + 2\, \underline{b}_2^T A_1^+ \underline{b}_1 - 4\, \underline{b}_2^T A_1^+ \underline{b}_1 = a_\mu - 4\beta_2 \beta_n \alpha\,. \qquad \blacksquare$$

We must remark that it is not possible to eliminate in Theorem 4.1 the hypothesis that $A$ is doubly nonnegative, since there are symmetric nonnegative matrices $A$ with $A_1$ positive semidefinite and $\underline{b} \in R(A_1)$, which fail to be positive semidefinite, and such that $A^-$ is positive semidefinite. This happens (by using the preceding notation and with $A$ in bordered form (2.1)) when $\alpha > 0$ and $\underline{b}^T A_1^+ \underline{b} > a > \underline{b}^T A_1^+ \underline{b} - 4\beta_2 \beta_n \alpha$. On the other hand, in the hypothsis of Theorema 4.1, if $\alpha \geqslant 0$, the condition $a \geqslant a_\mu - 4\beta_2\beta_n \alpha$ is automatically verified, since $a \geqslant a_\mu$. An immediate consequence of this remark is the following

COROLLARY 4.2. *Let $A$ be an excellent completely positive matrix od order $n > 1$ in bordered form* (1), *with $\underline{b} = (\beta_2\ 0\ 0\ \dots\ 0\ \beta_n)^T$, and let $\alpha$ be the element in the first row and last column of $A_1^+$. Then*

1) $a_e = \max(a_\mu, a_\mu - 4\beta_2\beta_n\alpha)$;

2) *$A$ is 1-minimizable if and only if $\alpha \geqslant 0$.*

PROOF. 1) is an immediate consequence of Theorem 4.1.

2) If $A$ is 1-minimizable, then $a_e = a_\mu$, hence $\alpha \geqslant 0$ by point 1). Conversely $\alpha \geqslant 0$ and point 1) imply $a_e = a_\mu$. $\blacksquare$

We give now some examples of excellent matrices with different behaviour with respect to 1-minimizability.

EXAMPLE 4.3. Consider the following symmetric matrix with excellent associated graph

$$A(a) = \begin{bmatrix} a & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 \\ 0 & 2 & 5 & 3 & 1 \\ 0 & 1 & 3 & 2 & 2 \\ 1 & 0 & 1 & 2 & 11 \end{bmatrix}. \tag{13}$$

The principal submatrix $A_1$ (obtained from the last four rows and columns) is singular and completely positive, and the vector $\underline{b} = (1\ 0\ 0\ 1)^T \in R(A_1)$, hence $A(a)$ is doubly nonnegative for $a \geqslant \underline{b}^T A_1^+ \underline{b}$. However, the matrix $A(a)$ is not completely positive for any value of $a$, since, according to Theorem 4 the vector $(1\ 0\ 0\ 0)^T$ does not belong to $R(A_1)$.

EXAMPLE 4.4. Consider the following symmetric matrix with excellent associated graph

$$(14) \qquad A(a) = \begin{bmatrix} a & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 & 2 \end{bmatrix}.$$

In this case the principal submatrix $A_1$ is non-singular and completely positive, so the vectors $\underline{b} = (1\ 0\ 0\ 1)^T$ and $\underline{b}_1 = (1\ 0\ 0\ 0)^T$ belong to $R(A_1)$, hence $A(a)$ is doubly nonnegative for $a \geqslant \underline{b}^T A_1^+ \underline{b} = 7$. Moreover, the element $\alpha$ in the first row and last column of $A_1^+$ equals 1, hence $A(a)$ is completely positive and 1-minimizable for all $a \geqslant 7$. If $7 > a \geqslant 3 = 7 - 4\beta_2 \beta_n \alpha$, $A$ is not doubly nonnegative and $A^-$ is positive semidefinite.

EXAMPLE 4.5. Consider the following symmetric matrix with excellent associated graph

$$(15) \qquad A(a) = \begin{bmatrix} a & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 3 & 6 \end{bmatrix}.$$

In this case $A_1$ is non-singular and completely positive, so $\underline{b} =$

$= (1\ 0\ 0\ 1)^T$ and $\underline{b}_1 = (1\ 0\ 0\ 0)^T$ belong to $R(A_1)$, hence $A(a)$ is doubly nonnegative for $a \geqslant \underline{b}^T A_1^+ \underline{b} = 3$. Moreover $\alpha = -1$, hence $A(a)$ is completely positive if and only if $a \geqslant a_\mu - 4\alpha = 7$ and one can conclude that $A$ is not 1-minimizable.

EXAMPLE 4.6. Consider the following completely positive cyclic matrix of order 5

$$C = \begin{bmatrix} 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 \end{bmatrix} \tag{16}$$

which is not $i$-minimizable for any $i$. A factorization of $C = VV^T$ is obtained for

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}. \tag{17}$$

Using Proposition 3.4, one can see that, setting $W^T = (V^T \underline{b})$ where $\underline{b} = (0\ 0\ 0\ 1\ 1)^T$, the following completely positive matrix

$$A = WW^T = \begin{bmatrix} 2 & 1 & 0 & 0 & 1 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 1 & 0 & 0 & 1 & 2 & 2 \end{bmatrix} \tag{18}$$

is singular, hence $i$-minimizable for some $i$, but it is not 1-minimizable.

## REFERENCES

[1] F. Barioli, *Completely positive matrices with a book-graph*, Linear Algebra and its Applications, **277** (1998), pp. 11-31.
[2] A. Berman - R. Grone, *Bipartite completely positive matrices*, Proc. Cambridge Phil. Soc., **103** (1988), pp. 269-276.
[3] A. Berman - D. Hershkowitz, *Combinatorial results on completely positive matrices*, Linear Algebra and its Applications, **95** (1987), pp. 111-125.
[4] M. Kaykobad, *On nonnegative factorization of matrices*, Linear Algebra and its Applications, **96** (1988), pp. 27-33.
[5] N. Kogan - A. Berman, *Characterization of completely positive graphs*, Discrete Math., **114** (1993), pp. 287-299.
[6] L. Salce, *Lezioni sulle Matrici*, Decibel-Zanichelli, Padova (1993).
[7] L. Salce - P. Zanardo, *Completely positive matrices and positivity of least square solutions*, Linear Algebra and its Applications, **178** (1993), pp. 201-216.

Manoscritto pervenuto in redazione il 9 maggio 1996.