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Extensions of Unbounded Topological Spaces.

ALESSANDRO CATERINO - STEFANO GUAZZONE (*)

Introduction.

A method of compactification of locally compact spaces has been proposed in [1]. This method is based on the concept of essential semilattice homomorphism (*ESH* for short). More precisely, let X be a locally compact (non-compact) Hausdorff space and K a compact Hausdorff space. Let \mathcal{B} be an (open) basis of K closed with respect to finite unions, and let \mathcal{N}_X be the family consisting of the empty set and the open subsets of X which are not relatively compact. A map $\pi: \mathcal{B} \rightarrow \mathcal{N}_X$, with $\pi(U) \neq \emptyset$ for every $U \neq \emptyset$, is an *ESH* if the following conditions hold:

ESH1) $X - \pi(K) \notin \mathcal{N}_X - \{\emptyset\}$;

ESH2) if $U, V \in \mathcal{B}$ then the symmetric difference

$$\pi(U \cup V) \Delta (\pi(U) \cup \pi(V)) \notin \mathcal{N}_X - \{\emptyset\};$$

ESH3) if $U, V \in \mathcal{B}$ and $\overline{U} \cap \overline{V} = \emptyset$ then $\pi(U) \cap \pi(V) \notin \mathcal{N}_X - \{\emptyset\}$.

If T_X is the topology of X and $S = \{U \cup (\pi(U) \setminus F) : U \in \mathcal{B}, F \subset X, F \text{ compact}\}$, then $T_X \cup S$ is a basis for a topology on the disjoint union $X \cup K$. This new space is a Hausdorff compactification of X with remainder K . It is denoted by $X \underset{\pi}{\cup} K$ and is called an *ESH*-compactification of X .

In this paper we present a natural generalization of the construction above. We say that a topological space X is locally bounded with respect to a family (of «bounded» sets) $\mathcal{F}_X \subset 2^X$ (which is closed under finite

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unions and subsets) if every point of X has a bounded neighborhood. We note that, if \mathcal{F}_X is the family of the relatively compact subsets of X (resp. the relatively Lindelöf subsets of a T_3 -space X), we have that local boundedness with respect to \mathcal{F}_X is equivalent to local compactness (resp. local Lindelöfness) of X .

We construct dense extensions of unbounded spaces, which we call B -extensions. By adding some requirements, mainly local boundedness of the space, we obtain Hausdorff B -extensions. This construction is obtained with a method similar to the one used to obtain ESH -compactifications. This method can be applied, for instance, to construct Lindelöf extensions of non-Lindelöf locally Lindelöf spaces. As a final remark, we mention that Theorem 2.3 of this paper appears to be a generalization of a Tkachuk's result (see [8], Proposition 1).

1. – B -extensions with respect to a boundedness.

An extension of a topological space will mean a dense extension.

We recall that a non-empty family \mathcal{F}_X of subsets of a space X is said to be a *boundedness* in X if \mathcal{F}_X is closed with respect to finite unions and subsets (see [5]). Elements of \mathcal{F}_X are called *bounded sets* of X . Every subset of X not in \mathcal{F}_X is called *unbounded*.

A space X with boundedness \mathcal{F}_X is said to be *locally bounded* if every point of X has a bounded neighborhood. If X is T_3 , this is equivalent to say that the family of the closed bounded neighborhoods of each point of X is a neighborhood base.

We remark that, for a given space X , the family $\mathcal{C}_X = \{A \subset X : \bar{A} \text{ is compact}\}$ is a boundedness in X , as well as $\mathcal{L}_X = \{A \subset X : \bar{A} \text{ is Lindelöf}\}$. Clearly, a space X is locally compact iff X is locally bounded with respect to \mathcal{C}_X .

A space X is said to be *locally Lindelöf* if every point of X has a Lindelöf neighborhood. If X is T_3 , it is equivalent to say that every point of X has a Lindelöf closed neighborhood (or to say that the family of the closed Lindelöf neighborhoods of every point of X is a neighborhood base). Hence, a T_3 -space is locally Lindelöf iff X is locally bounded with respect to \mathcal{L}_X .

In [8] Tkachuk defines a space X to be locally Lindelöf if every point of X has an open Lindelöf neighborhood. If X is $T_{3\frac{1}{2}}$, the two definitions are equivalent. In fact, if $x \in X$ and U is a Lindelöf neighborhood of x ,

then there exists $f \in C(X, [0, 1])$ such that $f(x) = 0$ and $f(X \setminus U) = 1$. Hence

$$x \in Z = f^{-1}(0) \subset W = f^{-1}([0, 1]) \subset U$$

and $f^{-1}([0, 1])$ is Lindelöf since it is an F_σ contained in a Lindelöf subspace.

We remark that Z can be chosen to be a zero-set neighborhood of the point x . In fact, it is sufficient to consider the map $g = (2f - 1) \vee 0$.

Therefore, a locally Lindelöf $T_{3\frac{1}{2}}$ -space X is locally bounded with respect to the boundedness

$$\mathcal{L}_X = \{A \subset X : A \subset f^{-1}(0), f^{-1}([0, 1]) \text{ is Lindelöf} \ \& \ f \in C(X, [0, 1])\}.$$

We note that, if X is locally bounded with respect to a boundedness \mathcal{F}_X , then $\mathcal{C}_X \subset \mathcal{F}_X$. If \mathcal{F}_X is also closed with respect to countable unions, then $\mathcal{L}_X \subset \mathcal{F}_X$ too.

If αX is an extension of X , then there is a natural boundedness in X associated to αX . In fact, if we define

$$\mathcal{X}_X(\alpha X) = \{A \subset X : Cl_X A = Cl_{\alpha X} A\},$$

then $\mathcal{X}_X(\alpha X)$ is a boundedness in X . We remark that if αX is T_3 and $\alpha X \setminus X$ is closed, or αX is T_2 and $\alpha X \setminus X$ is compact, then X is also locally bounded with respect to $\mathcal{X}_X(\alpha X)$.

Now, let X be an unbounded space with respect to \mathcal{F}_X and let \mathcal{N}_X be the collection consisting of the empty set and the unbounded open subsets of X . Let \mathcal{B} be a basis for the open subsets of a topological space Y , and assume that $Y \in \mathcal{B}$ and \mathcal{B} is closed with respect to finite unions.

We say that $\pi = \pi_{\mathcal{B}, \mathcal{F}_X} : \mathcal{B} \rightarrow \mathcal{N}_X$, with $\pi(U) \neq \emptyset$ for every $U \neq \emptyset$, is a B -map, if it satisfies the following conditions:

- B1) if $\{U_i\}_{i \in A} \subset \mathcal{B}$ is a cover of Y , then $X \setminus \bigcup_{i \in A} \pi(U_i) \in \mathcal{F}_X$;
- B2) if $U, V \in \mathcal{B}$ then

$$\overline{\pi(U \cup V) \Delta (\pi(U) \cup \pi(V))} \in \mathcal{F}_X;$$

- B3) if $U, V \in \mathcal{B}$ and $\overline{U} \cap \overline{V} = \emptyset$ then $\overline{\pi(U) \cap \pi(V)} \in \mathcal{F}_X$.

In the following, a B -map $\pi : \mathcal{B} \rightarrow \mathcal{N}_X$, with \mathcal{B} closed with respect to unions of cardinality $< \alpha$, will be also called an α - B map.

Now, a topological extension of X can be constructed by means of a B -map. If T_X is the topology of X and $S = \{U \cup (\pi(U) \setminus F) : U \in \mathcal{B}, F = \bar{F}, F \in \mathcal{F}_X\}$, then $T_X \cup S$ is a basis for a topology on the disjoint union $X \cup Y$. To prove this, it is sufficient to imitate the proof given in [1] (see p. 852).

The set $X \cup Y$, endowed with the topology generated by $T_X \cup S$, will be denoted by $X \underset{\pi}{\bigcup} Y$ and will be called a B -extension of X .

We observe that X is open in $X \underset{\pi}{\bigcup} Y$, and the topologies of the subspaces X, Y coincide with the original topologies. If $\emptyset \neq U \in \mathcal{B}$, then $\pi(U) \notin \mathcal{F}_X$. Hence $\pi(U) \setminus F \neq \emptyset$ for every $F \in \mathcal{F}_X$. It follows that X is dense in $X \underset{\pi}{\bigcup} Y$.

If X is a space with boundedness \mathcal{F}_X , we say that a continuous map $f: X \rightarrow Y$ is B -singular (with respect to \mathcal{F}_X) if, for every non-empty $U \in T_Y$, $f^{-1}(U)$ is unbounded in X . We note that, if $f: X \rightarrow Y$ is B -singular, then $\pi = f^{-1}: T_Y \rightarrow \mathcal{N}_X$ is a B -map.

If \mathcal{F}_X is a boundedness in X , then $\tilde{\mathcal{F}}_X = \{F \in \mathcal{F}_X : \bar{F} \in \mathcal{F}_X\}$ is also a boundedness and one has that $F \in \tilde{\mathcal{F}}_X$ iff $\bar{F} \in \mathcal{F}_X$.

A boundedness \mathcal{G}_X with the property that $F \in \mathcal{G}_X$ iff $\bar{F} \in \mathcal{G}_X$ (that is $\mathcal{G}_X = \tilde{\mathcal{G}}_X$) will be called a *closed boundedness*. Clearly, $\mathcal{C}_X, \mathcal{L}_X, \mathcal{Z}\mathcal{L}_X$ and $\mathcal{H}_X(aX)$ are closed boundednesses.

Now, if $\pi = \pi_{\mathcal{B}, \mathcal{F}_X}$ is a B -map, then $\tilde{\pi} = \pi_{\mathcal{B}, \tilde{\mathcal{F}}_X}$, defined by $\tilde{\pi}(U) = \pi(U)$ for every $U \in \mathcal{B}$, is also a B -map and we have that $X \underset{\pi}{\bigcup} Y = X \underset{\tilde{\pi}}{\bigcup} Y$. In fact, $\pi(U) \notin \mathcal{F}_X$ implies $\tilde{\pi}(U) \notin \tilde{\mathcal{F}}_X$ and the closed unbounded subsets with respect to \mathcal{F}_X are just the closed unbounded subsets with respect to $\tilde{\mathcal{F}}_X$. Hence, every B -extension of X can be considered as a B -extension with respect to a closed boundedness. Therefore, we can assume, without restriction from the stand point of B -extension, that every boundedness we consider is a closed boundedness.

Now, let aX be a B -extension of X . We show there is a maximal boundedness \mathcal{H}_X and a B -map $\pi' = \pi_{\mathcal{B}, \mathcal{H}_X}$ such that $aX = X \underset{\pi'}{\bigcup} Y$.

PROPOSITION 1.1. *Let $aX = X \underset{\pi}{\bigcup} Y$ be a B -extension of X , with B -map $\pi = \pi_{\mathcal{B}, \mathcal{F}_X}$. Then $\mathcal{F}_X \subset \mathcal{H}_X(aX)$ and there is a B -map $\pi' = \pi_{\mathcal{B}, \mathcal{H}_X(aX)}$ such that $aX = X \underset{\pi'}{\bigcup} Y$. If Y is compact, then $\mathcal{F}_X = \mathcal{H}_X(aX)$.*

PROOF. If $A \in \mathcal{F}_X$, then $Y \cup \pi(Y) \setminus \bar{A}$ is an open neighborhood of Y that does not meet A . Hence $Cl_X A = Cl_{aX} A$ and so $A \in \mathcal{H}_X(aX)$.

Now, we observe that if $\emptyset \neq U \in \mathcal{B}$ then $\pi(U) \notin \mathcal{D}_X(aX)$. Otherwise $U \cup \pi(U) \setminus Cl_{aX} \pi(U) = U \cup \pi(U) \setminus Cl_X \pi(U) = U$ would be a non-empty open subset of aX contained in $aX \setminus X$. Therefore, we can consider the B -map $\pi' = \pi_{\mathcal{B}, \mathcal{D}_X(aX)}$ defined by $\pi'(U) = \pi(U)$ for every $U \in \mathcal{B}$. Since $\mathcal{F}_X \subset \mathcal{D}_X(aX)$, then we have $T_{X \cup Y} \leq T_{X \cup Y}^{\pi'}$. On the other hand, every basic open of $X \cup Y$ of the form $U \cup \pi'(U) \setminus F = U \cup \pi(U) \setminus F$ is also open in $aX = X \cup Y^{\pi}$, because $F = Cl_X F = Cl_{aX} F$. Hence $T_{X \cup Y} = T_{X \cup Y}^{\pi}$.

Now, suppose that Y is compact and let $A \in \mathcal{D}_X(aX)$. For every $y \in Y$, there is a basic open $U_y \cup \pi(U_y) \setminus F_y$ containing y such that $(U_y \cup \pi(U_y) \setminus F_y) \cap A = \emptyset$. If $\{U_{y_1}, \dots, U_{y_n}\}$ is a finite subfamily of $\{U_y\}_{y \in Y}$ that covers Y , then

$$A \subset \left(X \setminus \bigcup_{i=1}^n \pi(U_{y_i}) \right) \cup F_{y_1} \cup \dots \cup F_{y_n} \in \mathcal{F}_X. \quad \blacksquare$$

We note that, if $\pi = \pi_{\mathcal{B}, \mathcal{F}_X}$ is a B -map and $X \cup Y$ is T_2 -compact, then $\mathcal{F}_X = \mathcal{D}_X(aX) = \mathcal{C}_X$. Moreover, if X is T_2 -locally compact and Y is T_2 -compact, then the B -maps (with respect to \mathcal{C}_X) are exactly the ESH 's as defined in [1]. In fact, let $\pi: \mathcal{B} \rightarrow \mathcal{N}_X$ be such that $X \setminus \pi(Y) \in \mathcal{C}_X$ and $\pi(U \cup V) \Delta (\pi(U) \cup \pi(V)) \in \mathcal{C}_X$ for every $U, V \in \mathcal{B}$. If $\mathcal{U} = \{U_i\}_{i \in A} \subset \mathcal{B}$ is a cover of Y and $\{U_{i_1}, \dots, U_{i_r}\}$ is a finite subcover of Y , then $X \setminus \pi(Y) = X \Delta \pi(Y) \in \mathcal{C}_X$ and $\pi(Y) \Delta \left(\bigcup_{k=1}^r \pi(U_{i_k}) \right) \in \mathcal{C}_X$ imply that $X \Delta \left(\bigcup_{k=1}^r \pi(U_{i_k}) \right) = X \setminus \left(\bigcup_{k=1}^r \pi(U_{i_k}) \right) \in \mathcal{C}_X$. Hence $X \setminus \bigcup_{i \in A} \pi(U_i) \in \mathcal{C}_X$ too.

If X is a space with boundedness \mathcal{F}_X , then the relation defined by $A \sim B$ iff $A \Delta B \in \mathcal{F}_X$ is an equivalence relation in 2^X . Finite unions and intersections are compatible with it. Moreover, if \mathcal{F}_X is closed under unions of cardinality γ , then one has that also unions of cardinality γ are compatible with \sim .

PROPOSITION 1.2 (see Prop. 1.1 in [2]). *Let X be a locally bounded space with respect to \mathcal{F}_X . If $\pi = \pi_{\mathcal{B}, \mathcal{F}_X}: \mathcal{B} \rightarrow \mathcal{N}_X$ is a B -map, then every map $\pi': \mathcal{B} \rightarrow \mathcal{N}_X$ such that $\pi(U) \Delta \pi'(U) \in \mathcal{F}_X$, for every $U \in \mathcal{B}$, is also a B -map such that $X \cup Y^{\pi} \cong X \cup Y^{\pi'}$.*

PROOF. It is easily seen that π' is a B -map such that $T_{X \cup Y} = T_{X \cup Y}^{\pi}$. \blacksquare

Now, we will see that the Hausdorff property of $X \bigcup_{\pi} Y$ is ensured under the assumption that X is locally bounded, namely that X is locally bounded with respect to a closed boundedness.

The proof of the following statement is straightforward (see the proof of Prop. 1 in [1]).

PROPOSITION 1.3. *Let X and Y be Hausdorff spaces. If X is locally bounded with respect to a (closed) boundedness \mathcal{F}_X and $\pi = \pi_{\mathcal{B}, \mathcal{F}_X}$ is a B -map, then $X \bigcup_{\pi} Y$ is a Hausdorff space containing X as a dense subspace.*

With an argument suggested by the last part of the proof of Prop. 1.1, one can also prove the following proposition.

PROPOSITION 1.4. *If $aX = X \bigcup_{\pi} Y$ is Hausdorff, with $\pi = \pi_{\mathcal{B}, \mathcal{F}_X}$, and Y is compact, then X is locally bounded with respect to \mathcal{F}_X .*

THEOREM 1.5. *Let aX be a T_4 -extension of X such that $aX \setminus X$ is closed and 0-dimensional. Then aX is a B -extension of X .*

PROOF. We denote by \mathcal{B} the basis of $Y = aX \setminus X$ consisting of the clopen subsets of Y . If $U \in \mathcal{B}$, by the normality of aX , we have that there is an open subset A of aX such that $A \cap (aX \setminus X) = U$ and $aX \setminus A$ is a neighborhood of $Y \setminus A$. In fact, U and $Y \setminus U$ are closed (and disjoint) in aX . For every $U \in \mathcal{B}$, choose such a set A . Observe that, if B is another subset of aX that satisfies the same conditions, then $(A \cap X) \Delta (B \cap X) \in \mathcal{C}_X(aX)$. Moreover, $(Cl_{aX}(A \cap X)) \setminus X = U$ implies that if $U \neq \emptyset$ then $A \cap X \notin \mathcal{C}_X(aX)$. Hence, we can define $\pi = \pi_{\mathcal{B}, \mathcal{C}_X(aX)}$ by setting $\pi(U) = A \cap X$. We note that X is locally bounded with respect to $\mathcal{C}_X(aX)$.

Now, we check the B -properties of π . First, suppose that $\mathcal{U} = \{U_i\}_{i \in I}$ is a cover of Y of members of \mathcal{B} . Since every $U_i \cup \pi(U_i)$ is open in aX , then $X \setminus \bigcup_{i \in I} \pi(U_i) = aX \setminus \left(\bigcup_{i \in I} U_i \cup \pi(U_i) \right)$ is closed in aX and so it belongs to $\mathcal{C}_X(aX)$.

Let $U, V \in \mathcal{B}$. If $O = (U \cup \pi(U)) \cup (V \cup \pi(V))$, then $O \cap (aX \setminus X) = U \cup V$ and $aX \setminus O$ is a neighborhood of $Y \setminus (U \cup V)$. Therefore,

$$(O \cap X) \Delta \pi(U \cup V) = (\pi(U) \cup \pi(V)) \Delta \pi(U \cup V) \in \mathcal{C}_X(aX).$$

Now, suppose $U, V \in \mathcal{B}$ and $\bar{U} \cap \bar{V} = \emptyset$, that is $U \cap V = \emptyset$. Then

$$\begin{aligned} Cl_{aX}(\pi(U) \cap \pi(V)) \setminus X &\subset (Cl_{aX} \pi(U)) \cap (Cl_{aX} \pi(V)) \setminus X = \\ &= (Cl_{aX} \pi(U)) \setminus X \cap (Cl_{aX} \pi(V)) \setminus X = U \cap V = \emptyset \end{aligned}$$

implies $Cl_{aX}(\pi(U) \cap \pi(V)) \subset X$ and so $\pi(U) \cap \pi(V) \in \mathcal{C}_X(aX)$.

Finally, we show that $aX = X \bigcup_{\pi} Y$. Obviously, $T_{X \bigcup Y} \leq T_{aX}$. Now, let T be an open subset of aX and suppose $y \in T \cap Y$. Choose $U_y \in \mathcal{B}$ such that $y \in U_y$ and $U_y \subset T$. Clearly, $\pi(U_y) \setminus T$ has no adherence points in T (that is open in aX). Also, $\pi(U_y) \setminus T$ has no adherence points in $Y \setminus T$. In fact,

$$(Cl_{aX}(\pi(U_y) \setminus T)) \setminus X \subset (Cl_{aX} \pi(U_y)) \setminus X = U_y \subset T.$$

Therefore, $F_y = \pi(U_y) \setminus T \in \mathcal{C}_X(aX)$ and we have $y \in U_y \cup \pi(U_y) \setminus \bar{F}_y \subset T$. Hence,

$$T = \bigcup_{y \in T \cap Y} (U_y \cup \pi(U_y) \setminus \bar{F}_y) \cup (T \cap X),$$

and so $T \in T_{X \bigcup Y}$. ■

2. – Lindelöf and other special extensions.

The proof of the next theorem is routine (see Proposition 1 in [1]).

THEOREM 2.1. *Let X be a locally bounded T_2 -space with respect to a (closed) boundedness $\mathcal{S}_X \subset \mathcal{L}_X$ (hence X is locally Lindelöf) and let Y be a Lindelöf T_2 -space. If $\pi = \pi_{\mathcal{B}, \mathcal{S}_X}$ is a B -map, then $X \bigcup_{\pi} Y$ is a Lindelöf T_2 -space.*

If X is a Hausdorff space, locally bounded with respect to \mathcal{C}_X , and Y is compact and T_2 , then $X \bigcup_{\pi} Y$ is a T_4 -space.

Now, we investigate regularity and normality of $X \bigcup_{\pi} Y$ when X is locally Lindelöf and Y is Lindelöf. Of course, we suppose that X and Y are T_3 -spaces.

THEOREM 2.2. *Let X be a locally bounded T_3 -space with respect to a (closed) boundedness $\mathcal{S}_X \subset \mathcal{L}_X$. Assume that every $F \in \mathcal{S}_X$ is contained in an open subset $A \in \mathcal{S}_X$. If Y is a Lindelöf T_3 -space and $\pi = \pi_{\mathcal{B}, \mathcal{S}_X}$ is an ω_1 - B map, then $X \bigcup_{\pi} Y$ is a T_4 -space.*

PROOF. In view of Proposition 1.3 and Theorem 2.1 we have only to show that $X \bigcup_{\pi} Y$ is regular.

Let I be a neighborhood of $x \in X$ in $X \bigcup_{\pi} Y$. If $U_1 = Cl_X U_1 \subset I \cap X$ and $U_2 = Cl_X U_2 \in \mathcal{G}_X$ are neighborhoods of x in X , then $V = U_1 \cap U_2 \subset I$ is a closed bounded neighborhood of x in X . Hence V is also a closed neighborhood of x in $X \bigcup_{\pi} Y$.

Now, let $x \in Y$ with $x \in U \cup \pi(U) \setminus F_1$. By hypotheses, there is an open subset A_1 of X such that $F_1 \subset A_1 \subset \overline{A_1}$ and $G_1 = \overline{A_1} \in \mathcal{G}_X$. By the regularity of Y , there exist $W, \overline{W}_1 \in \mathcal{B}$ such that $x \in W \subset \overline{W} \subset \overline{W}_1 \subset U$. If $y \in Y \setminus U \subset Y \setminus \overline{W}_1$, let $V_y \in \mathcal{B}$ be such that $x \in V_y \subset Y \setminus \overline{W}_1$. Since $Y \setminus U$ is closed and Y is Lindelöf, then the open cover $\{V_y\}_{y \in Y}$ of $Y \setminus U$ has a countable subcover $\{V_n\}_{n=1}^{\infty}$, and one has

$$Y \setminus U \subset V = \bigcup_{n=1}^{\infty} V_n \subset Y \setminus \overline{W}_1.$$

Now $U, V \in \mathcal{B}$ and $U \cup V = Y$ imply $F_2 = X \setminus (\pi(U) \cup \pi(V)) \in \mathcal{G}_X$. Denote by A_2 an open subset of X such that $F_2 \subset A_2 \subset \overline{A_2}$ and $G_2 = \overline{A_2} \in \mathcal{G}_X$.

Since $\overline{V} \cap \overline{W} = \emptyset$ we have that $\pi(V) \cap \pi(W) \in \mathcal{G}_X$. If we set $G_3 = \overline{\pi(V) \cap \pi(W)}$ then one has $\pi(V) \cap (\pi(W) \setminus G_3) = \emptyset$. Now, we claim that

$$\overline{W \cup \pi(W) \setminus (G_1 \cup G_2 \cup G_3)} \subset U \cup \pi(U) \setminus F_1.$$

Suppose $z \notin U \cup \pi(U) \setminus F_1$ and $z \in X$. Then

$$z \in \pi(V) \cup F_1 \cup F_2 \subset \pi(V) \cup A_1 \cup A_2 \subset \pi(V) \cup G_1 \cup G_2,$$

and $\pi(V) \cup A_1 \cup A_2$ is an open neighborhood of z that does not meet $W \cup \pi(W) \setminus (G_1 \cup G_2 \cup G_3)$.

Finally, let $z \notin U \cup \pi(U) \setminus F_1$ and $z \in Y$. Then $z \in V$ and $V \cup \pi(V)$ is an open neighborhood of z that does not meet $W \cup \pi(W) \setminus G_3$. ■

By the way, we note that, in the previous theorem, if \mathcal{G}_X is closed under countable unions, it is sufficient to assume that π is a B -map (in place of ω_1 - B map).

Let us see the theory at work in the following example. It also shows that Lindelöf extensions of locally Lindelöf spaces are not always compactifications.

EXAMPLE 2.3. Let E be an uncountable set, viewed as a discrete space, and consider $X = S \times E$, $f: S \times E \rightarrow S$ the canonical projection,

with S the Sorgenfrey line. Then X is a non-Lindelöf locally Lindelöf space, which has a Lindelöf non-compact B -extension $\alpha X = X \bigcup_{\pi} S$, with respect to the B -map

$$\pi = f^{-1}: T_S \rightarrow \mathcal{N}_X = \{\emptyset\} \cup \{M \subset X: M \text{ is not relatively Lindelöf}\}.$$

In fact, for every $U \in T_S \setminus \{\emptyset\}$, $f^{-1}(U) = U \times E$ has a closure $\overline{U} \times E = \bigcup \{\overline{U} \times q: q \in E\}$, namely $\overline{U} \times E$ has a partition in an uncountable family of non-empty open sets. Then $f^{-1}(U)$ is not relatively Lindelöf and f is B -singular. By definition S is closed as a subset of αX , therefore αX is not compact. On the other hand, X is obviously $T_{3\frac{1}{2}}$, therefore has T_2 -compactifications, and in each of them the remainder is not closed, since X is not locally compact. From Theorem 2.2, we know that $X \bigcup_{\pi} Y$ is a T_4 -space.

If a space X is $T_{3\frac{1}{2}}$, we have already observed that X is locally bounded with respect to $\mathcal{Z}\mathcal{L}_X$. Since for every $F \in \mathcal{Z}\mathcal{L}_X$ there is an open $A \in \mathcal{Z}\mathcal{L}_X$ containing F , we have the following result.

THEOREM 2.3. *Let X be a locally Lindelöf $T_{3\frac{1}{2}}$ -space and let Y be a Lindelöf T_3 -space. If $\pi = \pi_{\beta, \mathcal{Z}\mathcal{L}_X}$ is an ω_1 - B map then $X \bigcup_{\pi} Y$ is a Lindelöf T_4 -space.*

In the previous theorem, if X is a locally Lindelöf T_4 -space, $\mathcal{Z}\mathcal{L}_X$ can be replaced by \mathcal{L}_X . In fact, we have the following proposition.

PROPOSITION 2.4. *If X is a locally Lindelöf T_4 -space then $\mathcal{Z}\mathcal{L}_X = \mathcal{L}_X$.*

PROOF. Let $F \in \mathcal{L}_X$. For every $y \in \overline{F}$, let U_y be an open neighborhood of y with Lindelöf closure. The open cover $\{U_y\}_{y \in \overline{F}}$ of \overline{F} has a countable subcover $\{U_n\}_{n=1}^{\infty}$. Then the open subset $U = \bigcup_{n=1}^{\infty} U_n$ of X contains \overline{F} and is contained in $U' = \bigcup_{n=1}^{\infty} \overline{U_n}$, that is Lindelöf. Now, by the normality of X , there is $f \in C(X, [0, 1])$ such that

$$\overline{F} \subset f^{-1}(0) \subset f^{-1}([0, 1]) \subset U \subset U'.$$

Then $f^{-1}([0, 1])$ is Lindelöf because it is a cozero-set contained in the Lindelöf set U' . ■

Now, let X be non-Lindelöf locally Lindelöf, and $Y = \{\infty\}$. If X is $T_{3\frac{1}{2}}$, let $\pi = \pi_{T_Y, \mathcal{Z}\mathcal{L}_X}$ be the map defined by $\pi(\infty) = X$. Then $X \bigcup_{\pi} \{\infty\}$, that

is T_4 , is just the «single-point Lindelöfication» considered by Tkachuk in [8].

If X is T_3 , but not $T_{3\frac{1}{2}}$, and $\pi = \pi_{T_Y, \mathcal{L}_X}$ is defined by $\pi(\infty) = X$ then $X \bigcup_{\pi} \{\infty\}$ is a Lindelöf T_2 -extension of X , that is not T_4 .

We observe that, by Theorem 2.2, it follows that a T_3 -space X is locally Lindelöf and $T_{3\frac{1}{2}}$ if and only if X is locally bounded with respect to $\mathcal{Z}\mathcal{L}_X$.

If aX is a T_2 -extension of a space X with $|aX \setminus X| = n$, then X is locally bounded with respect to $\mathcal{H}_X(aX)$. If aX is also Lindelöf, then $\mathcal{H}_X(aX) \subset \mathcal{L}_X$ and so X is locally Lindelöf. Now, we show that aX is a B -extension of X . This result could be proved in a way similar to that of Theorem 1.5. Here, a slightly different proof is presented, which will be useful to prove the next theorem.

PROPOSITION 2.5. *Let aX be a T_2 -extension of a space X with $|aX \setminus X| = n$. Then X is locally bounded with respect to $\mathcal{H}_X(aX)$ and aX is a B -extension of X .*

PROOF. Let $Y = aX \setminus X = \{y_1, \dots, y_n\}$ and let U_1, \dots, U_n be mutually disjoint open neighborhoods of $\{y_1, \dots, y_n\}$ in aX . Since $U_i \cap X \notin \mathcal{H}_X(aX)$ for every i , we can define $\pi = \pi_{T_Y, \mathcal{H}_X(aX)}$ by setting

$$\pi(\{y_{i_1}, \dots, y_{i_r}\}) = (U_{i_1} \cap X) \cup \dots \cup (U_{i_r} \cap X).$$

It is easily seen that π is a B -map. Now, we show that $X \bigcup_{\pi} Y = aX$. Clearly $T_{X \bigcup_{\pi} Y} \leq T_{aX}$. Conversely, let A be an open subset of aX . If $A \subset X$, then A is obviously an open subset of $X \bigcup_{\pi} Y$. Suppose $y_k \in A$ for some k . Since $y_i \notin Cl_{aX}(U_k \setminus A)$ for every $i = 1, \dots, n$, we get $F_k = Cl_X(U_k \setminus A) = Cl_{aX}(U_k \setminus A) \in \mathcal{H}_X(aX)$. Hence, if $A \cap (aX \setminus X) = \{y_{k_1}, \dots, y_{k_s}\}$, then we have

$$A = \left(\bigcup_{j=1}^s (\{y_{k_j}\} \cup \pi(\{y_{k_j}\}) \setminus F_{k_j}) \right) \cup (X \cap A),$$

and A is open in $X \bigcup_{\pi} Y$. ■

Now, we characterize the spaces that have a T_2 -extension with finite remainder (compare [6]).

THEOREM 2.6. *A Hausdorff space X has a T_2 -extension aX with $|aX \setminus X| = n$ iff X is locally bounded with respect to a boundedness \mathcal{F}_X and there exist n mutually disjoint unbounded open sets $A_1, \dots, A_n \subset X$ and a bounded set $F \subset X$ such that*

$$X = A_1 \cup \dots \cup A_n \cup F.$$

PROOF. The proof of the above proposition shows that the condition is necessary. Conversely, suppose that the condition holds. If y_1, \dots, y_n are n points not in X and $Y = \{y_1, \dots, y_n\}$, we define $\pi = \pi_{\tau_Y, \mathcal{F}_X}$ by $\pi(\{y_{i_1}, \dots, y_{i_r}\}) = A_{i_1} \cup \dots \cup A_{i_r}$. Then π is a B -map and $aX = X \bigcup_{\pi} Y$ is a T_2 -extension with $|aX \setminus X| = n$. ■

From Prop. 1.1, it follows that in a Hausdorff space X , the existence of a boundedness \mathcal{F}_X with the properties as in in Theorem 2.6, implies that $\mathcal{F}_X = \mathcal{C}_X(aX)$ for a suitable T_2 -extension aX of X .

A consequence of Theorem 2.6 and Theorem 2.1 is the following.

COROLLARY 2.7. *A Hausdorff space X has a Lindelöf T_2 -extension aX with $|aX \setminus X| = n$ iff X is locally bounded with respect to a boundedness $\mathcal{G}_X \subset \mathcal{L}_X$ and there exist n mutually disjoint unbounded open sets $A_1, \dots, A_n \subset X$ and a bounded set $F \subset X$ such that*

$$X = A_1 \cup \dots \cup A_n \cup F.$$

Now, we give an example of a Lindelöf T_2 -extension of a locally Lindelöf space X that cannot be obtained as a B -extension with respect to \mathcal{L}_X .

EXAMPLE 2.8. Let E be a discrete non countable space, and let $E = S \cup T$ be a partition with S countably infinite. Consider further a two points set $Y = \{a, b\}$ disjoint from E , $\tilde{S} = S \cup \{a\}$ the one-point compactification of S , $\tilde{T} = T \cup \{b\}$ the Lindelöf one-point B -extension of T , with respect to $\pi = \pi_{\{b\}, \mathcal{L}_T}$ defined by $\pi(\{b\}) = T$. Then, the topological sum $L = \tilde{S} + \tilde{T}$ is T_2 Lindelöf and E is dense in L . But L , that is a B -extension of E with respect to the boundedness $\mathcal{C}_E(L)$, cannot be obtained as a B -extension of E with respect to the «Lindelöf» boundedness \mathcal{L}_E . In fact, $\mathcal{C}_E(L) \neq \mathcal{L}_E$, and, from Prop. 1.1, we have that if $\pi' = \pi_{\beta, \mathcal{F}_E}$ is a B -map such that $L = E \bigcup_{\pi'} Y$ then $\mathcal{F}_E = \mathcal{C}_E(L)$.

We recall that a space X is said to be $[\theta, \kappa]$ -compact if every open cover of X of cardinality $\leq \kappa$ has a subcover of cardinality $< \theta$. If $\theta = \omega$, then X is said to be initially κ -compact, and if $|X| \leq \kappa$, then X is said to be finally θ -compact. Lindelöf spaces are exactly the finally ω_1 -compact spaces.

For a space X , $\mathcal{C}_X(\theta, \kappa) = \{A \subset X: \bar{A} \text{ is } [\theta, \kappa]\text{-compact}\}$ is a boundedness in X . We say that a space X is locally $[\theta, \kappa]$ -compact if every point of X has a $[\theta, \kappa]$ -compact neighborhood. As in the Lindelöf case, if X is T_3 , then X is locally $[\theta, \kappa]$ -compact if and only if X is locally bounded with respect to $\mathcal{C}_X(\theta, \kappa)$.

Proposition 2.1 can be easily generalized to $[\theta, \kappa]$ -compact case.

PROPOSITION 2.9. *Let X be a locally bounded T_2 -space with respect to a (closed) boundedness $\mathcal{S}_X \subset \mathcal{C}_X(\theta, \kappa)$ (hence X is locally $[\theta, \kappa]$ -compact) and let Y be a $[\theta, \kappa]$ -compact T_2 -space. If $\pi = \pi_{\mathcal{B}, \mathcal{S}_X}$ is a B -map, then $X \bigcup_{\pi} Y$ is a $[\theta, \kappa]$ -compact T_2 -space.*

Theorem 2.2 can be generalized to finally θ -compact case in the following way.

THEOREM 2.10. *Let X be a locally bounded T_3 -space with respect to a (closed) boundedness $\mathcal{S}_X \subset \mathcal{C}_X(\theta, \kappa)$, with $|X| \leq \kappa$. Assume that every $F \in \mathcal{S}_X$ is contained in an open subset $A \in \mathcal{S}_X$. If Y is T_3 and $\pi = \pi_{\mathcal{B}, \mathcal{S}_X}$ is a θ - B map, then $X \bigcup_{\pi} Y$ is a T_3 -space.*

Finally, Theorem 2.3 and Corollary 2.7 have similar generalizations to the finally θ -compact case.

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