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Convolution in $(W^p_{M,a})'$-Space.

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ABSTRACT - A characterization of convolutors in $(W^p_{M,a})'$-space is given using the properties of the translate τ_h: $W^p_{M,a} \to W^p_{M,a}$. Using the theory of Fourier transform in these spaces, the Fourier transform of convolution is studied.

1. - Introduction.

A characterization of convolution operators on the $K\{M_p\}$ space was given by Swartz [9] generalizing the characterizations of the space O_c' of Schwartz [8] and of convolutors on the spaces of distributions of exponential growth by Hausmi [5]. This chracterization naturally yields a characterization for $W_{M,a}$-space, which is a special case of $K\{M_p\}$ space. A similar characterization of convolution operators in K_p' was given by Sampson and Zielezny [10]. All these results are related to L^∞-norms.

In terms of L^p norms the spaces W^p_M, $W^p_{M,a}$, $W^{\Omega,p}$, $W^{\Omega,b,p}$ were defined and their Fourier transforms were studied in [6]. We recall the definition of the spaces W^p_M, $W^p_{M,a}$, $W^{\Omega,p}$, $W^{\Omega,b,p}$. Let $\mu(\xi)$ be a continuous increasing function on $[0, \infty]$ such that $\mu(0)=0$, $\mu(\infty)=\infty$, and for $x \geq 0$ define an increasing convex continuous function M by

$$M(x) = \int_0^x \mu(\xi)\, d\xi, \qquad M(-x) = M(x).$$

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Then $M(0) = 0$, $M(\infty) = \infty$, and

$$M(x_1 + x_2) \leqslant M(x_1) + M(x_2). \tag{1}$$

Now the space $W_M^p(\mathbb{R})$ is defined as the set of all infinitely differentiable functions $\Phi(x)$ $(-\infty < x < \infty)$ satisfying

$$\left(\int_{-\infty}^{+\infty} |\exp[M(ax)]\, \Phi^{(k)}(x)|^p\, dx \right)^{1/p} \leqslant C_{k,p}, \qquad 1 \leqslant p < \infty \tag{2}$$

for each non-negative integer k where the positive constants a and $C_{k,p}$ depend upon Φ. Clearly W_M^p is a linear space. The space W_M^p can be regarded as the union of countably normed spaces $W_{M,a}^p$ of all complex valued C^∞-functions Φ which for any $\delta > 0$ satisfy

$$\left(\int_{-\infty}^{+\infty} |\exp[M(a-\delta)x]\, \Phi^{(k)}(x)|^p\, dx \right)^{1/p} \leqslant C_{k,\delta,p}, \qquad k = 0, 1, 2 \dots . \tag{3}$$

Let Ω be another increasing, continuous, convex function possessing properties similar to those of M. Then $W^{\Omega,p}$ is defined to be the set of all entire analytic functions $\Phi(z)$ $(z = x + iy)$ satisfying the inequalities

$$\left(\int_{-\infty}^{+\infty} |\exp[-\Omega(by)]\, z^k\, \Phi(z)|^p\, dx \right)^{1/p} \leqslant C_{k,p}, \qquad k = 0, 1, 2 \dots . \tag{4}$$

The space $W^{\Omega,b,p}$ is defined to be set of all those functions in $W^{\Omega,p}$ which satisfy the inequalities

$$\left(\int_{-\infty}^{+\infty} |\exp[-\Omega[(b+\varrho)y]]\, z^k\, \Phi(z)|^p\, dx \right)^{1/p} \leqslant C_{k,\varrho,p}. \tag{5}$$

In this paper the translate $\tau_h\colon W_{M,a}^p \to W_{M,a}^p$ defined by $\tau_h[\Phi(x)] = \Phi(x+h)$, $h \in \mathbb{R}$, is shown to be continuous, bounded and differentiable. A characterization of convolutors in $W_{M,a}^p$ is given. Furthermore, by using the theory of Fourier transform of $f \in (W_{M,a}^p)'$, $g \in (W_{M,a}^q)'$, $1/p + 1/q = 1$, we show that

$$F(f * g) = F(f) \cdot F(g) \text{ in } (W^{\Omega, 1/a, p})' .$$

2. – Characterization theorems.

THEOREM 2.1.

(i) *For each* $h \in \mathbb{R}$ *the function* $\Phi \to \tau_h \Phi$ *is continuous from* $W^p_{M,a}$ *into* $W^p_{M,a}$.

(ii) *For a bounded subset* A *of* $W^p_{M,a}$ *and* $\varepsilon > 0$, *the set* $\{\tau_h \Phi: |h| \leq \varepsilon, \Phi \in A\}$ *is bounded in* $W^p_{M,a}$.

PROOF. For $\Phi \in W^p_{M,a}$ and $h \in \mathbb{R}$, we have

$$\|\tau_h \Phi\|_p = \left(\int_{-\infty}^{+\infty} |\exp[M[(a-\delta)x]]\, \tau_h[\Phi^{(k)}(x)]|^p \, dx \right)^{1/p} =$$

$$= \left(\int_{-\infty}^{+\infty} |\exp[M[(a-\delta)x]]\, \Phi^{(k)}(x+h)|^p \, dx \right)^{1/p} =$$

$$= \left(\int_{-\infty}^{+\infty} |\exp[M[(a-\delta)x] - M[(a-\delta)(x+h)]] \times \right.$$

$$\left. \times \exp[M[(a-\delta)(x+h)]]\, \Phi^{(k)}(x+h)|^p \, dx \right)^{1/p}.$$

Now, using the convexity property (1), we get

$$(6) \quad \|\tau_h \Phi\|_p \leq$$

$$\leq \exp[-M[(a-\delta)h]] \left(\int_{-\infty}^{+\infty} |\exp[M[(a-\delta)(x+h)]]\, \Phi^{(k)}(x+h)|^p \, dx \right)^{1/p}$$

so that (i) and (ii) follow from inequality (6).

THEOREM 2.2. *For each* $\Phi \in W^p_{M,a}$ *the translate* $\tau_h \Phi$ *is differentiable in* $W^p_{M,a}$, $p \geq 1$.

PROOF. From [6, p. 734] we know that a function $\Phi \in W^p_{M,a}$ is differentiable in $W^p_{M,a}$ space. Since $\tau_h \Phi \in W^p_{M,a}$, it follows that $\tau_h \Phi$ is differentiable in $W^p_{M,a}$.

Now, we recall the definition of a convolute [3, p. 137].

DEFINITION 2.3. Let V be any test function space and V' be its dual. A generalized function $f \in V'$ is said to be a convolute if for each $\Phi \in V, f * \Phi \in V$, and $\Phi_\nu \to 0$ implies that $f * \Phi_\nu \to 0$ in the topology of V. If f is a convolute and $g \in V$, the convolution of f and g is given by

$$\langle f * g, \Phi \rangle = \langle g, f * \Phi \rangle .$$

THEOREM 2.4. *Let* $f \in (W^p_{M,a})'$ *and* $\Phi \in W^p_{M,a}$ *then* $f * \Phi \in W^r_{M,b}$, *where* $p, r \geqslant 1$ *and* $b > a > 0$.

PROOF. From [6, p. 734] we have

$$f = \sum_{j=0}^{n} D^{(j)}[\exp[M[(a-\delta)t]] f_j(t)], \qquad f_j \in L^q .$$

Therefore for $\Phi \in W^p_{M,a}$ we have

$$|(f * \Phi)(x)| = \left| \int_{-\infty}^{+\infty} f(t)\, \Phi(x+t)\, dt \right| \leqslant \int_{-\infty}^{+\infty} |f(t)\, \Phi(x+t)|\, dt \leqslant$$

$$\leqslant \int_{-\infty}^{+\infty} \left| \sum_{j=0}^{n} D^{(j)}[\exp[M[(a-\delta)t]] f_j(t)]\, \Phi(x+t) \right| dt \leqslant$$

$$\leqslant \int_{-\infty}^{+\infty} \left| \sum_{j=0}^{n} (-1)^j [\exp[M[(a-\delta)t]] f_j(t)]\, D_t^{(j)} \Phi(x+t) \right| dt \leqslant$$

$$\leqslant \sum_{j=0}^{n} \int_{-\infty}^{+\infty} |\exp[(a-\delta)t)] f_j(t)\, D_t^{(j)} \Phi(x+t)|\, dt .$$

So that for $1/p + 1/q = 1$, we have

$$|D_x^{(\beta)}(f * \Phi)(x)| \leqslant \sum_{j=0}^{n} \int_{-\infty}^{+\infty} |f_j(t) \cdot \exp[M[(a-\delta)t]] D_{x+t}^{(\beta+j)} \Phi(x+t)|\, dt \leqslant$$

$$\leqslant \sum_{j=0}^{n} \|f_j(t)\|_q \cdot \|\exp[M[(a-\delta)t]]\, D_{x+t}^{(\beta+j)} \Phi(x+t)\|_p \leqslant$$

$$\leqslant \sum_{j=0}^{n} \|f_j(t)\|_q \|\exp[M[-(a-\delta)(x+t)] + M[(a-\delta)t]] \cdot$$

$$\cdot \exp[M[(a-\delta)(x+t)]]\, D_{x+t}^{(\beta+j)} \Phi(x+t)\|_p \leqslant$$

$$\leqslant \exp[-M[a-\delta)x]] \sum_{j=0}^{n} \|f_j\|_q \|\exp[M[(a-\delta)(x+t)]] D_{x+t}^{(\beta+j)} \Phi(x+t)\|_p \leqslant$$

$$\leqslant \exp[-M[(a-\delta)x]] C_{q,n} \|\Phi\|_{\delta,p} .$$

Therefore, for $(b > a > 0)$, we have

$$\|\exp[M[(b-\delta)x]] D_x^{(\beta)}(f*\Phi)(x)\| \leqslant C_{q,n} \|\Phi\|_{\delta,p} \exp[-M[(b-a)x]] .$$

Hence for $r \geqslant 1$,

$$\|\exp[M[(b-\delta)x]] D_x^{(\beta)}(f*\Phi)(x)\|_r \leqslant C_{q,n} \|\Phi\|_{\delta,p} \|\exp - [-M[(b-a)x]]\|_r .$$

In particular, taking $r = p$ we have $f*\Phi \in W^p_{M,b}$, $b > a$. Therefore f is a convolute in $(W^p_M)'$.

THEOREM 2.5. *Assume that $b > a > 0$. Then $W^p_{M,b}$ is a dense subspace of $W^p_{M,a}$ for $1 \leqslant p < \infty$.*

PROOF. Let $u \in W^p_{M,a}$ and $\Phi \in D(\mathbb{R})$ such that $\Phi(x) \geqslant 0$, $\Phi(x) = 1$ for $|x| < 1$ and $\Phi(x) = 0$ for $|x| \geqslant 2$. Define $\Phi_\nu(x) = \Phi(x/\nu)$, $\nu \in \in \mathbb{N}$.

Set $u_\nu = \Phi_\nu \cdot u$. Then $u_\nu \in D(\mathbb{R})$. It can be easily seen that $u_\nu \to u$ in $W^p_{M,a}$. Therefore D is dense in $W^p_{M,a}$. Since $D \subset (W^p_{M,b})$, it follows that $W^p_{M,b}$ is dense in $W^p_{M,a}$. Consequently, $(W^p_{M,a})' \subset (W^p_{M,b})'$.

3. – Fourier transform.

THEOREM 3.1. *If $f \in (W^p_{M,a})'$, $g \in (W^q_{M,a})'$, $1/p + 1/q = 1$ then $f*g \in (W_{M,b})'$, $a \leqslant b$ and $F(f*g) = F(f) \cdot F(g)$ in $W^{(\Omega, 1/b)})'$.*

PROOF. From [6, p. 734] we again have

$$f = \sum_{j=0}^{n} D^j [\exp[M[(a-\delta)u]] f_j(u)], \qquad f_j \in L^q \tag{7}$$

and

$$g = \sum_{k=0}^{l} D^k [\exp[M[(a-\delta)u]] g_k(u)], \qquad g_k \in L^p . \tag{8}$$

Now,

$$[(\exp[M[(a-\delta)u]]f_j(u))*(\exp[M[(a-\delta)u]]g_k(u)](t)=$$

$$=\int_{-\infty}^{+\infty}\exp[M[(a-\delta)(t-u)]]f_j(t-u)\exp[M[(a-\delta)u]]g_k(u)\,du\leqslant$$

$$\leqslant\int_{-\infty}^{+\infty}\exp[M[(a-\delta)t]]f_j(t-u)g_k(u)\,du$$

which is known to be an element in L^r, $1/r=1/p+1/q-1$.

Since

$$L^r\subset(W_{M,b})',\ (\exp[M[(a-\delta)u]]f_j(u))*(\exp[M[(a-\delta)u]]g_k(u))$$

is an element of $(W_{M,b})'$, $(a\leqslant b)$. Also, since $(W_{M,b})'$ is closed with respect to differentiation, hence the distributional derivative $D^{j+k}[\exp[M[(a-\delta)u]]f_j(u)*\exp[M[(a-\delta)u]]g_k(u)]$ is also an element of $(W_{M,b})'$. Furthermore $f*g\in(W_{M,b})'$ implies that $F(f*g)\in\in(W^{\Omega,1/b})'$ by Gel'fand and Shilov [4].

Now, let $\Phi\in W^{\Omega,1/b}$. Then,

$$\langle F(f*g)(x),\Phi(x)\rangle=\langle(f*g)(u),F[\Phi](u)\rangle=$$

$$=\left\langle\sum_{j=0}^{n}\sum_{k=0}^{l}D^{j+k}[\exp[M[(a-\delta)u]]f_j(u)]*\right.$$

$$\left.*[\exp[M[(a-\delta)u]]g_k(u)],F[\Phi](u)\right\rangle=$$

$$=\left\langle\sum_{j=0}^{n}\sum_{k=0}^{l}(-1)^{j+k}[\exp[M[(a-\delta)u]]f_j(u)]*\right.$$

$$\left.*\exp[M[(a-\delta)u]]g_k(u),D^{j+k}F[\Phi](u)\right\rangle=$$

$$=\left\langle\sum_{j=0}^{n}\sum_{k=0}^{l}(-1)^{j+k}[\exp[M[(a-\delta)u]]f_j(u)]*\right.$$

$$\left.*\exp[M[(a-\delta)u]]g_k(u),F[(-i)^{j+k}x^{j+k}\Phi](u)\right\rangle=$$

$$= \left\langle \sum_{j=0}^{n} \sum_{k=0}^{l} (i)^{j+k} x^{j+k} F[\exp[M[(a-\delta)u]] f_j] \cdot \right.$$

$$\left. \cdot F[\exp[M[(a-\delta)]u] g_k], \Phi(x) \right\rangle =$$

$$= \left\langle \sum_{j=0}^{n} (i)^j x^j F[\exp[M[(a-\delta)u]] f_j] \cdot \right.$$

$$\left. \cdot \sum_{k=0}^{l} (i)^k x^k F[\exp[M[(a-\delta)u]] g_k], \Phi(x) \right\rangle =$$

$$= \left\langle F\left[\sum_{j=0}^{n} D^j [\exp[M[(a-\delta)u]] f_j(u)] \right] \cdot \right.$$

$$\left. \cdot F\left[\sum_{j=0}^{l} D^k [\exp[M[(a-\delta)u]] g_k(u)] \right], \Phi(x) \right\rangle = \langle F(f) \cdot F(g), \Phi \rangle .$$

DEFINITION 3.1. $f \in (W^p_{M,a})'$ is said to belong $(O^p_c)' \subset (W^p_{M,a})'$ if for all $g \in (W^p_{M,a})'$, $f * g \in (W^p_{M,a})'$.

THEOREM 3.2. *If $f \in (O^p_c)'$ and $g \in (W^p_{M,a})'$, then $F(f*g) = F(f) \cdot$ $\cdot F(g)$ in the sense of equality in $(W^{\Omega, 1/a, p})'$.*

PROOF. Let $\Phi \in W^{\Omega, 1/a, p}$, then we have

$$\langle F(f*g)(x), \Phi(x) \rangle = \langle (f*g)(t), F[\Phi](t) \rangle = \langle f(x), \langle g(t), F[\Phi](x+t) \rangle \rangle .$$

Since $f \in (O^p_c)' \subset (W^p_{M,a})' \subset (W^p_{M,b})'$, and $\langle g(t), F[\Phi](x+t) \rangle$ belongs to $W^p_{M,b}$ by Theorem 2.4, then right-hand side is meaningful. Now, using (8) we have

$$\langle g(t), F[\Phi](x+t) \rangle = \left\langle \sum_{j=0}^{n} D^j [\exp[M[(a-\delta)t]] g_j(t)], F[\Phi](x+t) \right\rangle =$$

$$= \left\langle \sum_{j=0}^{n} [(-1)^j \exp[M[(a-\delta)t]] g_j(t)], D^j F[\Phi](x+t) \right\rangle =$$

$$= \left\langle \sum_{j=0}^{n} (-1)^j \exp[M[(a-\delta)(u-x)]] g_j(u-x), D^j_u F[\Phi](u) \right\rangle =$$

$$= \left\langle \sum_{j=0}^{n} \exp[M[(a-\delta)(u-x)]]\, g_j(u-x),\ F[(i)^j y^j \Phi](u) \right\rangle =$$

$$= \left\langle \sum_{j=0}^{n} (i)^j \exp[M[(a-\delta)(u-x)]]\, g_j(u-x),\ F[y^j \Phi](u) \right\rangle =$$

$$= \sum_{j=0}^{n} (i)^j (\Psi * F[(y^j \Phi)])(x),$$

where $\Psi = \exp[M[(a-\delta)(u-x)]\, g_j(u-x)]$. Then the last expression equals

$$\sum_{j=0}^{n} (2\pi)^{-n} (i)^j (F(\check{F}[\Psi])) * F[y^j \Phi](x) = \sum_{j=0}^{n} (2\pi)^{-n} (i)^j F[\check{F}(\Psi) \cdot y^j \Phi](x).$$

Therefore,

$$\langle F(g*f), \Phi \rangle = \langle f(x), \langle g(t), F[\Phi](x+t) \rangle \rangle =$$

$$= \left\langle f(x), \sum_{j=0}^{n} (2\pi)^{-n} F(\check{F}[\Psi] \cdot (i)^j y^j \Phi)(x) \right\rangle =$$

$$= \left\langle F(f), \sum_{j=0}^{n} (i)^j (2\pi)^{-n} \check{F}[\Psi]\, y^j \Phi \right\rangle =$$

$$= \left\langle F(f), \sum_{j=0}^{n} (i)^j (2\pi)^{-n} y^j \check{F}[\exp[M[(a-\delta)x]]\, g_j(x)],\ \Phi \right\rangle =$$

$$= \left\langle F(f) \cdot F\left[\sum_{j=0}^{n} D^{(j)} [\exp[M[(a-\delta)x]\, g_j(x)]] \right],\ \Phi \right\rangle = \langle F(f) \cdot F(g), \Phi \rangle.$$

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