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Schauder Estimates for Steady Compressible Navier-Stokes Equations in Bounded Domains.

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ABSTRACT - Using the method of decomposition of the kinetic field to the compressible and incompressible parts, we prove existence of solutions and derive estimates in Hölder classes of continuous functions for steady compressible Navier-Stokes equations in bounded domains. The result is valid for small external data, e.g. we consider only subsonic flows, near the equilibrium.

1. - Introduction.

In the present paper, we investigate the steady compressible Navier-Stokes equations in bounded domains in Hölder classes of functions. These results are useful for many applications, especially for the investigation of large class of free boundary value problems (1). As far as the authors know, they have been missing in the matematical litterature (2). The reason of all difficulties in treating compressible Navier-

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(1) E.g. for the incompressible fluids, the results in Hölder spaces made possible to solve a series of free boundary value problems, see [So1], [So2], [So3], [Os].

(2) There exists a series of papers treating the compressible Navier-Stokes equations in Sobolev spaces; see [P1], [V1], [V2], [BV1], [NP2], ...

Stokes equations in Hölder spaces is not in the nature of the nonlinear terms (they are estimated more easily than in the Sobolev spaces), but in the character of the corresponding linearized systems.

For incompressible Navier-Stokes fluids, the linearized problem (which is the Stokes problem) is of elliptic type and therefore well investigated in Hölder classes of functions (see [ADN], [G1]).

The corresponding linearized system coming from compressible Navier-Stokes equations is of elliptic hyperbolic type and the adequate approach in Hölder spaces does not exist. For the sake of simplicity, we consider an isothermal motion of a compressible perfect gas filling a bounded domain $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ with sufficiently smooth boundary $\partial\Omega$, i.e. we consider the following system of equations

$$
(1.1) \qquad \begin{cases} -\mu_1 \Delta v - (\mu_1 + \mu_2) \nabla \operatorname{div} v + \nabla \varrho = \varrho f - \operatorname{div}(\varrho v \otimes v), & x \in \Omega, \\ \operatorname{div}(\varrho v) = 0, \quad x \in \Omega, \\ v|_{\partial\Omega} = 0. \end{cases}
$$

Here $v = (v_1, \dots, v_n)$ is the unknown velocity vector and $\varrho > 0$ is the unknown density while f is a given density of the external forces and μ_1, μ_2 are the viscosity coefficients which satisfy the conditions

$$
\mu_1 > 0, \qquad \mu_2 \geqslant -\frac{2}{n}\mu_1 .
$$

Since the gas is perfect and the motion is isothermal, the state equation for the pressure reduces to $p = k\varrho$, $k > 0$. Without loss of generality, we assume $k = 1$ (3). Due to the condition $(1.1)_3$, there is no outflow and inflow and therefore the conservation of mass yields

$$
(1.2) \qquad \int_\Omega \varrho(x)\, dx = M,
$$

where M is the total mass of the gas. For simplicity, and without loss of generality, we assume $M = meas(\Omega)$.

(3) The generalization of our appproach to the heat conductive fluids with the constitutive law for the pressure of the form $\Pi = \Pi(\varrho, \theta)$ and with the boundary conditions $v|_{\partial\Omega} = \overline{v}$ where $\overline{v}$ is such that $\overline{v} \cdot \nu = 0$ (ν is the outer normal to the boundary $\partial\Omega$) is standard; see [BV1], [NoPe].

2. – The method of decomposition-Heuristic approach.

We search for the solutions of the problem (1.1) as perturbations (σ, v) of the rest state $\varrho_o = 1$, $v_o = 0$, i.e.

$$\varrho = 1 + \sigma, \qquad \mathfrak{V} = v.$$

The equations for (σ, v) read

$$\begin{cases} -\mu_1 \Delta v - (\mu_1 + \mu_2) \nabla \operatorname{div} v + \nabla \sigma = F(\sigma, v), & x \in \Omega, \\ \operatorname{div} v + \operatorname{div}(\sigma v) = 0, \quad x \in \Omega, \\ v|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where

$$F(\sigma, v) = (1+\sigma) f - \operatorname{div}((1+\sigma) v \otimes v). \tag{2.2}$$

We are thus naturally led to investigate first the linearized system

$$\begin{cases} -\mu_1 \Delta v - (\mu_1 + \mu_2) \nabla \operatorname{div} v + \nabla \sigma = \mathcal{F}, & x \in \Omega, \\ \operatorname{div} v + \operatorname{div}(\sigma w) = 0, \quad x \in \Omega, \\ v|_{\partial\Omega} = 0, \end{cases} \tag{2.3}$$

with $(\mathcal{F}, w)$ given and (σ, v) the unknown functions.

In virtue of the approach of Novotny, Padula [NP1], the equations (2.3) are solved in the following way: we are looking for a solution in the form

$$v = u + \nabla \varphi, \qquad \operatorname{div} u = 0, \qquad \frac{\partial \varphi}{\partial \nu}\Big|_{\partial\Omega} = 0, \qquad u \cdot \nu|_{\partial\Omega} = 0. \tag{2.4}$$

Then the system (2.3) is equivalent to the following system of equations

$$\begin{cases} -\mu_1 \Delta u + \nabla p = \mathcal{F}, & x \in \Omega, \\ \operatorname{div} u = 0, \quad x \in \Omega, \\ u|_{\partial\Omega} = -\nabla\varphi|_{\partial\Omega}, \end{cases} \tag{2.5}$$

$$\sigma + (2\mu_1 + \mu_2) \operatorname{div}(\sigma w) = p, \qquad x \in \Omega, \tag{2.6}$$

$$\begin{cases} \Delta \varphi = -\operatorname{div}(\sigma w), & x \in \Omega, \\ \dfrac{\partial \varphi}{\partial \nu}\Big|_{\partial\Omega} = 0. \end{cases} \tag{2.7}$$

Hence the triplet (σ, φ, u) can be found (formally) as a fixed point of a linear operator

$$L: \xi \to \varphi \tag{2.8}$$

which is defined as follows:

i) For a given ξ, we solve the Stokes problem

$$\begin{cases} -\mu_1 \Delta u + \nabla p = \mathcal{F}, & x \in \Omega, \\ \operatorname{div} u = 0, \quad x \in \Omega, \\ u|_{\partial\Omega} = -\nabla\xi|_{\partial\Omega}. \end{cases} \tag{2.9}$$

ii) Once p is known, we find σ as solution of the transport equation (2.6).

iii) When σ is known, φ is a solution of the Neumann problem (2.7).

To solve the nonlinear problem (2.1)-(2.2), we proceed as follows: we show the existence of solutions of the nonlinear problem (2.1)-(2.2) by contraction principle applied to the composite map

$$N: (\tau, w) \to F(\tau, w) \to (\sigma, v) \tag{2.10}$$

where (σ, v) is a solution of the linear system (2.3) corresponding to $\mathcal{F} = F(\tau, w)$.

3. - Functional spaces.

Let $l = 0, 1, 2, \ldots$, $k = 0, 1, 2, \ldots$, $0 < \alpha < 1$. Let $Q \subset \mathbb{R}^n$ $(n = 2, 3)$ be a bounded domain with the boundary ∂Q of class $C^{l,\alpha}$ and with the outer normal ν. We need the following functional spaces:

- $C_0^\infty(Q)$ is a set of all infinitely differentiable functions with compact support in Q.
- $C^k(\overline{Q})$ is the Banach space of continuous functions, k-times differentiable up to the boundary equipped with the norm

$$\|u\|_{C^k, Q} = \sum_{m=0}^{k} \sup_{x \in \overline{Q}} |\nabla^m u(x)|.$$

- $C^{l,\alpha}(\overline{Q})$ is the usual space of Hölder continuous functions i.e. a completion of $C^\infty(\overline{Q})$ in the norm

$$\|u\|_{C^{l,\alpha}, Q} = \|u\|_{C^l, Q} + \mathcal{H}_{\alpha, Q}(\nabla^l u),$$

where

$$\|u\|_{C^l, Q} = \sum_{i=0}^{l} \|\nabla^i u\|_{C^0, Q}$$

and

$$\mathcal{H}_{\alpha, Q}(\nabla^l u) = \sup_{\substack{x, y \in \overline{Q} \\ x \neq y}} \frac{|\nabla^l u(x) - \nabla^l u(y)|}{|x - y|^\alpha} .$$

- $\widehat{C}^{l, \alpha}(Q) = \left\{z;\ z \in C^{l, \alpha}(\overline{Q}), \int_Q z(x)\, dx = 0\right\}$ is a subspace of $C^{l, \alpha}(Q)$.

- We also define in the standard way the space of Hölder continuous funtions on $\partial Q \in C^{l, \alpha}$ and denote them by $C^{l, \alpha}(\partial Q)$.

- $L^2(Q)$ is the classical Lebesgue space with the norm $\|\cdot\|_{0, 2, Q}$ and $\widehat{L}^2(Q)$ its subspace of functions with the zero mean value, i.e. $\widehat{L}^2(Q) = L^2(Q) \cap \left\{z: \int_Q z(x)\, dx = 0\right\}$.

- Similarly, $W^{k, 2}(Q)$ (and $W_0^{k, 2}(Q)$), $k = 1, 2, \ldots$, is the usual Sobolev space equipped with norm $\|\cdot\|_{k, 2, Q}$ (index zero denotes zero traces) and $\widehat{W}^{k, 2}(Q)$ is its subspace: $\widehat{W}^{k, 2}(Q) = W^{k, 2}(Q) \cap \cap \left\{z: \int_Q z(x)\, dx = 0\right\}$.

- Dual space to $W_0^{1, 2}(Q)$ is denoted by $W^{-1, 2}(Q)$ and equipped with the usual duality norm $\|\cdot\|_{-1, 2, Q}$.

- If the norms refer to the domain Ω, we omit Ω as the further indice at the norm: e.g. $\|\cdot\|_{0, 2}$, $\|\cdot\|_{C^l}$, $\|\cdot\|_{C^{l, \alpha}}$ mean $\|\cdot\|_{0, 2, \Omega}$, $\|\cdot\|_{C^l, \Omega}$, $\|\cdot\|_{C^{l, \alpha}, \Omega}$.

- In order to simplify the notation, we use in the sequel, the following composed Banach spaces:

$$G = \widehat{C}^{k+1, \alpha}(\overline{\Omega}),$$

$$V = \{v \in C^{k+2, \alpha}(\overline{\Omega}), \operatorname{div} v|_{\partial\Omega} = 0, v|_{\partial\Omega} = 0\},$$

$$U = \{u \in C^{k+2, \alpha}(\overline{\Omega}), \operatorname{div} u = 0, u \cdot \nu|_{\partial\Omega} = 0\},$$

$$\Phi = \left\{ \varphi \in \hat{C}^{k+3,\alpha}(\overline{\Omega}),\ \frac{\partial \varphi}{\partial \nu}\,|_{\partial\Omega} = 0,\ \Delta\varphi\,|_{\partial\Omega} = 0 \right\},$$

$$GV = G \times V, \qquad \Phi U = \Phi \times U, \qquad G\Phi U = G \times \Phi \times U.$$

These are Banach spaces equipped with the norms:

$$\|\cdot\|_G = \|\cdot\|_{C^{k+1,\alpha}}, \qquad \|\cdot\|_V = \|\cdot\|_U = \|\cdot\|_{C^{k+2,\alpha}}, \qquad \|\cdot\|_\Phi = \|\cdot\|_{C^{k+3,\alpha}},$$

$$\|(\sigma, v)\|_{GV} = \|\sigma\|_{C^{k+1,\alpha}} + \|v\|_{C^{k+2,\alpha}}, \qquad \|(\varphi, u)\|_{\Phi U} = \|\varphi\|_{C^{k+3,\alpha}} + \|u\|_{C^{k+2,\alpha}},$$

$$\|(\sigma, \varphi, u)\|_{G\Phi U} = \|\sigma\|_{C^{k+1,\alpha}} + \|\varphi\|_{C^{k+3,\alpha}} + \|u\|_{C^{k+2,\alpha}}.$$

Spaces G, Φ, U and V are used for the description of the density field (space G), the compressible part of the velocity field (space Φ), the incompressible part of the velocity field (space U) and the velocity field (space V).

- In the notation, we do not distinguish between the spaces of scalar and vector valued functions: e.g. $C^{k,\alpha}(\Omega)$ denotes either $C^{k,\alpha}(\Omega; \mathbb{R})$ or $C^{k,\alpha}(\Omega; \mathbb{R}^n)$; the difference is always clear from the context.

- Some remarks on the notation in estimates. In what follows, c is a generic positive constant dependent only of k, α, $\partial\Omega$, μ_1 and μ_2; it can have different values even in the same formulas.

4. - Main theorems.

In this chapter, we give the statements of the main theorems. First theorem deals with the linearized system (2.3) while the second theorem with the fully nonlinear system (2.1)-(2.2).

THEOREM 4.1. *Let $k = 1, \ldots, 0 < \alpha < 1$, Ω be a bounded domain, $\partial\Omega \in C^{k+3,\alpha}$, $\mathcal{F} \in C^{k,\alpha}(\overline{\Omega})$,*

$$w \in V = \{v \in C^{k+2,\alpha}(\overline{\Omega}),\ \operatorname{div} v\,|_{\partial\Omega} = 0,\ v\,|_{\partial\Omega} = 0\}$$

and let

$$G\Phi U = \hat{C}^{k+1,\alpha}(\overline{\Omega}) \times \left\{ \varphi \in \hat{C}^{k+3,\alpha}(\overline{\Omega}),\ \frac{\partial\varphi}{\partial\nu}\,|_{\partial\Omega} = 0,\ \Delta\varphi\,|_{\partial\Omega} = 0 \right\} \times$$

$$\times \{u \in C^{k+2,\alpha}(\overline{\Omega}),\ \operatorname{div} u = 0,\ u\cdot\nu\,|_{\partial\Omega} = 0\},$$

$$G = \hat{C}^{k+1,\alpha}(\overline{\Omega}).$$

Then there exists γ_1 *such that if*

(4.1) $$\|w\|_{C^{k+2,\alpha}} < \gamma_1$$

then there exists just one solution

(4.2) $$(\sigma, \varphi, u) \in G\Phi U, \qquad p \in G$$

of problem (2.5)-(2.7), *which satisfies the estimate*

(4.3) $$\|p\|_{C^{k+1,\alpha}} + \|\sigma\|_{C^{k+1,\alpha}} + \|\varphi\|_{C^{k+3,\alpha}} + \|u\|_{C^{k+2,\alpha}} \leq c\|\mathcal{F}\|_{C^{k,\alpha}} .$$

CONSEQUENCE 4.1. Let (σ, φ, u) be a triplet from Theorem 4.1. Put $v = u + \nabla\varphi$. Then the couple

$$(\sigma, v) \in GV = \widehat{C}^{k+1,\alpha}(\overline{\Omega}) \times \{v \in C^{k+2,\alpha}(\overline{\Omega}),\ \operatorname{div} v|_{\partial\Omega} = 0,\ v|_{\partial\Omega} = 0\}$$

solves the system (2.3) and satisfies the estimate:

(4.4) $$\|\sigma\|_{C^{k+1,\alpha}} + \|v\|_{C^{k+2,\alpha}} \leq c\|\mathcal{F}\|_{C^{k,\alpha}} .$$

THEOREM 4.2. *Let* $k = 1, \ldots,$ $0 < \alpha < 1$, Ω *be a bounded domain,* $\partial\Omega \in C^{k+3,\alpha}$, $f \in C^{k,\alpha}(\overline{\Omega})$, *and let*

$$GV = \widehat{C}^{k+1,\alpha}(\overline{\Omega}) \times \{v \in C^{k+2,\alpha}(\overline{\Omega}),\ \operatorname{div} v|_{\partial\Omega} = 0,\ v|_{\partial\Omega} = 0\}.$$

Then there exists $\gamma_0, \gamma_1 > 0$ *such that if*

(4.5) $$\|f\|_{C^{k,\alpha}} < \gamma_1$$

then in the ball

(4.6) $$B_{\gamma_0} = \{(\sigma, v) \in GV,\ \|(\sigma, v)\|_{GV} = \|\sigma\|_{C^{k+1,\alpha}} + \|v\|_{C^{k+2,\alpha}} \leq \gamma_0\}$$

there exists just one couple (σ, v) *which solves the problem* (2.1)-(2.2). *This couple satisfies the estimate*:

(4.7) $$\|\sigma\|_{C^{k+1,\alpha}} + \|v\|_{C^{k+2,\alpha}} \leq c\|f\|_{C^{k,\alpha}} .$$

5. – Auxiliary linear problems.

In the first part of this section, we recall several estimates for the Dirichlet and the Neumann problems for the Laplace operator and for the Stokes problem in Hölder spaces, when Ω is bounded. These results are standard, see [ADN]. They will be used in the proofs of Theorem 4.1 and 4.2. More detailed investigation of the Stokes problem can be found e.g. in [G1]. Last but not least, we give the Schauder estimates for the transport equation which are due to Novotny [No2]. In the second part, we recall the results for the same problems in Sobolev spaces with L^2 structure. This part is only used to prove that the nonlinear operator N is a contraction. We would be able to avoid it, if we would suppose more regularity at the r.h.s. in the Theorem 4.2, namely $f \in C^{2,\alpha}(\overline{\Omega})$ instead of $C^{1,\alpha}(\overline{\Omega})$.

Consider the Dirichlet problem for Laplacian:

$$\begin{cases} \Delta\theta = f, & x \in \Omega, \\ \theta|_{\partial\Omega} = 0. \end{cases} \tag{5.1}$$

LEMMA 5.1. *Let $k = 0, 1, \ldots$, $0 < \alpha < 1$, Ω a bounded domain in $\mathbb{R}^n$, $\partial\Omega \in C^{k+2,\alpha}$, $f \in C^{k,\alpha}(\overline{\Omega})$. Then there exists just one solution*

$$\theta \in C^{k+2,\alpha}(\overline{\Omega}) \tag{5.2}$$

of the problem (5.1), *which satisfies the estimate*

$$\|\theta\|_{C^{k+2,\alpha}} \leq c\|f\|_{C^{k,\alpha}}. \tag{5.3}$$

For the Neumann problem:

$$\begin{cases} \Delta\varphi = f, & x \in \Omega, \\ \left.\dfrac{\partial\varphi}{\partial\nu}\right|_{\partial\Omega} = 0, \end{cases} \tag{5.4}$$

we have:

LEMMA 5.2. *Let $k = 0, 1, \ldots$, $0 < \alpha < 1$, Ω a bounded domain in $\mathbb{R}^n$, $\partial\Omega \in C^{k+3,\alpha}$, $f \in \widehat{C}^{k+1,\alpha}(\overline{\Omega})$. Then there exists just one solution*

$$\varphi \in \widehat{C}^{k+3,\alpha}(\overline{\Omega}) \tag{5.5}$$

of the problem (5.4), *which satisfies the estimate*

$$\|\varphi\|_{C^{k+3,\alpha}} \leq c\|f\|_{C^{k+1,\alpha}}. \tag{5.6}$$

Further consider the nonhomogenous Stokes problem:

$$\begin{cases} -\mu_1\Delta u + \nabla p = f, & x \in \Omega, \\ \operatorname{div} u = g, & x \in \Omega, \\ u|_{\partial\Omega} = \psi. \end{cases} \tag{5.7}$$

We have:

LEMMA 5.3. *Let $k = 0, 1, \ldots, 0 < \alpha < 1$, Ω a bounded domain in* $\mathbb{R}^n$, $\partial\Omega \in C^{k+2,\alpha}$, $f \in C^{k,\alpha}(\overline{\Omega})$, $g \in C^{k+1,\alpha}(\overline{\Omega})$, $\psi \in C^{k+2,\alpha}(\partial\Omega)$ *such that*

$$\int_{\Omega} g(x)\,dx = \int_{\partial\Omega} \psi\cdot\nu\,dS\,.$$

Then there exists just one solution

$$u \in C^{k+2,\alpha}(\overline{\Omega}), \qquad p \in \widehat{C}^{k+1,\alpha}(\overline{\Omega}) \tag{5.8}$$

of the problem (5.7) *which satisfies the estimate*

$$\|u\|_{C^{k+2,\alpha}} + \|p\|_{C^{k+1,\alpha}} \leq c(\|f\|_{C^{k,\alpha}} + \|\psi\|_{C^{k+2,\alpha}} + \|g\|_{C^{k+1,\alpha}})\,. \tag{5.9}$$

Last equation to be investigated is the transport equation

$$\sigma + w\cdot\nabla\sigma + a\sigma = f \tag{5.10}$$

with the characteristics remaining in Ω, i.e. with the condition

$$(w\cdot\nu)\,|_{\partial\Omega} = 0\,.$$

We have:

LEMMA 5.4. *Let $k = 0, 1, \ldots, 0 < \alpha < 1$, Ω a bounded domain in* $\mathbb{R}^n$, $\partial\Omega \in C^{k+1}$, $w \in C^{k+1}(\overline{\Omega})$, $a \in C^{k,\alpha}(\overline{\Omega})$, $f \in C^{k,\alpha}(\overline{\Omega})$, $(w\cdot\nu)\,|_{\partial\Omega} = 0$. *Then there exists $\gamma > 0$ depending only on k, α, $\partial\Omega$ such that if*

$$\|w\|_{C^{k+1}} + \|a\|_{C^{k,\alpha}} \leq \gamma \tag{5.11}$$

then there exists just one solution

$$\sigma \in C^{k,\alpha}(\overline{\Omega}) \tag{5.12}$$

of the problem (5.10), *which satisfies the estimate*

$$\|\sigma\|_{C^{k,\alpha}} + \|\mathrm{div}\,(\sigma w)\|_{C^{k,\alpha}} \leq c\|f\|_{C^{k,\alpha}}\,. \tag{5.13}$$

Here c depends of k, α, $\partial\Omega$. *Moreover if $f \in \widehat{C}^{k,\alpha}(\overline{\Omega})$ then $\sigma \in$ $\in \widehat{C}^{k,\alpha}(\overline{\Omega})$.*

For the estimates and the regularity in L^2 spaces, for the Neumann problem, we refer to [SiSo1], for the Stokes problem, we refer to [Ca] and for the transport equation to [BV2], [BV3], [No1]. As far as the Neumann problem (5.4) is concerned, we have:

LEMMA 5.5. *Let Ω a bounded domain in* $\mathbb{R}^n$, $\partial\Omega \in C^2$ *and* $f \in$

$\in \widehat{L}^2(\Omega)$. *Then there exists just one (strong) solution*

$$\varphi \in \widehat{W}^{2,2}(\Omega) \tag{5.14}$$

of problem (5.4), *which satisfies the estimate*

$$\|\varphi\|_{2,2} \leqslant c\|f\|_{0,2}\,. \tag{5.15}$$

For the Stokes problem (5.7) with $\psi = 0$, we get:

LEMMA 5.6. *Let* Ω *a bounded domain in* $\mathbb{R}^n$, $\partial\Omega \in C^2$, $f \in W^{-1,2}(\Omega)$, $g \in \widehat{L}^2(\Omega)$ *and* $\psi = 0$. *Then there exists just one (strong) solution*

$$(u, p) \in W_0^{1,2}(\Omega) \times \widehat{L}^2(\Omega) \tag{5.16}$$

of the problem (5.7), *which satisfies the estimate*

$$\|u\|_{1,2} + \|p\|_{0,2} \leqslant c(\|f\|_{-1,2} + \|g\|_{0,2})\,. \tag{5.17}$$

Finally, for the transport equation we have:

LEMMA 5.7. *Let assumptions of Lemma* 5.4 *be satisfied and let* $\sigma \in C^{k,\alpha}$ *be a solution of the problem* (5.10) *in* Ω. *Then*

$$\|\sigma\|_{0,2} + \|\operatorname{div}(\sigma w)\|_{0,2} \leqslant c\|f\|_{0,2}\,. \tag{5.18}$$

6. – Proof of Theorem 4.1.

Let $w \in V$ and $\mathcal{F} \in C^{k,\alpha}(\overline{\Omega})$. It is clear from Lemmas 5.1-5.5, that for a given $\xi \in \Phi$, the familly of problems (2.9), (2.6)-(2.7) posseses a solution (p, σ, φ, u), $p \in G$, $(\sigma, \varphi, u) \in G\Phi U$ provided $\|w\|_V$ is sufficiently small. Therefore, the operator L (see (2.8)) is, for $w \in V$, $\|w\|_V \leqslant \gamma_1$, defined on Φ and maps Φ onto itself. We show that L is a contraction on Φ. If it is so, then L posseses a unique fixed point $\xi = \varphi \in \Phi$. Obviously, φ and the corresponding p, σ, u solve the linearised system (2.5)-(2.7). This is equivalent to say that σ, $v = u + \nabla\varphi$ solve the problem (2.3).

Proof of the contraction:

We start with a priori estimates. Lemma 5.3 applied to the Stokes problem (2.9) yields existence of $u \in C^{k+2,\alpha}$, $p \in \widehat{C}^{k+1,\alpha}$ satisfying

$$\|u\|_{C^{k+2,\alpha}} + \|p\|_{C^{k+1,\alpha}} \leqslant c(\|\mathcal{F}\|_{C^{k,\alpha}} + \|\nabla\xi\|_{C^{k+2,\alpha}}) \leqslant \leqslant c(\|\mathcal{F}\|_{C^{k,\alpha}} + \|\xi\|_{C^{k+3,\alpha}})\,. \tag{6.1}$$

Taking div of (2.9), we get

$$\Delta p = \operatorname{div} \mathcal{F} \tag{6.2}$$

which yields the estimate

$$\|\Delta p\|_{C^{k-1,\alpha}} \leqslant c\|\mathcal{F}\|_{C^{k,\alpha}} . \tag{6.3}$$

Lemma 5.4 applied to the equation (2.6) yields existence of $\sigma \in \widehat{C}^{k+1,\alpha}$ and the estimate

$$\|\sigma\|_{C^{k+1,\alpha}} + \|\operatorname{div}(\sigma w)\|_{C^{k+1,\alpha}} \leqslant c\|p\|_{C^{k+1,\alpha}} \leqslant c(\|\mathcal{F}\|_{C^{k,\alpha}} + \|\xi\|_{C^{k+3,\alpha}}), \tag{6.4}$$

provided $\|w\|_V \leqslant \gamma_1$ (w «sufficiently small»). Taking the Laplacian of (2.6), we have

$$\Delta\sigma + (2\mu_1 + \mu_2)\operatorname{div}(\Delta\sigma w) = \tag{6.5}$$

$$= \Delta p - 2(2\mu_1 + \mu_2)\operatorname{div}(\nabla\sigma \cdot \nabla w) - (2\mu_1 + \mu_2)\operatorname{div}(\sigma\Delta w).$$

Using the estimate $\|ab\|_{C^{k,\alpha}} \leqslant c\|a\|_{C^{k,\alpha}}\|b\|_{C^{k,\alpha}}$, we see that the last two terms on the r.h.s. of (6.5) satisfy the estimate

$$\|\operatorname{div}(\nabla\sigma \cdot \nabla w)\|_{C^{k-1,\alpha}} + \|\operatorname{div}(\sigma\Delta w)\|_{C^{k-1,\alpha}} \leqslant c\|\sigma\|_{C^{k+1,\alpha}}\|w\|_{C^{k+2,\alpha}} . \tag{6.6}$$

Applying Lemma 5.4 to the equation (6.5), we have, in virtue of (6.3), (6.6)

$$\|\Delta\sigma\|_{C^{k-1,\alpha}} + \|\operatorname{div}(\Delta\sigma w)\|_{C^{k-1,\alpha}} \leqslant c(\|\mathcal{F}\|_{C^{k,\alpha}} + \|\sigma\|_{C^{k+1,\alpha}}\|w\|_{C^{k+2,\alpha}})$$

which gives in turn with (6.6)

$$\|\Delta\sigma\|_{C^{k-1,\alpha}} + \|\Delta\operatorname{div}(\sigma w)\|_{C^{k-1,\alpha}} \leqslant c(\|\mathcal{F}\|_{C^{k,\alpha}} + \|\sigma\|_{C^{k+1,\alpha}}\|w\|_{C^{k+2,\alpha}}). \tag{6.7}$$

Lemma 5.2 applied to the equation (2.7) yields

$$\|\varphi\|_{C^{k+3,\alpha}} \leqslant c\|\operatorname{div}(\sigma w)\|_{C^{k+1,\alpha}} . \tag{6.8}$$

Further, considering the obvious problem

$$\Delta \operatorname{div}(\sigma w) = \Delta \operatorname{div}(\sigma w),$$

$$\operatorname{div}(\sigma w)|_{\partial\Omega} = 0,$$

we deduce by Lemma 5.1 the estimate

$$\|\operatorname{div}(\sigma w)\|_{C^{k+1,\alpha}} \leqslant \|\Delta \operatorname{div}(\sigma w)\|_{C^{k-1,\alpha}} . \tag{6.9}$$

Therefore, by (6.8), (6.9) and (6.7),

$$\|\varphi\|_{C^{k+3,\alpha}} \leq c(\|\mathcal{F}\|_{C^{k,\alpha}} + \|\sigma\|_{C^{k+1,\alpha}}\|w\|_{C^{k+2,\alpha}}) \tag{6.10}$$

and finally by (6.4)

$$\|\varphi\|_{C^{k+3,\alpha}} \leq c((\|\mathcal{F}\|_{C^{k,\alpha}} + \|\xi\|_{C^{k+3,\alpha}})\|w\|_{C^{k+2,\alpha}} + \|\mathcal{F}\|_{C^{k,\alpha}}). \tag{6.11}$$

Since L is a linear operator, (6.11) yields the contraction of L for

$$c\|w\|_{C^{k+2,\alpha}} < 1 .$$

The existence of the fixed point $\xi = \varphi \in \Phi$ is thus proved. After this, the proof of the estimate (4.3) is easy; we only add the inequalities (6.1), (6.4) and (6.11) written in the fixed point $\xi = \varphi$.

7. - Proof of Theorem 4.2.

Denote by $X = L^2(\Omega) \times W^{1,2}(\Omega)$ the Banach space with the norm $\|(\sigma, v)\|_X = \|\sigma\|_{0,2} + \|v\|_{1,2}$. Then $GV \subset X$ and B_{γ_0} (see (4.6)) is a closed subset of X. We prove that:

i) for $\|f\|_{C^{k,\alpha}}$ and γ_0 sufficiently small, the operator N (see (2.10)) maps B_{γ_0} into itself.

ii) N is a contraction in B_{γ_0} in the topology of X, provided $\|f\|_{C^{k,\alpha}}$ and γ_0 are sufficiently small. If it is so, then according to the Banach fixed point principle, N possesses a unique fixed point $(\sigma, v) \in B_{\gamma_0}$ which obviously solves the fully nonlinear system (2.1)-(2.2).

Proof of i): Take $(\tau, w) \in GV$ and $\|f\|_{C^{k,\alpha}} \leq \gamma_1$. We start with the obvious estimates

$$\|F(\tau, w)\|_{C^{k,\alpha}} \leq c(1 + \|\tau\|_{C^{k+1,\alpha}})(\|w\|^2_{C^{k+2,\alpha}} + \|f\|_{C^{k,\alpha}}). \tag{7.1}$$

Applying Theorem 4.1 to the system (2.5)-(2.7) with $F = F(\tau, w)$, we get the estimate (cf. (4.3)-(4.4)):

$$\|\sigma\|_{C^{k+1,\alpha}} + \|v\|_{C^{k+2,\alpha}} \leq c(1 + \|\tau\|_{C^{k+1,\alpha}})(\|w\|^2_{C^{k+2,\alpha}} + \|f\|_{C^{k,\alpha}}). \tag{7.2}$$

This means that N maps B_{γ_0} into itself provided

$$c(1 + \gamma_0)(\gamma_0^2 + \gamma_1) < \gamma_0 .$$

Proof of ii): Let $(\tau_i, w_i) \in B_{\gamma_0}$ $(i = 1, 2)$ and let (σ_i, v_i) be a solution of (2.5)-(2.7) with $\mathcal{F}_i = F(\tau_i, w_i)$. Put $\tilde{a} = a_1 - a_2$ where a stands for

σ, v, τ, w and $\tilde{F} = F(\tau_1, w_1) - F(\tau_2, w_2)$. We compute

$$(7.3) \quad \|\tilde{F}\|_{-1,2} \leq \|\tilde{\tau} f\|_{-1,2} + \|(1+\tau_1)\tilde{w} \otimes w_1\|_{0,2} +$$

$$+ \|\tilde{\tau} w_1 \otimes w_2\|_{0,2} + \|(1+\tau_2)\tilde{w} \otimes w_2\|_{0,2} \leq$$

$$\leq c(1 + \|\tau_1\|_G + \|\tau_2\|_G)(\|f\|_{C^{k,\alpha}} + \|(\tau_1, w_1)\|_{GV} + \|(\tau_2, w_2)\|_{GV}) \|(\tilde{\tau}, \tilde{w})\|_X .$$

Writting the system (2.3) with $\mathcal{F} = F(\tau, w)$ for the differences, we get

$$(7.4) \qquad \begin{cases} -\mu_1 \Delta \tilde{v} - (\mu_1 + \mu_2)\nabla \operatorname{div} \tilde{v} + \nabla \tilde{\sigma} = \tilde{F}, \\ \operatorname{div} \tilde{v} + \operatorname{div}(w_2 \tilde{\sigma}) = -\operatorname{div}(\tilde{w}\sigma_1), \\ \tilde{v}\,|_{\partial\Omega} = 0 . \end{cases}$$

Multiply $(7.4)_1$ by $\tilde{v}$ and $(7.4)_2$ by $\tilde{\sigma}$, integrate over Ω and add together. We obtain, after some calculation:

$$(7.5) \quad \|\tilde{v}\|_{1,2}^2 \leq$$

$$\leq c(\|\tilde{F}\|_{-1,2}\|\tilde{v}\|_{1,2} + (\|\tilde{\sigma}\|_{0,2}^2 + \|\tilde{w}\|_{1,2}^2)(\|(\sigma_1, w_1)\|_{GV} + \|(\sigma_2, w_2)\|_{GV})) .$$

Here we have used several facts, e.g.

$$\int_\Omega \operatorname{div} \tilde{v}\tilde{\sigma}\, dx + \int_\Omega \nabla\tilde{\sigma}\tilde{v}\, dx = 0,$$

$$\int_\Omega w_2 \cdot \nabla\tilde{\sigma}\tilde{\sigma}\, dx = -\frac{1}{2}\int_\Omega \operatorname{div} w_2\, |\tilde{\sigma}|^2\, dx \leq \|w_2\|_V \|\tilde{\sigma}\|_{0,2}^2 ,$$

$$\int_\Omega \tilde{w} \cdot \nabla\sigma_1 \tilde{\sigma}\, dx \leq \|\tilde{w}\|_{0,6}\|\nabla\sigma_1\|_{0,3}\|\tilde{\sigma}\|_{0,2} \leq \|\tilde{w}\|_{1,2}\|\sigma_1\|_G\|\tilde{\sigma}\|_{0,2} .$$

Estimates (7.3) and (7.5) give

$$(7.6) \qquad \|\tilde{v}\|_{1,2}^2 \leq c(\gamma_0 + \gamma_1)(\|\tilde{\tau}\|_{0,2}^2 + \|\tilde{\sigma}\|_{0,2}^2 + \|\tilde{w}\|_{1,2}^2).$$

Further consider for $(\tilde{\sigma}, \tilde{v})$ the Stokes problem

$$(7.7) \qquad \begin{cases} -\mu_1 \Delta \tilde{v} + \nabla \tilde{\sigma} = \tilde{F} + (\mu_1 + \mu_2)\nabla \operatorname{div} \tilde{v}, \\ \operatorname{div} \tilde{v} = \operatorname{div} \tilde{v}, \\ \tilde{v}\,|_{\partial\Omega} = 0 . \end{cases}$$

Applying to (7.7) Lemma 5.6, one gets

$$\|\tilde{v}\|_{1,2}^2 + \|\tilde{\sigma}\|_{0,2}^2 \leq c(\|\tilde{v}\|_{1,2}^2 + \|\tilde{F}\|_{-1,2}^2) . \tag{7.8}$$

This yields in turn with (7.3) and (7.6)

$$\|(\tilde{\sigma}, \tilde{v})\|_X^2 \leq c(\gamma_0 + \gamma_1)\|(\tilde{\tau}, \tilde{w})\|_X^2 .$$

This means that the operator $N: B_{\gamma_0} \to B_{\gamma_0}$ is a contraction in the topology of X provided

$$c(\gamma_0 + \gamma_1) < 1 .$$

The fixed point (σ, v) (which exists due to the Banach contraction principle) solves the nonlinear problem (2.1)-(2.2). Estimate (4.7) follows directly from (7.2) with $(\tau, w) = (\sigma, v)$. The proof of Theorem 4.2 is thus complete.

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