RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

FRANCESCA ALESSIO
MARTA CALANCHI

Homoclinic-type solutions for an almost periodic semilinear elliptic equation on R^n

Rendiconti del Seminario Matematico della Università di Padova, tome 97 (1997), p. 89-111
<http://www.numdam.org/item?id=RSMUP_1997__97__89_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 97 (1997)

Homoclinic-Type Solutions for an Almost Periodic Semilinear Elliptic Equation on $\boldsymbol{R}^n$.

FRANCESCA ALESSIO - MARTA CALANCHI (*)

1. – Introduction and basic properties.

In a recent paper, [17], the authors studied the existence of a homoclinic solution for a second order system where the potential is an almost periodic function of time.

The existence of homoclinic orbits has been deeply investigated by people working with variational methods. We refer, for example, to works concerning first order systems, as [5], [15] and [16], and second order systems, as [1], [2], [10], [11] and the above mentioned [17]; see also [6], [9], [14] for semilinear elliptic equations.

In this paper we try to extend the idea of [17] to a semilinear elliptic equation where the potential involves an almost periodic function on $\boldsymbol{R}^n$ in a suitable sense. In this context, we point out [6] for the periodic case and [9] for the asymptotically periodic case.

Our aim is to prove the existence of at least one (homoclinic-type) solution for the problem

$$\text{(P)} \qquad \begin{cases} -\Delta u + u = a(x) g(u), \\ u(x) \to 0 \text{ as } |x| \to \infty, \end{cases}$$

where $n \geqslant 2$ and a and g satisfy the following assumptions:

(α_1) $a \in \mathcal{C}^{0,\gamma}(\boldsymbol{R}^n, \boldsymbol{R})$ for some $\gamma \in (0, 1)$, with

$$\inf_{x \in \boldsymbol{R}^n} a(x) := \underline{a} > 0,$$

(*) Indirizzo degli AA.: Dipartimento di Matematica del Politecnico di Torino, Corso Duca degli Abruzzi, 24, I-10129 Torino.

(α_2) $a(x) = a(x^1, \dots, x^n)$ is almost periodic in x^1 uniformly with respect to the others variables (see Definition 1.1) and T_i-periodic in x^i for $i = 2, \dots, n$,

(g_1) $g \in \mathcal{C}^1(\boldsymbol{R}, \boldsymbol{R})$,

(g_2) there is a $\theta > 2$ such that

$$0 < \theta G(\xi) := \theta \int_0^\xi g(t)dt \leq g(\xi)\,\xi$$

for all $\xi \in \boldsymbol{R} \setminus \{0\}$,

(g_3) there are costants a_1, $a_2 > 0$ such that

$$|g'(\xi)| \leq a_1 + a_2 |\xi|^{s-1}$$

for all $\xi \in \boldsymbol{R}$, where $s \in (1, (n+2)/(n-2))$ if $n > 2$ and s is not restricted if $n = 2$.

More precisely, we will prove the following result.

THEOREM 1.1. *If a satisfies* (α_1), (α_2) *and g satisfies* (g_1), (g_2) *and* (g_3) *then* (P) *possesses a nontrivial classical solution* $u \in H^1(\boldsymbol{R}^n, \boldsymbol{R})$.

Standard arguments, as in [14], show that $u \in \mathcal{C}^{2,\gamma}(\boldsymbol{R}^n, \boldsymbol{R})$ and that $u(x) \to 0$ as $|x| \to \infty$ in a $\mathcal{C}^{2,\gamma}$ sense.

We seek solutions of (P) as critical points of the functional

$$f(u) = \frac{1}{2} \int_{\boldsymbol{R}^n} (|\nabla u|^2 + |u|^2)\,dx - \int_{\boldsymbol{R}^n} a(x)\,G(u)\,dx$$

in the Hilbert space $H := H^1(\boldsymbol{R}^n, \boldsymbol{R})$, equipped with the usual norm:

$$\|u\|^2 = \int_{\boldsymbol{R}^n} (|\nabla u|^2 + |u|^2)\,dx.$$

It is known, see e.g. [6] and [9], that if (α_1)-(α_2) and (g_1)-(g_3) hold then $f \in \mathcal{C}^1(H, \boldsymbol{R})$ and

$$\nabla f(u) \cdot v = \int_{\boldsymbol{R}^n} (\nabla u \nabla v + uv)\,dx - \int_{\boldsymbol{R}^n} a g(u)\,v\,dx \qquad \forall v \in H,$$

so that critical points of f are weak and—by regularity arguments—classical solutions of problem (P).

We briefly discuss the main features of the problem. For our purposes we use a classical minimax procedure. Indeed the functional f satisfies the geometric assumptions of the Mountain Pass Theorem, so that we can construct a Palais-Smale sequence (u_m) at a positive level c. The form of the functional and the assumptions imply that Palais-Smale sequences are bounded and that their weak limits are weak solutions. The main difficulty is to make sure that these weak limits are not zero. To overcome this problem we use the Concentration-Compactness Principle of P. L. Lions that allows us to find a sequence (x_m) in $\boldsymbol{R}^n$ such that no subsequence of the translates $v_m = u_m(\cdot + x_m)$ can converge weakly to zero. The trouble is that, in general, (v_m) in not a Palais-Smale sequence for f. If a is periodic in each component we can overcome this problem using the invariance of f under the action of $\boldsymbol{Z}^n$: we can consider, without loss of generality, each component of x_m to be a multiple of the period so that (v_m) also results a Palais-Smale sequence. On the other hand, if a is almost periodic in the first component and periodic in the others, we show that the translated sequence (v_m) still results a Palais-Smale sequence if we choose the last $n-1$ components of x_m as multiples of the periods, as in the periodic case, and the first component «near» a $1/m$-period. The main part of the paper is devoted to prove that this choice is possible if the Palais-Smale sequence satisfies the aditional property $\|u_m - u_{m-1}\| \to 0$ as $m \to \infty$. The existence of this type of sequences has been proved in [4] and [15].

Finally, we emphasize that although we assume almost periodicity in only one variable, the proof is not a direct application of that in the one-dimensional case ([17]). The technique we use to solve the problem is quite different from that of [17]: it is essentially based on the Maximum Principle and on an estimate of the exponential decay at infinity for the solutions of problems of type (P). Actually, we remark that our method also applies to the one-dimensional case, so that it could be used to slightly simplify some proofs in [17].

For sake of completeness, we recall some definitions and properties concerning almost periodic functions depending on a parameter (see [3] or [18] for further details and proofs).

DEFINITION 1.1. *A continuous function $a(x, X)$ is called almost periodic in $x \in \boldsymbol{R}$, uniformly with respect to $X \in \boldsymbol{R}^{n-1}$ if to any $\varepsilon > 0$ there corresponds a number $l(\varepsilon)$ such that any interval of the real line of length $l(\varepsilon)$ contains at least a real σ for which*

$$\sup_{(x, X) \in \boldsymbol{R} \times \Omega} |a(x+\sigma, X) - a(x, X)| \leq \varepsilon .$$

The number σ is called an ε-period for a. The uniform dependence on parameters follows from the fact that $l(\varepsilon)$ and σ are independent of X.

The following two theorems contain classical results on almost periodic functions depending on parameters.

THEOREM 1.2. *If Ω is a closed and bounded set of $\mathbf{R}^{n-1}$, then an almost periodic function $a(x, X)$ in x uniformly with respect to $X \in \Omega$ is uniformly continuous and bounded on $\mathbf{R} \times \Omega$.*

In particular, from (α_2) it follows that

$$\sup_{x \in \mathbf{R}^n} a(x) := \overline{a} < \infty .$$

THEOREM 1.3. *A necessary and sufficient condition for a function $a(x, X)$ to be almost periodic in x uniformly with respect to $X \in \Omega$, where Ω is a closed and bounded set of $\mathbf{R}^{n-1}$, is that the family of its translates $\{a(x+\tau, X),\ \tau \in \mathbf{R}\}$ is (uniformly) precompact in $\mathcal{C}(\mathbf{R} \times \Omega, \mathbf{R})$.*

2. - Preliminary properties.

In this section we state some properties concerning the functional f that we will use to prove the main result. Some of these properties are classical results on homoclinic solutions, see e.g. [6] and [9], and therefore their proofs will be omitted.

Note first that (g_1) and (g_2) imply that

$$G(\xi) = o(|\xi|^2) \text{ and } g(\xi) = o(|\xi|) \text{ for } |\xi| \to 0 .$$

These properties will be often used in the sequel. It is well known that $u \equiv 0$ is a strict local minimum for f and that f satisfies the geometric assumption of the Mountain Pass Theorem. Indeed, by (g_1)-(g_3) and (α_2) it follows that

$$f(u) = \frac{1}{2}\|u\|^2 + o(\|u\|^2) \qquad \text{as } u \to 0$$

and that, by (g_2), if $u \in H \setminus \{0\}$ then $f(\lambda u) \to -\infty$ as $|\lambda| \to \infty$.

From (α_1), (α_2) we have that $0 < \underline{a} \leq a(x) \leq \overline{a}$, $\forall x \in \mathbf{R}^n$. In the sequel we will denote by A_a the set

$$A_a = \{\beta \in \mathcal{C}^{0,\gamma}(\mathbf{R}^n, \mathbf{R}) / \underline{a} \leq \beta(x) \leq \overline{a}, \quad \forall x \in \mathbf{R}^n\},$$

and for every $\beta \in A_a$ we define a functional $f(\beta, \cdot) \in \mathcal{C}^1(H, \mathbf{R})$ by setting

$$f(\beta, u) = \frac{1}{2}\int_{\mathbf{R}^n} (|\nabla u|^2 + |u|^2)\, dx - \int_{\mathbf{R}^n} \beta G(u)\, dx = \frac{1}{2}\|u\|^2 - \int_{\mathbf{R}^n} \beta G(u)\, dx .$$

The expression $\nabla f(\beta, u)$ stands for $\nabla_u f(\beta, u)$. Note that in particular we have $f(\alpha, u) = f(u)$.

In the proofs that follow, we will take $n > 2$, the case $n = 2$ being not more difficult to prove.

As first results concerning $f(\beta, \cdot)$ we have

LEMMA 2.1. *For* $\beta \in A_\alpha$, *let* $\mathcal{K}_\beta = \{u \in H / u \not\equiv 0, \ \nabla f(\beta, u) = 0\}$. *Then we have:*

(i) $\inf_{\beta \in A_\alpha} \inf_{u \in \mathcal{K}_\beta} \|u\| > 0$,

(ii) $\inf_{\beta \in A_\alpha} \inf_{u \in \mathcal{K}_\beta} f(\beta, u) > 0$.

PROOF. In [6] it is proved that there is a $\mu > 0$ such that if $u \in \mathcal{K}_\beta$ then $\|u\| \geqslant \mu$. A simple uniformity argument, as in [17], shows that μ is independent of $\beta \in A_\alpha$. To prove (ii), using (g_2), for $u \in \mathcal{K}_\beta$ we have

$$f(\beta, u) = f(\beta, u) - \frac{1}{\theta} \nabla f(\beta, u) \cdot u \geqslant \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u\|^2 .$$

Then, using (i), the proof is complete. ■

PROPOSITION 2.1. *For every* $\beta \in A_\alpha$, $\nabla f(\beta, \cdot)$ *is weakly continuous, in the sense that if* $u_m \rightharpoonup u$ *weakly in* H *then* $\nabla f(\beta, u_m) \rightharpoonup \nabla f(\beta, u)$ *weakly in* H.

PROOF. Let

$$J_\beta(u) := \int_{\mathbf{R}^n} \beta G(u) \, dx .$$

Then, since

$$f(\beta, u) = \frac{1}{2} \|u\|^2 - J_\beta(u)$$

and the quadratic part has the desired property, we only have to check that the same holds for J_β. To this aim, first note that (g_1)-(g_2) imply that there exist positive constants a_3 and a_4 such that

$$|g(\xi)| \leqslant a_3 |\xi| + a_4 |\xi|^s \tag{2.1}$$

for all $\xi \in \boldsymbol{R}$, where $s \in (1, (n+2)/(n-2))$. Pick $v \in H$, fix $\varepsilon > 0$ and let $R_\varepsilon > 0$ be so large that $\int_{|x| > R_\varepsilon} |v|^2 dx \leq \varepsilon^2$. Then, if $u_m \rightharpoonup u$, we have, using Hölder inequality,

$$|(\nabla J_\beta(u_m) - \nabla J_\beta(u)) \cdot v| = \left| \int_{\boldsymbol{R}^n} \beta(g(u_m) - g(u))\, v\, dx \right| \leq$$

$$\leq \bar{a} \int_{|x| \leq R_\varepsilon} |g(u_m) - g(u)|\, |v|\, dx + \varepsilon \bar{a} \left(\int_{|x| > R_\varepsilon} |g(u_m) - g(u)|^2 dx \right)^{1/2}.$$

The second integral is bounded independently of m (using (2.1), Sobolev inequality and the fact that (u_m) is bounded in H) while for the first one, from (2.1) and Hölder inequality, we have

$$\int_{|x| \leq R_\varepsilon} |g(u_m) - g(u)|\, |v|\, dx \leq$$

$$\leq \int_{|x| \leq R_\varepsilon} [a_3 + a_4(|u| + |u_m|)^{4/(n-2)}]\, |u_m - u|\, |v|\, dx \leq$$

$$\leq \left(\int_{|x| \leq R_\varepsilon} [a_3 + a_4(|u| + |u_m|)^{4/(n-2)}]^p\, dx \right)^{1/p} \left(\int_{|x| \leq R_\varepsilon} |u_m - u|^q\, |v|^q dx \right)^{1/q}$$

with $q \in (2, 2^*)$ where $2^* = 2n/(n-2)$. Since $u_m \to u$ in L^q_{loc} for all $q \in [1, 2^*)$, using Sobolev inequality and the fact that (u_m) is bounded in H, we obtain that the first integral tends to zero. This completes the proof. ■

We now begin the study of the Palais-Smale sequences for the functionals $f(\beta, \cdot)$. To this aim we first prove some technical results.

LEMMA 2.2. *Let $(v_m)_m \subset H$ be a sequence such that $v_m \rightharpoonup v_0$ weakly in H. Then*

(i) $\sup_{\beta \in A_a} |f(\beta, v_m - v_0) - f(\beta, v_m) + f(\beta, v_0)| \to 0$,

(ii) $\sup_{\beta \in A_a} \|\nabla f(\beta, v_m - v_0) - \nabla f(\beta, v_m) + \nabla f(\beta, v_0)\| \to 0$,

as $m \to \infty$.

PROOF. Since

$$f(\beta, v_m - v_0) - f(\beta, v_m) + f(\beta, v_0) =$$

$$= \frac{1}{2}(\|v_m - v_0\|^2 - \|v_m\|^2 + \|v_0\|^2) - \int_{\boldsymbol{R}^n} \beta[G(v_m - v_0) - G(v_m) + G(v_0)]\, dx =$$

$$= o(1) - \int_{\boldsymbol{R}^n} \beta[G(v_m - v_0) - G(v_m) + G(v_0)] dx$$

for $m \to \infty$, and likewise, for all $v \in H$,

$$[\nabla f(\beta, v_m - v_0) - \nabla f(\beta, v_m) + \nabla f(\beta, v_0)] \cdot v =$$

$$= - \int_{\boldsymbol{R}^n} \beta[g(v_m - v_0) - g(v_m) + g(v_0)]\, v\, dx\,,$$

it suffices to prove that

$$\text{(i)}' \quad \sup_{\beta \in A_\alpha} \left| \int_{\boldsymbol{R}^n} \beta[G(v_m - v_0) - G(v_m) + G(v_0)]\, dx \right| \to 0$$

and

$$\text{(ii)}' \quad \sup_{\beta \in A_\alpha} \sup_{\|v\| = 1} \left| \int_{\boldsymbol{R}^n} \beta[g(v_m - v_0) - g(v_m) + G(v_0)]\, v\, dx \right| \to 0$$

as $m \to \infty$.

To this aim, first note that (g_1)-(g_3) imply that there are two positive constants a_5 and a_6 such that

$$G(\xi) \leqslant a_5 |\xi|^2 + a_6 |\xi|^{s+1} \tag{2.2}$$

for all $\xi \in \boldsymbol{R}$. Fix $\varepsilon > 0$ and let $R > 0$ be free for the moment. Split the integral in (i)′ as $\int_{|x| \leqslant R} + \int_{|x| > R}$ and note that for any fixed R the first integral tends to zero as $m \to \infty$. Indeed, by (2.1)-(2.2) and the Mean

Value Theorem, we have

$$\left| \int_{|x| \leqslant R} \beta[G(v_m - v_0) - G(v_m) + G(v_0)]\, dx \right| \leqslant$$

$$\leqslant \bar{\alpha} \int_{|x| \leqslant R} G(v_m - v_0)\, dx + \bar{\alpha} \int_{|x| \leqslant R} \left| \int_0^1 g(v_m + t(v_0 - v_m))(v_0 - v_m)\, dt \right| dx \leqslant$$

$$\leqslant \bar{\alpha} \int_{|x| \leqslant R} (a_5(|v_m - v_0|^2 + a_6 |v_m - v_0|^{s+1})\, dx +$$

$$+ \bar{\alpha} \int_{|x| \leqslant R} (a_3(|v_m| + |v_m - v_0|) + a_4(|v_m| + |v_m - v_0|)^s) |v_m - v_0|\, dx\,.$$

Using Hölder inequality and the fact that $v_m \to v_0$ in L^q_{loc} for all $q \in [1, 2^*)$, we have that the first integral tends to zero. Therefore, to prove (i)$'$, it is now enough to show that, given $\varepsilon > 0$, we can find $R = R_\varepsilon$ such that the second integral is less than ε for all m. Let R_ε be so large that $\|v_0\|_{H^1(|x| > R_\varepsilon)} \leqslant \varepsilon$. Then, using Sobolev inequality and (2.2), we obtain

$$\left| \int_{|x| > R_\varepsilon} G(v_0)\, dx \right| \leqslant c(\|v_0\|^2_{H^1(|x| > R_\varepsilon)} + \|v_0\|^{s+1}_{H^1(|x| > R_\varepsilon)}) \leqslant c\varepsilon\,.$$

Moreover, using the Mean Value Theorem and (2.1), we have

$$\left| \int_{|x| > R_\varepsilon} G(v_m - v_0) - G(v_m)\, dx \right| \leqslant \int_{|x| > R_\varepsilon} \left(\int_0^1 |g(v_m - tv_0)|\, dt \right) |v_0|\, dx \leqslant$$

$$\leqslant \int_{|x| > R_\varepsilon} (a_3(|v_m| + |v_0|) + a_4(|v_m| + |v_0|)^s)\, |v_0|\, dx$$

and, using Hölder and Sobolev inequalities and the fact that (v_m) is bounded in H, the conclusion follows. The proof of (ii)$'$ is analogous. ■

We can now begin to describe the Palais-Smale sequences for the functional f. The next result is the first step in this direction.

PROPOSITION 2.2. *Let $(u_m)_m \subset H$ be a Palais-Smale sequence for f at level $c \in \boldsymbol{R}$, that is,*

$$f(u_m) \to c \text{ and } \nabla f(u_m) \to 0 \text{ in } H, \quad \text{as } m \to \infty .$$

Then there exists a subsequence (still denoted u_m) and $u_0 \in H$ such that

(i) $u_m \rightharpoonup u_0$ *weakly in H,*

(ii) $\nabla f(u_0) = 0$,

(iii) $(u_m - u_0)$ *is a Palais-Smale sequence for f at level $c - f(u_0)$.*

PROOF. To prove (i), note that by (g_2), for m large we have

$$c + o(1) + \frac{1}{\theta}\|u_m\| \geq f(u_m) - \frac{1}{\theta}\nabla f(u_m)\cdot u_m \geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_m\|^2 ,$$

and hence (u_m) is bounded in H. Then there exist a subsequence, still denoted with (u_m), and $u_0 \in H$ such that $u_m \rightharpoonup u_0$ weakly in H.

Now, since $u_m \rightharpoonup u_0$ weakly in H and ∇f is weakly continuous, we obtain (ii).

Finally, to prove (iii) it is enough to use Lemma 2.2, with $\beta = \alpha$. Indeed we have

$$f(u_m - u_0) = f(u_m) - f(u_0) + o(1) = c - f(u_0) + o(1)$$

and

$$\nabla f(u_m - u_0) = \nabla f(u_m) - \nabla f(u_0) + o(1) = o(1)$$

as $m \to \infty$. This completes the proof. ∎

REMARK 2.1. Since f satisfies the geometric assumptions of the Mountain Pass Theorem, we can find a Palais-Smale sequence (u_m) for f at some level $c > 0$. By Proposition 2.2, there is a subsequence (u_m) converging weakly to some $u_0 \in H$, which is a critical point for f. If $u_0 \not\equiv 0$ we have found a solution of (P). Therefore, in the sequel, we will always consider Palais-Smale sequences converging weakly to zero.

We now turn to a non vanishing property of Palais-Smale sequences.

Using the Concentration-Compacteness Principle (see [8]), we can prove, exactly as in [6], the following result:

PROPOSITION 2.3. *Let $(u_m) \subset H$ be a Palais-Smale sequence for f at level $c \in \boldsymbol{R}$. Then either*

(i) *$u_m \to 0$ strongly in H, or*

(ii) *there is a sequence $(y_m) \subset \boldsymbol{R}^n$ and $R > 0$ such that*

$$\liminf_{m \to \infty} \int_{B_R(y_m)} |u_m|^2 \, dx > 0 ,$$

where $B_R(y_m)$ denotes the open ball of radius R centered in y_m.

In our case, Proposition 2.3 takes the following form:

PROPOSITION 2.4. *Let (u_m) $(u_m \rightharpoonup 0)$ be a Palais-Smale sequence for f at level $c > 0$, then there exist a sequence $(y_m) \in \boldsymbol{R}^n$ and $R > 0$ such that*

$$\liminf_{m \to \infty} \int_{B_R(y_m)} |u_m|^2 \, dx > 0 . \tag{2.3}$$

and $|y_m| \to \infty$.

PROOF. The first part follows immediately from Proposition 2.3, since $f(u_m) \to c > 0$ and then (u_m) cannot tend to zero in H.

Next we have $|y_m| \to \infty$ since $u_m \rightharpoonup 0$ weakly in H. ■

Now we turn to some continuity properties concerning the functional $f(\beta, u)$ with respect to $\beta \in A_a$ on bounded subsets of H.

LEMMA 2.3. *Let $\beta_1, \beta_2 \in L^\infty(\boldsymbol{R}^n, \boldsymbol{R})$ and let $\mathcal{B}$ be a bounded subset of H. Then there is a constant $C > 0$, depending only on $\mathcal{B}$, such that for all $u \in \mathcal{B}$ there results*

(i) $|f(\beta_1, u) - f(\beta_2, u)| \leq C \|\beta_1 - \beta_2\|_\infty$,

(ii) $\|\nabla f(\beta_1, u) - \nabla f(\beta_2, u)\| \leq C \|\beta_1 - \beta_2\|_\infty$.

PROOF. Let us start with (i). We have

$$|f(\beta_1, u) - f(\beta_2, u)| = \left| \int_{\boldsymbol{R}^n} (\beta_1 - \beta_2) G(u) \, dx \right| \leq$$

$$\leq \|\beta_1 - \beta_2\|_\infty \sup_{u \in \mathcal{B}} \int_{\boldsymbol{R}^n} G(u) \, dx ,$$

and, using (2.2), we obtain

$$\int_{\mathbf{R}^n} G(u)\,dx \leqslant a_5 \|u\|^2_{L^2(\mathbf{R}^n)} + a_6 \|u\|^{2^*}_{L^{2^*}(\mathbf{R}^n)} .$$

Now, since $\mathcal{B}$ is bounded, by the Sobolev inequality, this means that

$$\sup_{u \in \mathcal{B}} \int_{\mathbf{R}^n} G(u)\,dx \leqslant C$$

where $C = C(\mathcal{B})$.

For the second part, with a similar calculation we have

$$\|\nabla f(\beta_1, u) - \nabla f(\beta_2, u)\| \leqslant \|\beta_1 - \beta_2\|_\infty \sup_{u \in B} \left(\int_{\mathbf{R}^n} |g(u)|^2\,dx \right)^{1/2}$$

and the conclusion follows as above, by (2.1). ■

REMARK 2.2. Note that if (β_m) is a sequence in L^∞ such that $\beta_m \to \beta$ in L^∞ and (u_m) is a bounded sequence in H, then

$$|f(\beta_m, u_m) - f(\beta, u_m)| \to 0$$

and

$$\|\nabla f(\beta_m, u_m) - \nabla f(\beta, u_m)\| \to 0 .$$

For every $\tau \in \mathbf{R}^n$ we define an isometry $T_\tau : H \to H$ by setting for a.e. $x \in \mathbf{R}^n$

$$T_\tau u(x) = u(x + \tau), \qquad \forall u \in H .$$

With some trivial changes of variable, it is immediate to see that

$$f(\beta, T_\tau u) = f(T_{-\tau}\beta, u),$$

in particular $f(T_\tau \beta, T_\tau u) = f(\beta, u)$, and that

$$\nabla f(\beta, T_\tau u) \cdot v = \nabla f(T_{-\tau}\beta, u) \cdot T_{-\tau} v , \qquad \forall v \in H ,$$

which also yields $\nabla f(T_\tau \beta, T_\tau u) \cdot T_\tau v = \nabla f(\beta, u) \cdot v$.

The next lemma is the last step that allows us to completely describe the Palais-Smale sequences for f.

LEMMA 2.4. *Let $(u_m) \subset H$ $(u_m \rightharpoonup 0)$ be a Palais-Smale sequence for f at level $c > 0$. Then there exist a function $\beta_1 \in A_\alpha$, a function $v_1 \in H$, $v_1 \not\equiv 0$ and a sequence $(\tau_m) \subset \boldsymbol{R}^n$ such that for a subsequence of $(T_{\tau_m} u_m)$, still denoted $T_{\tau_m} u_m$, the following properties are satisfied:*

(i) $T_{\tau_m} u_m \rightharpoonup v_1$ *weakly in* H,

(ii) $\nabla f(\beta_1, v_1) = 0$,

(iii) $|\tau_m| \to \infty$,

(iv) $(u_m - T_{-\tau_m} v_1)$ *is a Palais-Smale sequence for f at level* $c - f(\beta_1, v_1)$.

PROOF. Let $(y_m) \subset \boldsymbol{R}^n$ be the sequence given by Proposition 2.4. For each $m \in \boldsymbol{N}$, choose $k_m = (k_m^2, \dots, k_m^n) \in \boldsymbol{Z}^{n-1}$ such that

$$|(y_m^1, k_m^2 T_2, \dots, k_m^n T_n)| =$$

$$= \min \{ |(y_m^1, k^2 T_2, \dots, k^n T_n) - y_m| / k = (k^2, \dots, k^n) \in \boldsymbol{Z}^{n-1} \},$$

where $y_m = (y_m^1, \dots, y_m^n)$, and set

$$\tau_m := (y_m^1, k_m^2 T_2, \dots, k_m^n T_n) \in \boldsymbol{R}^n .$$

Then, by Proposition 2.4, (iii) is satisfied. Since the sequence $(T_{\tau_m} u_m)$ is bounded in H it contains a subsequence, still denoted $T_{\tau_m} u_m$, such that $T_{\tau_m} u_m \rightharpoonup v_1$ weakly, for some $v_1 \in H$. By Proposition 2.4, $v_1 \not\equiv 0$. Indeed, by definition, we have

$$|\tau_m - y_m| \leqslant \frac{1}{2} \left(\sum_{j=2}^{n} T_j^2 \right)^{1/2} := T$$

and then

$$\int_{B_{R+T}(0)} |T_{\tau_m} u_m|^2 \, dx \geqslant \int_{B_R(y_m)} |u_m|^2 \, dx \geqslant a > 0, \qquad \forall m .$$

This proves that no subsequence of $(T_{\tau_m} u_m)$ tends weakly to zero and, in particular, that $v_1 \not\equiv 0$. Therefore (i) is satisfied.

Consider now the (sub) sequence $T_{\tau_m} \alpha$: by (α_2) and the definition of τ_m, for all $x \in \boldsymbol{R}^n$ we have

$$T_{\tau_m} \alpha(x) = \alpha(x^1 + y_m^1, x^2 + k_m^2 T_2, \dots, x^n + k_m^n T_n) =$$

$$= \alpha(x^1 + y_m^1, x^2, \dots, x^n) .$$

Since we can assume, by periodicity, $(x^2, \dots, x^n) \in [0, T_2] \times \dots \times [0, T_n]$, by Theorem 1.3 there exist a subsequence of (τ_m), still denoted τ_m, and a function $\beta_1 \in \mathcal{C}(\boldsymbol{R}^n, \boldsymbol{R})$ such that $\tau_m a \to \beta_1$ uniformly in $\boldsymbol{R}^n$ (which also implies $\beta_1 \in A_a$).

Let us prove that (ii) and (iv) hold. For all $v \in H$, by Proposition 2.1, we have

$$\nabla f(\beta_1, v_1) \cdot v = \lim_m \nabla f(\beta_1, T_{\tau_m} u_m) \cdot v =$$

$$= \lim_m [\nabla f(\beta_1, T_{\tau_m} u_m) \cdot v - \nabla f(T_{\tau_m} a, T_{\tau_m} u_m) \cdot v] +$$

$$+ \lim_m \nabla f(T_{\tau_m} a, T_{\tau_m} u_m) \cdot v = 0 .$$

Indeed the first term vanishes by Lemma 2.3, while the second is zero since

$$\nabla f(T_{\tau_m} a, T_{\tau_m} u_m) \cdot v = \nabla f(a, u_m) \cdot T_{-\tau_m} v ,$$

(u_m) is a Palais-Smale sequence for $f = f(a, \cdot)$ and $T_{-\tau_m} v$ is bounded. Then (ii) is proved. Let us turn to (iv). We have

$$f(u_m - T_{-\tau_m} v_1) - f(u_m) + f(\beta_1, v_1) =$$

$$= [f(T_{\tau_m} a, T_{\tau_m} u_m - v_1) - f(T_{\tau_m} a, T_{\tau_m} u_m) +$$

$$+ f(T_{\tau_m} a, v_1)] - [f(T_{\tau_m} a, v_1) - f(\beta_1, v_1)] = o(1)$$

as one can immediately see by using Lemmas 2.2 and 2.3. Then $f(u_m - T_{-\tau_m} v_1) \to c - f(\beta_1, v_1)$ as $m \to \infty$. Finally, note that for all $v \in H$ it results, using Lemma 2.2,

$$\nabla f(u_m - T_{-\tau_m} v_1) \cdot v = \nabla f(u_m - T_{-\tau_m} v_1) \cdot v - \nabla f(u_m) \cdot v +$$

$$+ \nabla f(T_{\tau_m} a, v_1) \cdot T_{\tau_m} v + o(1) = \nabla f(T_{\tau_m} a, T_{\tau_m} u_m - v_1) \cdot T_{\tau_m} v -$$

$$- \nabla f(T_{\tau_m} a, T_{\tau_m} u_m) \cdot T_{\tau_m} v + \nabla f(T_{\tau_m} a, v_1) \cdot T_{\tau_m} v + o(1) = o(1) . \quad \blacksquare$$

We can now prove the main result of this section.

PROPOSITION 2.5 (Representation Lemma). *Let $(u_m) \subset H$ $(u_m \rightharpoonup 0)$ be a Palais-Smale sequence for f at level $c > 0$. Then there exist a number $q \in \boldsymbol{N}$, depending only on c, q functions $\beta_i \in A_a$, q functions $v_i \in H$,*

$v_i \not\equiv 0$, a subsequence of (u_m), still denoted u_m, and q sequences $(\theta_m^i) \subset \mathbf{R}^n$ such that

(i) $u_m - \sum_{i=1}^{q} T_{\theta_m^i} v_i \to 0$ *in H as $m \to \infty$,*

(ii) $\nabla f(\beta_i, v_i) = 0$, *for all $i = 1, \dots, q$,*

(iii) $c = \sum_{i=1}^{q} f(\beta_i, v_i)$,

(iv) $|\theta_m^i| \to \infty$ *for all $i = 1, \dots, q$.*

PROOF. The proof uses a standard technique. Being very short, we give it for completeness.

Applying Lemma 2.4 we find a subsequence of (u_m), a function $\beta_1 \in A_\alpha$, $v_1 \in H$ with $v_1 \not\equiv 0$ and a sequence $(\tau_m) \subset \mathbf{R}^n$ such that, setting $\theta_m^1 = -\tau_m$, we have

$$u_m - T_{\theta_m^1} v_1 \rightharpoonup 0 \text{ weakly in } H,$$

$$\nabla f(\beta_1, v_1) = 0,$$

$$|\theta_m^1| \to \infty,$$

$$f(u_m - T_{\theta_m^1} v_1) \to c - f(\beta_1, v_1),$$

$$\nabla f(u_m - T_{\theta_m^1} v_1) \to 0 \text{ in } H,$$

as $m \to \infty$. Therefore $(u_m - T_{\theta_m^1} v_1)$ is a Palais-Smale sequence for f at level $c - f(\beta_1, v_1) \geq 0$ (Lemma 2.1), this implies that $f(\beta_1, v_1) \leq c$. Now two different cases must be considered. If $f(\beta_1, v_1) = c$, then $f(u_m - T_{\theta_m^1} v_1) \to 0$, which implies that $\|u_m - T_{\theta_m^1} v_1\| \to 0$, so that the proposition is proved with $q = 1$. If $f(\beta_1, v_1) = c_1 < c$ then $u_m^1 = u_m - T_{\theta_m^1} v_1$ is a Palais-Smale sequence for f at level $c - c^1 > 0$. In this case we iterate the application of Lemma 2.4, starting with the Palais-Smale sequence (u_m^1).

To prove that this procedure ends, it is enough to show that for some $q \in \mathbf{N}$ we obtain $f(\beta_q, v_q) = c_q = c - c_1 - \dots - c_{q-1}$. But this follows from Lemma 2.1: indeed for all $i = 1, 2, \dots$ we have

$$c_i = f(\beta_i, v_i) \geq \inf_{\beta \in A_\alpha} \inf_{v \in \mathcal{K}_\beta} f(\beta, v) := b > 0$$

so that after at most $q := [c/b] + 1$ we obtain $c - c_1 - \dots - c_q = 0$. This completes the proof. ■

In order to conclude this section we recall a result, which can be found essentially in [4] and that will be used in the end of the proof. We state this result in a particular (and useful for our purposes) form that the reader can find, with its proof, in [17].

THEOREM 2.1. *Let $f \in \mathcal{C}^1(H, \boldsymbol{R})$ and let Γ be a class of subsets of H for which*

$$c = \inf_{A \in \Gamma} \sup_{x \in A} f(x) \in \boldsymbol{R}.$$

Assume that there exists $\varepsilon_0 > 0$ such that Γ is invariant for all deformations η with the property that $\eta(\cdot, t)$ is the identity in $\{u \in H \mid f(u) \geq c + \varepsilon_0$ or $f(u) \leq c - \varepsilon_0\}$ (i.e. if $A \in \Gamma$ then $\eta(A, t) \in \Gamma$ for all $t \in [0, 1]$). Then, for all $\varepsilon \in (0, \varepsilon_0)$, there exists a sequence (u_m) in H such that

(i) $\lim_{m \to \infty} f(u_m) \in [c - \varepsilon, c + \varepsilon]$,

(ii) $\lim_{m \to \infty} \nabla f(u_m) = 0$,

(iii) $\lim_{m \to \infty} \|u_m - u_{m-1}\| = 0$.

REMARK 2.3. Since the functional f satisfies the geometric assumptions of the Mountain Pass Theorem, by the previous result we can find a Palais-Smale sequence (u_m) for f at some level $c > 0$ with the further property $\|u_m - u_{m-1}\| \to 0$. This property will be essential for our aim.

3. – The main result.

In this section we examine some qualitative properties of the Palais-Smale sequences for f which are the fundamental arguments to prove the existence of a solution to problem (P). First, we need to fix some notation.

Define

$$\mathcal{K}_\infty := \{v \in H / v \not\equiv 0,\ \exists \beta \in A_\alpha,\ \nabla f(\beta, v) = 0\},$$

where we recall that $A_\alpha = \{\beta \in \mathcal{C}^{0,\gamma}(\boldsymbol{R}^n, \boldsymbol{R}) / \underline{\alpha} \leq \beta(x) \leq \overline{\alpha},\ \forall x \in \boldsymbol{R}^n\}$, and for all $q \in \boldsymbol{N}$ define

$$\mathcal{Q}_\infty := \{\varphi \in H / \varphi(x) = \sum_{i=1}^{r} T_{\theta_i} v_i,\ v_i \in \mathcal{K}_\infty,\ \theta_i \in \boldsymbol{R}^n,\ \forall i = 1, \ldots, r, \text{ and } r \leq q\}.$$

REMARK 3.1. Note that the Representation Lemma implies that the Palais-Smale sequences for f are, up to negligible quantities in H, sums of translates of solutions to problems $\nabla f(\beta, \cdot) = 0$, where β is a uniform limit of translates of α in $\mathbf{R}^n$ (and then $\beta \in A_\alpha$). Therefore, using the previous notations, we obtain that if (u_m) is a Palais-Smale sequence for f at level $c > 0$ (with $u_m \rightharpoonup 0$) then there exist $q \in \mathbf{N}$, depending only on $c > 0$, a subsequence (u_{m_k}) and a sequence $(\varphi_k) \subset \mathcal{Q}_\infty$ such that $\|u_{m_k} - \varphi_k\| \to 0$, i.e. $\mathrm{dist}(u_{m_k}, \mathcal{Q}_\infty) \to 0$. This shows that there are no subsequences of (u_m) which do not converge to $\mathcal{Q}_\infty$ and therefore that

$$\mathrm{dist}(u_m, \mathcal{Q}_\infty) \to 0, \qquad \text{as } m \to \infty .$$

REMARK 3.2. By classical regularity results (see e.g. [7] and [14] or [6]), it can be proved that if $v \in \mathcal{K}_\infty$ then $v \in \mathcal{C}^2(\mathbf{R}^n)$ and, if $\nabla f(\beta, v) = 0$, v solves

$$\begin{cases} -\Delta v + v = \beta(x) g(v), \\ v(x) \to 0 \quad \text{as } |x| \to \infty . \end{cases}$$

Moreover, we can assume, without loss of generality, that every $v \in \mathcal{K}_\infty$ is positive (see e.g. [6]) and a fortiori every $\varphi \in \mathcal{Q}_\infty$.

The next result contains an uniform qualitative property of the elements in $\mathcal{Q}_\infty$.

PROPOSITION 3.1. *There exists $\delta > 0$ such that*

$$\varphi(x) < \delta \textit{ implies } \Delta\varphi(x) > \frac{1}{2}\varphi(x), \qquad \textit{for all } \varphi \in \mathcal{Q}_\infty .$$

PROOF. Choose any $\varphi \in \mathcal{Q}_\infty$, that is

$$\varphi = \sum_{i=1}^{r} T_{\theta_i} v_i$$

for some $v_i \in \mathcal{K}_\infty$ and $\theta_i \in \mathbf{R}^n$. Then, since $\nabla f(\beta_i, v_i) = 0$, for some $\beta_i \in A_\alpha$, using Remark 3.2 we obtain

$$\Delta\varphi = \sum_{i=1}^{r} T_{\theta_i} \Delta v_i = \sum_{i=1}^{r} [T_{\theta_i} v_i - T_{\theta_i} \beta_i \cdot g(T_{\theta_i} v_i)] \geq \varphi - \overline{\alpha} \sum_{i=1}^{r} g(T_{\theta_i} v_i).$$

Now let $\varepsilon > 0$ be so small that $1 - \varepsilon\overline{\alpha} > (1/2)$ and take $\delta > 0$ such that $|\xi| < \delta$ implies $|g(\xi)| \leq \varepsilon|\xi|$ (this is possible by (g_1)-(g_2)). Let $x \in \boldsymbol{R}^n$ such that $\varphi(x) < \delta$. Then, by Remark 3.2, $T_{\theta_i} v_i(x) < \delta$ for each $i = 1, \dots, r$, so that

$$\Delta\varphi(x) \geq (1 - \varepsilon\overline{\alpha})\,\varphi(x) > \frac{1}{2}\varphi(x).$$

Since δ does not depend on the choice of $\varphi \in \mathcal{Q}_\infty$, the proof is complete. ■

The following definition gives the essential ingredient to complete the proof.

DEFINITION 3.1. *For every $\varphi \in \mathcal{Q}_\infty$ let*

$$X^1(\varphi) := \max\{x^1 \in \boldsymbol{R} \mid \varphi(x^1, x') = \delta \textit{ for some } x' \in \boldsymbol{R}^{n-1}\},$$

where $\delta > 0$ is the real number given by Proposition 3.1.

Note that $X^1 \colon \mathcal{Q}_\infty \to \boldsymbol{R}$ is well defined (by Lemma 2.1 and the continuity of every $\varphi \in \mathcal{Q}_\infty$).

REMARK 3.3. Note that by definition, for every $\varphi \in \mathcal{Q}_\infty$ there exists at least some $x'(\varphi) \in \boldsymbol{R}^{n-1}$ such that $\varphi(X^1(\varphi), x'(\varphi)) = \delta$. We then set $x(\varphi) = (X^1(\varphi), x'(\varphi))$.

We next prove that the function X^1 is uniformly continuous on bounded subsets of $\mathcal{Q}_\infty$. To this aim some remarks are in order.

REMARK 3.4. From classical regularity arguments (see e.g. [7] and [14]), using (α_1) and (g_1), we obtain that for all $v \in \mathcal{K}_\infty$ and for every open ball $B \subset \boldsymbol{R}^n$ there results

$$\|v\|_{C^{2,\gamma}(B)} \leq C\|v\|,$$

where $C > 0$ depends on n, $\overline{\alpha}$, $\gamma \in (0, 1)$ and on diam(B). Then it follows that for all $\varphi \in \mathcal{Q}_\infty$ we have

$$\|\varphi\|_{C^{2,\gamma}(B)} \leq C_q\|\varphi\|$$

where C_q depends on the same quantities as C. Consider now a bounded subset $\mathcal{B}$ in $\mathcal{Q}_\infty$. From the previous estimates it follows that

$$\|\nabla\varphi\|_{L^\infty(\boldsymbol{R}^n)} \leq K$$

for every $\varphi \in \mathcal{B}$, where $K > 0$ does not depend on $\varphi \in \mathcal{B}$. Therefore, for every bounded subset $\mathcal{B} \subset \mathcal{Q}_\infty$ there results:

(i) there exists $K = K(\mathcal{B}) > 0$ such that $|\varphi(x) - \varphi(y)| \leqslant K|x-y|$ for every $x, y \in \boldsymbol{R}^n$ and $\varphi \in \mathcal{B}$, and in particular

(ii) for all $\varepsilon > 0$ there exists $r = r(\varepsilon, \mathcal{B})$ such that $|x - y| < r$ implies $|\varphi(x) - \varphi(y)| \leqslant \varepsilon$, for every $\varphi \in \mathcal{B}$.

Using the previous properties we can give an estimate of the L^∞-distance between elements in $\mathcal{Q}_\infty$.

LEMMA 3.1. *For every bounded subset $\mathcal{B}$ in $\mathcal{Q}_\infty$ there exists a constant $C = C(\mathcal{B}) > 0$ such that*

$$\|\varphi - \psi\|_{L^\infty(\boldsymbol{R}^n)} \leqslant C\|\varphi - \psi\|^{2/(n+2)}, \qquad \forall \varphi, \psi \in \mathcal{B}.$$

PROOF. Let $\varphi, \psi \in \mathcal{B}$ and $x_0 \in \boldsymbol{R}^n$. From Remark 3.4 (i) there exists $K > 0$ such that

$$|\varphi(x) - \varphi(x_0)| \leqslant K|x - x_0| \text{ and } |\psi(x) - \psi(x_0)| \leqslant K|x - x_0|, \quad \forall x \in \boldsymbol{R}^n.$$

Suppose that $\psi(x_0) > \varphi(x_0)$, and let $\varrho = (\psi(x_0) - \varphi(x_0))/4K$. Then, by the previous inequalities, for every $x \in \boldsymbol{R}^n$ such that $|x - x_0| \leqslant \varrho$ we have

$$\psi(x) - \varphi(x) \geqslant \psi(x_0) - \varphi(x_0) - 2K\varrho = \frac{\psi(x_0) - \varphi(x_0)}{2}.$$

It follows that

$$\|\psi - \varphi\|^2 \geqslant \int_{B_\varrho(x_0)} |\psi(x) - \varphi(x)|^2\, dx \geqslant \left(\frac{\psi(x_0) - \varphi(x_0)}{2}\right)^2 \omega_n \varrho^n =$$

$$= (\psi(x_0) - \varphi(x_0))^{n+2} \frac{\omega_n}{2^{2n+2} K^n},$$

where ω_n is the volume of unit ball in $\boldsymbol{R}^n$. Then, if $\psi(x_0) > \varphi(x_0)$ we find

$$\psi(x_0) - \varphi(x_0) \leqslant C\|\psi - \varphi\|^{2/(n+2)},$$

where $C > 0$ depends only on $\mathcal{B}$ and n, and likewise, if $\varphi(x_0) > \psi(x_0)$. Since x_0 is arbitrary in $\boldsymbol{R}^n$, the proof is complete. ■

We can now prove, using the previous result, the continuity of $X^1: \mathcal{Q}_\infty \to \boldsymbol{R}$.

PROPOSITION 3.2. *The function $X^1: \mathcal{Q}_\infty \subset H \to \boldsymbol{R}$ is uniformly continuous on bounded subsets of $\mathcal{Q}_\infty$.*

PROOF. First we prove, as in [12], an exponential estimate for the functions in $\mathcal{Q}_\infty$. Let $\varphi \in \mathcal{Q}_\infty$ and for every $x = (x^1, \dots, x^n) \in \boldsymbol{R}^n$ with $x^1 \geq X^1(\varphi)$ set

$$v(x) = \delta e^{-\omega(x^1 - X^1(\varphi))},$$

where $\omega \in (0, 1/\sqrt{2}]$ and let

$$\Omega = \{x \in \boldsymbol{R}^n \mid x^1 > X^1(\varphi)\}.$$

Then we have

$$\Delta v = \omega^2 v \leq \frac{1}{2} v, \quad \text{in } \Omega \quad \text{and} \quad v = \delta \quad \text{on } \partial\Omega.$$

Now, note that by definition we have $\varphi \leq \delta$ on $\partial\Omega$, and $\varphi < \delta$ in Ω so that, by Proposition 3.1, $\Delta\varphi > (1/2)\varphi$ in Ω. Then, setting $L = -\Delta + (1/2)I$, we obtain

$$L(v - \varphi) \geq 0 \quad \text{in } \Omega \quad \text{and} \quad v - \varphi \geq 0 \quad \text{on } \partial\Omega.$$

Moreover we have

$$\liminf_{\substack{|x| \to \infty \\ x \in \Omega}} (v(x) - \varphi(x)) = \liminf_{\substack{|x| \to \infty \\ x \in \Omega}} v(x) \geq 0.$$

Therefore, by the Maximum Principle applied to the unbounded domain Ω (see [13]), we obtain $v - \varphi \geq 0$ in Ω, that is,

$$\varphi(x^1, x') \leq \delta e^{-\omega(x^1 - X^1(\varphi))}, \quad \forall x^1 \geq X^1(\varphi),\ x' \in \boldsymbol{R}^{n-1}. \tag{3.1}$$

Using this estimate, we can now prove the continuity of X^1.

Let $\mathcal{B}$ be a bounded subset of $\mathcal{Q}_\infty$. Take $\varphi, \psi \in \mathcal{B}$ with $\|\varphi - \psi\|_{L^\infty(\boldsymbol{R}^n)} \leq \mu$, and let $x(\psi)$, $x(\varphi)$ be defined according to Remark 3.3. Then, if $X^1(\psi) > X^1(\varphi)$, by (3.1) we have

$$\mu > \psi(x(\psi)) - \varphi(x(\psi)) \geq \delta - \delta e^{-\omega(X^1(\psi) - X^1(\varphi))}$$

and then

$$X^1(\psi) - X^1(\varphi) \leq \frac{1}{\omega} \log \frac{\delta}{\delta - \mu},$$

and likewise if $X^1(\varphi) > X^1(\psi)$. Using this fact and Lemma 3.1 we conclude the proof. ■

Using this result we can prove the following proposition which contains the last property we need.

PROPOSITION 3.3. *Let (u_m) be a Palais-Smale sequence for f at level $c > 0$ $(u_m \rightharpoonup 0)$ such that*

$$\|u_m - u_{m-1}\| \to 0, \qquad \text{as } m \to \infty . \tag{3.2}$$

Then there exists a sequence (x_m) in $\mathbf{R}^n$ such that

(i) $\lim_{m\to\infty} |x_m^1 - x_{m-1}^1| = 0$, *where* $x_m = (x_m^1, \dots, x_m^n)$,

(ii) *there exists $R > 0$ such that* $\liminf_{m\to\infty} \int_{B_R(x_m)} |u_m|^2 dx > 0$,

(iii) $\lim_{m\to\infty} |x_m| = \infty$.

PROOF. By the Representation Lemma and as we note in Remark 3.1, we know that there is a sequence (φ_m) in $\mathcal{Q}_\infty$ such that

$$\|u_m - \varphi_m\| \to 0, \qquad \text{as } m \to \infty .$$

This implies, using (3.2), that

$$\|\varphi_m - \varphi_{m-1}\| \to 0 \qquad \text{as } m \to \infty . \tag{3.3}$$

Let $x_m = x(\varphi_m) \in \mathbf{R}^n$, where $x(\varphi_m)$ is defined as in Remark 3.3. Then $x_m^1 = X^1(\varphi_m)$ and since (φ_m) is bounded, by Proposition 3.2 and (3.3) we prove (i). Moreover, since by definition $\varphi_m(x_m) = \delta$ for every m, using the equicontinuity of (φ_m) (see Remark 3.4 (ii)), there exists $R > 0$ such that

$$\varphi_m(x) \geqslant \frac{\delta}{2}, \quad \forall x \in B_R(x_m). \tag{3.4}$$

Now, since we have

$$\int_{B_R(x_m)} |u_m|^2 dx = \int_{B_R(x_m)} |\varphi_m|^2 dx - \int_{B_R(x_m)} (|\varphi_m|^2 - |u_m|^2)\, dx =$$

$$= \int_{B_R(x_m)} |\varphi_m|^2 dx + o(1), \qquad \text{as } m \to \infty ,$$

by (3.4) we obtain (ii). Finally, (iii) follows (like in Proposition 2.4) from (ii) and the fact that $u_m \rightharpoonup 0$ as $m \to \infty$. ■

We can now prove the main result.

PROOF OF THEOREM 1.1. Since f satisfies the geometric assumptions of the Mountain Pass Theorem, using Theorem 2.1 we can find a Palais-Smale sequence (u_m) for f at some level $c > 0$ such that

$$\|u_m - u_{m-1}\| \to 0, \quad \text{as } m \to \infty .$$

Moreover, we know that (u_m) admits a subsequence converging weakly in H to a critical point u_0 for f (Proposition 2.1). If $u_0 \not\equiv 0$ we have found a solution of our problem (P). Otherwise, (u_m) satisfies the assumptions of the Proposition 3.3 and then there exists a sequence $(x_m) \subset \boldsymbol{R}^n$ such that

$$|x_m^1 - x_{m-1}^1| \to 0, \qquad \text{as } m \to \infty , \tag{3.5}$$

$$\exists R > 0, \exists a > 0 \quad \text{such that} \int_{B_R(x_m)} |u_m|^2 \, dx \geqslant a, \quad \forall m , \tag{3.6}$$

$$|x_m| \to \infty, \qquad \text{as } m \to \infty . \tag{3.7}$$

Now, let $k_m = (k_m^2, \dots, k_m^n) \in \boldsymbol{Z}^{n-1}$ such that

$$|(x_m^1, k_m^2 T_2, \dots, k_m^n T_n)| =$$

$$= \min \{ |(x_m^1, k^2 T_2, \dots, k^n T_n) - x_m| / k = (k^2, \dots, k^n) \in \boldsymbol{Z}^{n-1} \}$$

and set

$$\tau_m := (x_m^1, k_m^2 T_2, \dots, k_m^n T_n) \in \boldsymbol{R}^n .$$

Then, by definition we have

$$|\tau_m - x_m| \leqslant \frac{1}{2} \left(\sum_{j=2}^{n} T_j^2 \right)^{1/2} := T ,$$

therefore, setting $v_m = T_{\tau_m} u_m$, by (3.6) we have

$$\int_{B_{R+T}(0)} |v_m|^2 dx \geqslant \int_{B_R(x_m)} |u_m|^2 dx \geqslant a > 0, \qquad \forall m ,$$

and no subsequences of (v_m) converges weakly to zero in H.

Note also that by (3.7) we have $|\tau_m| \to \infty$. Moreover we can suppose that $|\tau_m^1| \to \infty$. Indeed, if there exists some subsequence $(\tau_{m_k}^1)$ which is bounded, then, by the choice of τ_m and the periodicity of $a(x^1, \cdot)$, the sequence $w_k(x) = u_{m_k}(x^1, x^2 + \tau_{m_k}^2, \dots, x^n + \tau_{m_k}^n)$ results a Palais-Smale sequence for f at level $c > 0$ which has no subsequence weakly convergent to zero.

Finally, by the almost periodicity of a in the first variable, there exists a sequence (σ_k) such that $|\sigma_k| \to \infty$ and

$$|a(x^1 + \sigma_k, x') - a(x^1, x')| \to 0, \qquad \text{as } k \to \infty,$$

uniformly in $\boldsymbol{R}^n$. Then, since $|\tau_m^1 - \tau_{m-1}^1| \to 0$ and $|\tau_m^1| \to \infty$ as $m \to \infty$, there exists a subsequence $(\tau_{m_k}^1)$ such that

$$|\tau_{m_k}^1 - \sigma_k| \to 0, \qquad \text{as } k \to \infty.$$

Moreover, since (v_{m_k}) is bounded, it contains a subsequence, still denoted v_{m_k}, such that

$$v_{m_k} \rightharpoonup v_0 \not\equiv 0, \qquad \text{weakly in } H.$$

We claim that v_0 is the desired solution of problem (P).

Indeed, first note that by the periodicity of a we have

$$\|T_{\tau_{m_k}} a - a\|_\infty \leqslant$$

$$\leqslant \|a(\cdot + \tau_{m_k}) - a(\cdot + \sigma_k, \cdot + \tau_{m_k}^2, \dots, \cdot + \tau_{m_k}^n)\|_\infty + \|a(\cdot + \sigma_k, \cdot, \dots, \cdot) - a\|_\infty$$

and therefore $\|T_{\tau_{m_k}} a - a\|_\infty \to 0$, since the first term tends to zero by the uniform continuity of a (Theorem 1.2) while the second by definition of σ_k. This implies that for every $w \in H$, by Proposition 2.1 and Lemma 2.3,

$$\nabla f(v_0) \cdot w = \lim_{k \to \infty} \nabla f(a, v_{m_k}) \cdot w = \lim_{k \to \infty} \nabla f(T_{\tau_{m_k}} a, v_{m_k}) \cdot w =$$

$$= \lim_{k \to \infty} \nabla f(a, u_{m_k}) \cdot T_{-\tau_{m_k}} w = 0,$$

because (u_{m_k}) is a Palais-Smale sequence for $f = f(a, \cdot)$. This proves that $v_0 \not\equiv 0$ is a critical point of f. ■

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Manoscritto pervenuto in redazione il 28 agosto 1995.