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The α-Approximation Property and the Weak Topology in Tensor Products of Banach Spaces.

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ABSTRACT - We use the α-approximation property for Banach spaces in order to extend particular results about the weak topology in tensor products. We also define and apply the α-density property for a couple of Banach spaces (E, F). We say that (E, F) has the α-density property if the equality between the (E, F')-component of the minimal and the maximal operator ideals associated to a tensor norm α holds. This property is closely related to the Radon-Nikodym property.

The aim of this paper is to study sufficient conditions to assure weak sequential completeness of tensor products of Banach spaces and weak relative compactness of bounded subsets in them. Heinrich [3] obtained conditions of weak sequential completeness under certain restrictions over the Banach spaces E and F involved in the tensor products $E \widehat{\otimes}_{g_p} F$ and $E \widehat{\otimes}_{\varepsilon_p} F$, where g_p is the Saphar's tensor norm, $1 \leq p < \infty$. Similar questions were treated by Lewis [5]. These results admit generalization to the family α_{pq} of Laprestés' tensor norms that can be found in [2]. Rivera [7] studied weak relative compactness of bounded subsets in some right-injective tensor products (also under restrictions over the Banach spaces E and F). The results of our paper provided characterizations of both properties in a broad class of topological tensor products under some further restrictions of their dual spaces. The basic idea of this paper is to use the following properties A) and B) of Banach spaces in order to obtain these results. We denoted by (B) the class of

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all Banach spaces and we write E' for the topological dual of $E \in (B)$. If α is a tensor norm, α' stands for the dual of α, $E \otimes_\alpha F$ for the α-tensor product and $E \hat{\otimes}_\alpha F$ for its completion. Properties A) and B) are

A) Let α be a finitely generated tensor norm. $F \in (B)$ is said to have the *α-approximation property* if for all $E \in (B)$ the natural mapping

$$E \hat{\otimes}_\alpha F \to E \hat{\otimes}_\varepsilon F$$

is injective. This definition can be found in [1], and generalizes the classical approximation property, the approximation property of order p of Saphar [9] and the approximation property of order (p, q) of Diaz, López Molina and Rivera [2].

B) Let α be a tensor norm. If $E, F \in (B)$, we say that the couple (E, F) has the *α-density property* if $E' \otimes F' \subset (E \hat{\otimes}_\alpha F)'$ is norm dense.

Sufficient conditions for both properties are known. For the first one, if $F \in (B)$ has the approximation property, then it has the α-approximation property for each finitely generated tensor norm α([1], Proposition 21.7). Other relations can be found between the α-approximation property and the accessibility of α. For the second property, well known sufficient conditions are related to the Radon-Nikodym property and approximation properties. The basic results in this direction were obtained by Lewis [5] (see also [1], [9] for Saphars' tensor norms and [1] for Laprestés' tensor norms).

Throughout this paper, the notation concerning Banach spaces and tensor products is standard. We begin with a general result on weak conditional compactness in topological tensor products.

THEOREM 1. *Let α be a finitely generated tensor norm and E, $F \in (B)$ such that F has the α-approximation property and (E, F) has the α-density property. Then a bounded subset $M \subset E \hat{\otimes}_\alpha F$ is weakly conditionally compact if*

1) *$\{z(x'), z \in M\} \subset F$ is weakly relatively compact for each $x' \in E'$.*

2) *$\{z^t(y'), z \in M\} \subset E$ is weakly conditionally compact for each $y' \in F'$.*

PROOF. Our proof starts with the observation that the injective tensor norm ε satisfies the assumptions of Proposition 1 in [7] over the tensor norm, since every bounded $\sigma(E \hat{\otimes}_\varepsilon F, E' \otimes F')$-convergent sequence in $E \widehat{\otimes}_\varepsilon F$ is weakly convergent (this was proved by Lewis in [4]).

Let $M \subset E \widehat{\otimes}_\alpha F$ a bounded subset satisfying conditions 1) and 2). Since F has the α-approximation property we can consider M as a bounded subset in $E \widehat{\otimes}_\varepsilon F$. Moreover, the first observation (Proposition 1 in [7]) shows that M is also weakly conditionally compact in $E \widehat{\otimes}_\varepsilon F$.

If (z_n) is a sequence in M we can assume, by passing to a subsequence if necessary, that (z_n) is weak Cauchy. Thus, for every $\sum_{i=1}^{h} x_i' \otimes y_i' \in E' \otimes F' \subset (E \widehat{\otimes}_\varepsilon F)'$, if $r \leqslant n \leqslant m$, r, n, $m \in \boldsymbol{N}$, $\left\langle z_n - z_m, \sum_{i=1}^{h} x_i' \otimes y_i' \right\rangle$ converges to 0 if $r \to \infty$.

Consider C such that $\alpha(z_n) < C$ for each $n \in \boldsymbol{N}$. If $\mu > 0$ and $f \in (E \widehat{\otimes}_\alpha F)'$, there exists a tensor $\sum_{i=1}^{k} x_i' \otimes y_i'$ such that $\alpha'\left(f - \sum_{i=1}^{k} x_i' \otimes y_i'\right) < \mu/4C$.

On the other hand, there is $n_0 \in \boldsymbol{N}$ such that for all n and m: $n_0 \leqslant n, m$

$$\left| \left\langle z_n - z_m, \sum_{i=1}^{k} x_i' \otimes y_i' \right\rangle \right| < \frac{\mu}{2}.$$

The following inequalities conclude the proof

$$|\langle z_n - z_m, f\rangle| \leqslant$$

$$\leqslant \left| \left\langle z_n - z_m, f - \sum_{i=1}^{k} x_i' \otimes y_i' \right\rangle \right| + \left| \left\langle z_n - z_m, \sum_{i=1}^{k} x_i' \otimes y_i' \right\rangle \right| < \mu. \qquad \blacksquare$$

REMARK 2. Theorem 1 is a generalization of Proposition 1 in [7] for any finitely generated tensor norm α. Since the proof depends on the injectivity of the map $E \widehat{\otimes}_\alpha F \to E \widehat{\otimes}_\varepsilon F$, the α-approximation property for F can be replaced by the approximation property for either E or F. Moreover, if E and F are dual spaces and α is totally accessible no approximation property is needed (see e.g. 33.1 [11]). Particular results about the α_{pq} tensor norms—applying the p-approximation property, the (p, q)-approximation property and the Radon-Nikodym property—that assure the α_{pq}-density property can be found in [2] and [9].

The reader can obtain in this way a broad class of applications of Theorem 1 and the next corollaries.

COROLLARY 3. *If α is a finitely generated tensor norm, $E, F \in (B)$, F is reflexive and has the α-approximation property, and (E, F) has the α-density property, then*

$$l_1 \not\subset E \widehat{\otimes}_\alpha F \quad \textit{iff } l_1 \not\subset E\,.$$

PROOF. If $l_1 \not\subset E \widehat{\otimes}_\alpha F$, then $l_1 \not\subset E$ since E is isometrically imbedded in the tensor product. Let M be a bounded subset in $E \widehat{\otimes}_\alpha F$. Condition 2) of Theorem 1 follows from Rosenthal's theorem: l_1 is not contained in E iff every bounded set of E is weakly conditionally compact [8]. If F is reflexive, then F is weakly sequentially complete and $l_1 \not\subset F$, and condition 1) is also verified. Then M is weakly conditionally compact and $l_1 \not\subset E \widehat{\otimes}_\alpha F$. ∎

COROLLARY 4. *If α is a finitely generated tensor norm, $E, F \in (B)$, $l_1 \not\subset E$, F is reflexive and has the α-approximation property, and (E, F) has the α-density property, then $E \widehat{\otimes}_\alpha F$ is reflexive iff it is weakly sequentially complete.*

PROOF. By Corollary 3 the unit ball of $E \widehat{\otimes}_\alpha F$ is weakly conditionally compact. Thus, $E \widehat{\otimes}_\alpha F$ is weakly sequentially compact since it is weakly sequentially complete. ∎

In the following we use the α-density property in order to obtain characterization of weak sequential completeness in certain topological tensor products. We also use the fact that under any restrictions on the Banach space E the natural map $E \widehat{\otimes}_\alpha F \to E \widehat{\otimes}_\varepsilon F$ is injective.

REMARK 5. Consider the tensor product $E \widehat{\otimes}_\alpha F$. If E has an unconditional basis, then the space E has the approximation property and then it has the α-approximation property [1]. Therefore, the canonical map $E \widehat{\otimes}_\alpha F \to E \widehat{\otimes}_\varepsilon F$ is injective for each finitely generated tensor norm. So, although this property is used in the proof of the following theorem, the α-approximation property for F is not required.

Next theorem is a generalization of Theorem 1 [3]. We use in its proof two lemmas from [3]. We write them here for the sake of completeness.

LEMMA 1 [3]. *Let B be a Banach space with an unconditional Schauder decomposition $\{Q_k\}$. If for any weak Cauchy sequence $\{z_n\} \subset \subset B$ there exists a z_0 such that for all k we have* w-lim $Q_k(z_n) = Q_k(z_0)$, *then B is weakly sequentially complete.*

If $\{e_k\}$ is an unconditional basis for E, we denote by $\{P_k\}$ the sequence of one-dimensional projection operators onto the unit vectors e_k.

LEMMA 2 [3]. *For any uniform crossnorm α on the tensor product $E \otimes F$ the sequence $\{P_k \widehat{\otimes}_\alpha I_F\}$ defines an unconditional Schauder decomposition of the space $E \widehat{\otimes}_\alpha F$.*

THEOREM 6. *Let α be a finitely generated tensor norm and $E, F \in \in (B)$ such that (E, F') has the α'-density property. Suppose that E has an unconditional basis (or E' has an unconditional retro-basis). Then the space $E' \widehat{\otimes}_\alpha F$ is weakly sequentially complete if and only if both spaces E' and F are weakly sequentially complete.*

PROOF. We prove the sufficiency of the conditions. Let $\{e_k\}$ be an unconditional basis of the space E. Let $\{e_k'\} \subset E'$ the sequence of coefficient functionals corresponding to the basis $\{e_k\}$. Since E' is weakly sequentially complete, it follows that $\{e_k'\}$ is an unconditional basis of E' ([6], p. 41). (If E' has by assumption an unconditional basis, we denote it $\{e_k'\}$).

Let $\{z_n\} \subset E' \widehat{\otimes}_\alpha F$ be a weak Cauchy sequence. The natural mapping

$$E' \widehat{\otimes}_\alpha F \to E' \widehat{\otimes}_\alpha F''$$

is always injective by the embedding Lemma 13.3 [1]—it's even an isometry—. Since E' has an unconditional basis we can apply Remark 5. Therefore, the map

$$E' \widehat{\otimes}_\alpha F'' \to (E \widehat{\otimes}_{\alpha'} F')'$$

is continuous, one to one and onto – since (E, F') has the α'-density property – and then it's an isomorphism. So we can also consider $\{z_n\}$ as a weak Cauchy sequence in $(E \widehat{\otimes}_{\alpha'} F')'$ and then as weak* Cauchy sequence. Since the unit ball is compact in the weak* topology, there exists a $z_0 \in E' \widehat{\otimes}_\alpha F''$ such that w*-lim $z_n = z_0$. Thus,

$$\lim_n \langle z_n, e_k \otimes y' \rangle = \langle z_0, e_k \otimes y' \rangle \tag{1}$$

for each e_k, $k \in \mathbb{N}$, and each $y' \in F'$. (If E' has by assumption an unconditional retro-basis $\{e_k'\}$, then $e_k'' \in E$ for each k and we can write $e_k'' = e_k$).

Now we apply Lemma 2 from [3], and we obtain that the sequence $\{P_k' \widehat{\otimes}_\alpha I_F\}$ defines an unconditional Schauder decomposition of the space $E' \otimes_\alpha F$. This decomposition is $\{P_k' \widehat{\otimes}_\alpha I_{F''}\}$ for $E' \widehat{\otimes}_\alpha F''$.

Therefore we can expand each z_n as

$$z_n = \sum_{k=1}^{\infty} (P_k' \widehat{\otimes}_\alpha I_F) z_n .$$

Fix a representation of z_n as $\sum_{i=1}^{\infty} x_i' \otimes y_i'$. Each x_i' can be written as $\sum_{j=1}^{\infty} \lambda_j^i e_j'$, and for every k,

$$(P_k' \otimes_\alpha I_F) z_n = \sum_{i=1}^{\infty} P_k'(x_i') \otimes y_i = \sum_{i=1}^{\infty} \lambda_k^i e_k' \otimes y_i = e_k' \otimes \left(\sum_{i=1}^{\infty} \lambda_k^i y_i \right).$$

Last equality only can be written if $\sum_{i=1}^{\infty} \lambda_k^i y_i \in F$. But if we consider the injective map $\widehat{I}$: $E' \widehat{\otimes}_\alpha F \to E' \widehat{\otimes}_\varepsilon F$, the following inequalities hold

$$\left\| \sum_{i=1}^{\infty} \lambda_k^i y_i \right\| = \sup_{y' \in B_{F'}} \left| \left\langle \sum_{i=1}^{\infty} \lambda_k^i y_i , y' \right\rangle \right| \leqslant$$

$$\leqslant \sup_{x \in B_E \ y' \in B_{F'}} \left| \left\langle e_k' \otimes \left(\sum_{i=1}^{\infty} \lambda_k^i y_i \right), x \otimes y' \right\rangle \right| =$$

$$= \sup_{x \in B_E \ y' \in B_{F'}} |\langle (P_k' \otimes_\alpha I_F) z_n , x \otimes y' \rangle| =$$

$$= \varepsilon((P_k' \otimes_\alpha I_F) z_n) \leqslant \|P_k' \otimes_\alpha I_F\| \alpha(z_n) < \infty .$$

So we can write $\sum_{i=1}^{\infty} \lambda_k^i y_i = y_k^n \in F$ and $z_n = \sum_{k=1}^{\infty} e_k' \otimes y_k^n$. In the same way we can see that $z_0 = \sum_{k=1}^{\infty} e_k' \otimes y_k^0$, $y_k^0 \in F''$.

From (1) and this we obtain

$$\lim_n \sum_{k=1}^{\infty} \langle e_k', e_j \rangle \langle y_k^n, y' \rangle = \sum_{k=1}^{\infty} \langle e_k', e_j \rangle \langle y_k^0, y' \rangle$$

and consequently $\lim_n \langle y_j^n, y' \rangle = \langle y_j^0, y' \rangle$. Assuming the fact that F is weakly sequentially complete we can conclude that $y_j^0 \in F$ for all $j \in \mathbb{N}$ and $z_0 \in E' \widehat{\otimes}_\alpha F$. Now, consider $f \in (E' \widehat{\otimes}_\alpha F)'$. For every $n \in \mathbb{N}$

$$\langle (P_k' \widehat{\otimes}_\alpha I_F) z_n , f \rangle = \langle e_k' \otimes y_k^n , f \rangle = \langle y_k^n , f e_k' \rangle, \quad f e_k' \in F' .$$

Since for every $k \in \mathbb{N}$ we have

$$\lim_n \langle (P_k' \widehat{\otimes}_\alpha I_F) z_n, f \rangle = \lim_n \langle y_k^n, fe_k' \rangle = \langle y_k^0, fe_k' \rangle = \langle (P_k' \widehat{\otimes}_\alpha I_F) z_0, f \rangle$$

it follows that $\text{w-}\lim_n (P_k' \widehat{\otimes}_\alpha I_F) z_n = (P_k' \widehat{\otimes}_\alpha I_F) z_0$.

According to the facts that $\{P_k' \widehat{\otimes}_\alpha I_F\}$ is an unconditional Schauder decomposition and $\{z_n\}$ is an arbitrary weak Cauchy sequence, we conclude from Lemma 1 [3] the weak sequential completeness of $E' \widehat{\otimes}_\alpha F$. ■

Let $B = \{e_k\}$ be a basis for the Banach space E. Let $\{e_k'\} \subset E'$ the sequence of coefficient functionals corresponding to the basis $\{e_k\}$. We define E_B^{pre} as the closure of the linear expansion of $\{e_k'\}$.

If B is monotone and boundedly complete, the equality $(E_B^{\text{pre}})' = E$ holds (see e.g. [6], p. 37).

COROLLARY 5. *Let α be a finitely generated tensor norm and $E, F \in (B)$. Suppose that E has an unconditional basis and (E_B^{pre}, F') has the α'-density property. Then the space $E \widehat{\otimes}_\alpha F$ is weakly sequentially complete if and only if both spaces E and F are weakly sequentially complete.*

PROOF. We can assume that B is also monotone. Since E is weakly sequentially complete, B is boundedly complete. Thus, $(E_B^{\text{pre}})' = E$ and $\{e_k'\} \subset E_B^{\text{pre}}$. It follows that $\{e_k\}$ is a retro-basis for $(E_B^{\text{pre}})'$. Applying Theorem 6 we conclude the proof. ■

COROLLARY 6. *Let α be a finitely generated tensor norm and $E, F \in (B)$. Suppose that E is reflexive and it has an unconditional basis. If (E', F') has the α'-density property, then the space $E \widehat{\otimes}_\alpha F$ is weakly sequentially complete if and only if F is weakly sequentially complete.*

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