Rendiconti
del
Seminario Matematico
della
Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 97 (1997), p. 235-267

<http://www.numdam.org/item?id=RSMUP_1997__97__235_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 97 (1997)

Strong Solutions of the Steady Nonlinear Navier-Stokes System in Domains with Exits to Infinity.

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ABSTRACT - The stationary Navier-Stokes equations of a viscous incompressible fluid are considered in domains Ω with $m > 1$ exits to infinity, which have in some coordinate system the following form

$$\Omega_i = \{x \colon |x'| < g_i(x_n),\ x_n > 0\},$$

where g_i are functions satisfying the global Lipschitz condition and $g_i'(x_n) \to \to 0$ as $x_n \to \infty$. In the paper we prove the solvability of the Navier-Stokes system with prescribed fluxes in weighted Sobolev and Hölder spaces and we show the pointwise decay of the solutions. For three-dimensional domains Ω the obtained results are true for arbitrary large data, while in the case of two-dimensions the results are proved only for small data.

Introduction.

The solvability of the boundary value problems for stationary Stokes and Navier-Stokes equations has been studied in many papers and monographs (e.g. [10], [35], [8]). The existence theory which is developed there concerns mainly the domains with compact boundaries (bounded or exterior). Although some of these results do not depend on the shape of the boundary, many problems of scientific interest, conserning the flow of a viscous incompressible fluid in domains with noncompact boundaries, were unsolved and, therefore, it is not suprising that during the last 17 years the special attention was given to problems in such domains.

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In this paper we consider the class of domains $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, having $m > 1$ exits to infinity Ω_i of the form

$$(0.1)\qquad \Omega_i = \{x \in \mathbb{R}^n : |x'| < g_i(x_n), \, x_n > 1\}$$

and we study in such domains the nonlinear stationary Navier-Stokes system

$$(0.2)\qquad \begin{cases} -\nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega, \end{cases}$$

supplimented by the additional flux conditions

$$(0.3)\qquad \int_{\sigma_i} \boldsymbol{u}\cdot\boldsymbol{n}\, ds = F_i, \qquad i = 1, \dots, m, \quad \sum_{i=1}^{m} F_i = 0,$$

where $\sigma_i(t) = \{x \colon x \in \Omega_i, \, x_n = t, \, t = \text{const}\}$.

The weak solvability of problem (0.2), (0.3) was studied in [11], [12], [29], [30], [32], [9]). In [29] was found that for solutions with the finite Dirichlet integral the nonzero fluxes can be prescribed only in exits to infinity Ω_i blowing up not to slowly. This means that the corresponding functions g_i satisfy the following condition

$$(0.4)\qquad \int_0^\infty g_i(t)^{-n-1}\, dt < \infty .$$

Prescribing arbitrary fluxes F_i in exits to infinity, satisfying (0.4), and zero fluxes if

$$(0.5)\qquad \int_0^\infty g_i(t)^{-n-1}\, dt = \infty ,$$

the weak solvability of (0.2), (0.3) was proved in [29] (for arbitrary large data) in a class of functions having the finite Dirichlet integral

$$\int_\Omega |\nabla \boldsymbol{u}(x)|^2\, dx < \infty .$$

The physically natural problem with prescribed fluxes also in «narrow» (subjected to (0.5)) exits to infinity (for example, in pipes) can not be solved in a class of divergence-free vectors with a finite Dirichlet integral. This is related to the fact that in such exits to infinity each divergence free vector field $\boldsymbol{u}$, having zero trace on $\partial\Omega$ and the finite

Dirichlet norm, admits only zero fluxes. In the basic paper of O. A. Ladyzhenskaya and V. A. Solonnikov (1980) [13] the weak solvability of the Navier-Stokes equations with prescribed fluxes through all exits to infinity is proved for arbitrary data in a class of functions with an infinite Dirichlet integral. The estimates of the growth of the Dirichlet integral are given there in terms of the functions g_i. These estimates are proved by mean of differential inequalities techniques (so called «techniques of the Saint-Venant's principle»).

The similar results for domains with layer-like exits to infinity are obtained by K. Pileckas (1981) [20].

In [31], [32], [33] V. A. Solonnikov developed the existence theory of weak solutions for stationary and nonstationary Navier-Stokes equations in a very general class of domains with exits to infinity. He avoided to make the assumptions on the shape of exits to infinity and only impose certain general restrictions. Roughly speaking, in [31], [32], [33] the axiomatic approach is developed. The methods used in [31], [32], [33] are closed to that of [13].

In this work we develop the existence theory of strong solutions of the stationary nonlinear Navier-Stokes problem (0.2), (0.3) in domains with $m > 1$ exits to infinity of form (0.1). As we have mentioned above, remaining in a class of solutions with the bounded Dirichlet integral, we can prescribe the fluxes only in «wide» exits to infinity, satisfying (0.4). On the other hand, in K. Pileckas (1980, 1983) [18], [22] (see also V. N. Maslennikova, M. E. Bogovskii (1981) [15]) it is shown that solenoidal vector fields, possesing the finite «L^q-Dirichlet norm»

$$\left(\int_\Omega |\nabla \boldsymbol{u}(x)|^q\, dx\right)^{1/q},$$

can have nonzero fluxes, even if condition (0.4) is violated. In this case (0.4) is changed to

$$\int_0^\infty g_i(t)^{-(n-1)(q-1)-q}\, dt < \infty\,. \tag{0.6}$$

Therefore, it is natural to suppose that, if the functions $g_i(t)$ grow as $t \to \infty$ and if we can find the numbers $q_i > 1$ such that (0.6) is valid, the Navier-Stokes system has solutions $\boldsymbol{u}$ with prescribed fluxes and $\nabla \boldsymbol{u} \in L^q(\Omega_i)$.

In [27], [28] the analogous questions were studied for the linear

Stokes problem

$$(0.7) \qquad \begin{cases} -\nu\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega\,, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega\,, \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega\,, \end{cases}$$

$$(0.8) \qquad \int_{\sigma_i} \boldsymbol{u}\cdot\boldsymbol{n}\, ds = F_i\,, \qquad i = 1, \ldots, m\,, \qquad \sum_{i=1}^{m} F_i = 0\,.$$

In the case of zero fluxes F_i, $i = 1, \ldots, m$, the unique solvability of problem (0.7), (0.8) was proved in [27], [28] in weighted Sobolev and Hölder spaces $V^{l,q}_{(\boldsymbol{æ},\boldsymbol{\beta})}(\Omega)$ and $C^{l,\delta}_{(\boldsymbol{æ},\boldsymbol{\beta})}(\Omega)$ with $\boldsymbol{\beta}) = (\beta_1, \ldots, \beta_m) \neq \boldsymbol{0}$ and $\boldsymbol{æ} = (æ_1, \ldots, æ_m)$. Notice that elements of $V^{l,q}_{(\boldsymbol{æ},\boldsymbol{\beta})}(\Omega)$ and $C^{l,\delta}_{(\boldsymbol{æ},\boldsymbol{\beta})}(\Omega)$ exponentially vanish as $|x| \to \infty$, $x \in \Omega_i$, if $\beta_i > 0$, and they can exponentially grow if $\beta_i < 0$. In particular, from [27], [28] it follows that if the right-hand side $\boldsymbol{f}$ has compact support, then the solution $(\boldsymbol{u}, \nabla p)$ of (0.7), (0.8) with $F_i = 0$, $i = 1, \ldots, m$, exponentially vanishes as $|x| \to \infty$. In the case of nonzero fluxes ($F_i \neq 0$, $i = 1, \ldots, m$) the solvability of the linear Stokes problem (0.7), (0.8) was proved in [27], [28] in the spaces $V^{l,q}_{(\boldsymbol{æ},0)}(\Omega)$ and $C^{l,\delta}_{(\boldsymbol{æ},0)}(\Omega)$ (1).

In this paper we prove the analogous results for the nonlinear Navier-Stokes problem (0.2), (0.3). The paper is organized as follows. In Section 1 we present the main notations, definitions of function spaces and certain weighted imbedding theorems in domains with exits to infinity.

In Sections 2 and 3 we formulate the results from [27], [28], concerning the solvability of the linear Stokes problem (0.7), (0.8) in weighted function spaces and the results from [29], [13] about the weak solutions to the nonlinear Navier-Stokes problem (0.2), (0.3).

In Section 4 we consider the nonlinear Navier-Stokes problem (0.2), (0.3). First, in Subsection 4.1 the problem (0.2), (0.3) is studied in the case of zero fluxes (i.e. $F_i = 0$, $i = 1, \ldots, m$). For sufficiently small data we prove the unique solvabilyty of it in spaces of exponentially vanishing functions $V^{l,q}_{(\boldsymbol{æ},\boldsymbol{\beta})}(\Omega)$, $q_0 = q_1 = \ldots = q_m$, and $C^{l,\delta}_{(\boldsymbol{æ},\boldsymbol{\beta})}(\Omega)$, $\beta_i > 0$. These results are true for the space dimensions $n = 2, 3$ and are based on the Banach contraction principle.

For arbitrary fluxes F_i problem (0.2), (0.3) is studied in Subsection 4.2. Using the weighted imbedding theorems and results conserning the linear Stokes problem (0.7), (0.8), we prove by bootstrap arguments that in three dimensional domains Ω the weak solution $\boldsymbol{u}$ with the un-

(1) The definitions of function spaces are given in Section 1.

bounded Dirichlet integral (such solution exists due to O. A. Ladyzhenskaya, V. A. Solonnikov (1980) [13]) is regular and belongs to $V^{l+2,q}_{(æ,0)}(\Omega)$ or $C^{l+2,\delta}_{(æ,0)}(\Omega)$ with appropriate $q_i > 1$ and $æ_i$. The gradient of the corresponding pressure function p belongs to $V^{l,q}_{(æ,0)}(\Omega)$ or $C^{l,\delta}_{(æ,0)}(\Omega)$. Moreover, we derive the pointwise estimates of the solution. These results hold true for arbitrary data F_i. For simplicity, we present the proof only in the case when the right-hand side $\boldsymbol{f}$ is equal to zero. However, the results remain valid also for nonzero right-hand sides $\boldsymbol{f}$ which vanish at infinity sufficiently rapidly. Notice that the decay estimates for the solutions of the nonlinear Navier-Stokes problem (0.2), (0.3) have the same character as those of the linear Stokes problem (0.7), (0.8).

The case of two dimensional domains Ω is more complicated than the three dimensional one. For such domains we can prove the solvability of (0.2), (0.3) with nonzero fluxes only for small data (Subsection 4.3), applying the Banach contraction principle, and we do not know, if the results are true for arbitrary large data.

Finally, in Section 5 we specify the obtained results for domains with cylindrical exits to infinity and we show that for small data the solution $(\boldsymbol{u}, p)$ of (0.2), (0.3) approaches exponentially (in the norms of weighted Sobolev or Hölder spaces) the corresponding exact Poiseuille solution. Notice that the results concerning the existence of the solutions, approaching for small data the Poiseuille flow, are well known (see e.g. [13], [3], [4] for the Dirichlet boundary conditions and [19], [21], [23], [24], [25], [26], [17], [1] for the stress free boundary conditions). The analogous results for the aperture domain were obtained in [6], [7].

1. – Main notations and preliminary results.

1.1. *Function spaces.*

We indicate by $C_0^\infty(\Omega)$ the set of all infinitely differentiable real vector functions with compact supports in Ω and by $J_0^\infty(\Omega)$ the subset of all solenoidal (i.e., satisfying the condition $\operatorname{div} \boldsymbol{u}(x) = \sum_{k=1}^{n} \partial u_k(x)/\partial x_k = 0$) vector functions from $C_0^\infty(\Omega)$; $W^{l,q}(\Omega)$ is the usual Sobolev space with the norm

$$\|\boldsymbol{u}; W^{l,q}(\Omega)\| = \left(\sum_{\alpha=0}^{l} \int_\Omega |D^\alpha u|^q \, dx \right)^{1/q},$$

where $D_x^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \ldots \alpha_n$; $C^{l,\delta}(\Omega)$, l being an integer, $0 < \delta < 1$, is a Hölder space of continuous in Ω functions u which have continuous derivatives $D^\alpha u = \partial^{|\alpha|} u/\partial x_1^{\alpha_1} \dots x_n^{\alpha_n}$ up to the order l and the finite norm

$$\|u;\, C^{l,\delta}(\Omega)\| = \sum_{|\alpha| \leq l} \sup\{|D^\alpha u(x)|\} + \sum_{|\alpha| = l} \sup\{[D^\alpha u]_\delta(x)\},$$

where the supremum is taken over $x \in \Omega$ and

$$[u]_\delta(x) = \sup_{\substack{0 < |x-y| < |x|/2 \\ y \in \Omega}} \frac{|u(x) - u(y)|}{|x-y|^\delta}.$$

$W^{l,q}_{\mathrm{loc}}(\Omega)$ and $C^{l,\delta}_{\mathrm{loc}}(\Omega)$ are the spaces of functions which belong $W^{l,q}(\Omega')$ and $C^{l,\delta}(\Omega')$ for every strictly interior subdomain Ω' of Ω.

Denote by $D_0^q(\Omega)$ the completions of $C_0^\infty(\Omega)$ in the norm

$$\|\nabla \boldsymbol{u}\|_{q,\Omega} = \left(\int_\Omega \sum_{i=1}^n \left|\frac{\partial \boldsymbol{u}}{\partial x_i}\right|^q dx\right)^{1/q}. \tag{1.1}$$

For simplicity in notations we put $L^q(\Omega) = W^{0,q}(\Omega)$, $\|\cdot;\, L^q(\Omega)\| = \|\cdot\|_{q,\Omega}$, $D_0(\Omega) = D_0^2(\Omega)$.

Let $D^{-1}(\Omega)$ be the dual space to $D_0(\Omega)$ with the norm

$$\|\boldsymbol{f}\|_{-1,2,\Omega} = \sup_{\boldsymbol{\eta} \in D_0(\Omega)} \frac{\left|\int_\Omega \boldsymbol{f} \cdot \boldsymbol{\eta}\, dx\right|}{\|\nabla \boldsymbol{\eta}\|_{2,\Omega}}.$$

Let us consider now a domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, having m exits to infinity. We suppose that outside the sphere $|x| = R_0$ the domain Ω splits into m connected components Ω_i (exits to infinity) which in some coordinate systems $x^{(i)}$ are given by the relations

$$\Omega_i = \{x^{(i)}:\ |x^{(i)\prime}| < g_i(x_n^{(i)}),\ x_n^{(i)} > 0\}, \tag{1.1}$$

where $|x^{(i)\prime}| \equiv |x_1^{(i)}|$ if $n = 2$ and $|x^{(i)\prime}| \equiv \sqrt{x_1^{(i)2} + x_2^{(i)2}}$ if $n = 3$, and $g_i(t)$ are functions satisfying the conditions

$$|g_i(t) - g_i(t')| \leq M_i\,|t - t'|, \quad \forall t, t' > 0; \qquad g_i(t) \geq g_0 > 0, \tag{1.2}$$

$$\lim_{t\to\infty} g_i'(t) = 0, \quad |g_i'(t)| \leq M_i, \qquad i = 1, \dots, m. \tag{1.3}$$

Below we omit the index i in the notations for lokal coordinates. In what follows we use the notations:

$$\sigma_i(t) = \{x \in \Omega_i : x_n = t = \text{const}\};$$

$$R_{i0} = 0, \quad R_{ik+1} = R_{ik} + (2M_i)^{-1} g_{ik}, \quad g_{ik} = g_i(R_{ik}), \quad i = 1, \ldots, m,$$

$$\Omega_{ik} = \{x \in \Omega_i : x_n < R_{ik}\}, \quad \omega_{ik} = \Omega_{ik+1} \setminus \Omega_{ik}, \quad \Omega_{(k)} = \Omega_0 \cup \left(\bigcup_{i=1}^{m} \Omega_{ik} \right);$$

$$N_i(x_n) = g_i(x_n) g_{ik_0}^{-1}, \quad i = 1, \ldots, m,$$

$$N(x)^{\text{æ}} = \begin{cases} N_i(x)^{\text{æ}_i}, & x = \Omega_i \setminus \Omega_{(k_0)}, \quad i = 1, \ldots, m, \\ 1, & x \in \Omega_{(k_0)}, \end{cases}$$

$$\text{æ} = (\text{æ}_1, \ldots, \text{æ}_m);$$

$$\alpha_i(t) = \int_0^t g_i(\tau)^{-1} \, d\tau, \qquad \alpha_{ik} = \alpha_i(R_{ik}), \qquad i = 1, \ldots, m.$$

LEMMA 1.1 [27]. *There hold the following relations*

$$\frac{1}{2} g_{ik} \leqslant g_i(t) \leqslant \frac{3}{2} g_{ik}, \qquad t \in [R_{ik}, R_{ik+1}], \tag{1.4}$$

$$\int_0^\infty g_i(\tau)^{-1} \, d\tau = \infty, \tag{1.5}$$

$$\mu_* |l - k| \leqslant |\alpha_{il} - \alpha_{ik}| \leqslant \mu^* |l - k|, \qquad i = 1, \ldots, m, \tag{1.6}$$

where μ_* and μ^* are positive constants independent of k and l.

Let us introduce the weighted function spaces in the domain Ω with $m > 1$ exits to infinity. $L^{\boldsymbol{q}}_{(\text{æ}, \boldsymbol{\beta})}(\Omega)$, $\boldsymbol{q} = (q_0, q_1, \ldots, q_m)$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m)$, is the space of functions with the finite norm

$$\|\boldsymbol{f}; L^{\boldsymbol{q}}_{(\text{æ}, \boldsymbol{\beta})}(\Omega)\| = \left(\int_{\Omega_{(k_0+1)}} |\boldsymbol{f}|^{q_0} \, dx \right)^{1/q_0} + \\ + \sum_{i=1}^{m} \left(\int_{\Omega_i \setminus \Omega_{(k_0)}} N_i^{q_i \text{æ}_i} \exp(q_i \beta_i \alpha_i(x_n)) |\boldsymbol{f}|^{q_i} \, dx \right)^{1/q_i}$$

and $\widetilde{L}^{q}_{(\mathfrak{X},\beta)}(\Omega)$ is the space of functions with the norm

$$\|f;\widetilde{L}^{q}_{(\mathfrak{X},\beta)}(\Omega)\| = \left(\int\limits_{\Omega_{(k_0+2)}} |f|^{q_0}\,dx\right)^{1/q_0} + \\ + \sum_{i=1}^{m}\left(\int\limits_{\Omega_i\setminus\Omega_{(k_0-1)}} N_i^{q_i \mathfrak{X}_i}\exp(q_i\beta_i\alpha_i(x_n))\,|f|^{q_i}\,dx\right)^{1/q_i}.$$

Notice that for $q_0 = q_1 = \ldots = q_m = q$ the spaces $\widetilde{L}^{q}_{(\mathfrak{X},\beta)}(\Omega)$ and $L^{q}_{(\mathfrak{X},\beta)}(\Omega)$ are equivalent.

Let $V^{l,q}_{(\mathfrak{X},\beta)}(\Omega)$, $l \geqslant 1$, be the completion of $C_0^\infty(\Omega)$ in the norm

$$\|f;\, V^{l,q}_{(\mathfrak{X},\beta)}(\Omega)\| = \sum_{|\alpha|=0}^{l} \|D^\alpha f;\, L^{q}_{(\mathfrak{X}+|\alpha|-l,\,\beta)}(\Omega)\|$$

and $V^{-1,q}_{(\mathfrak{X},\beta)}(\Omega)$ be the space of functions f which can be represented in the form

$$f = f^{(0)} + (\operatorname{div} f^{(1)}, \ldots, \operatorname{div} f^{(n)}) \tag{1.7}$$

with $f^{(0)} \in L^{q}_{(\mathfrak{X}+1,\,\beta)}(\Omega)$, $f^{(j)} \in L^{q}_{(\mathfrak{X},\beta)}(\Omega)$, $j = 1, \ldots, n$, and

$$\|f;\, V^{-1,q}_{(\mathfrak{X},\beta)}(\Omega)\| = \|f^{(0)};\, L^{q}_{(\mathfrak{X}+1,\,\beta)}(\Omega)\| + \sum_{j=1}^{n}\|f^{(j)};\, L^{q}_{(\mathfrak{X},\beta)}(\Omega)\|.$$

Here $\mathfrak{X} + b = (\mathfrak{X}_1 + b, \ldots, \mathfrak{X}_m + b)$.

The spaces $\widetilde{V}^{l,q}_{(\mathfrak{X},\beta)}(\Omega)$, $l \geqslant -1$, are defined by the same formulas with the only difference that $L^{q}_{(\cdot,\,\beta)}(\Omega)$ in the definitions of the norms are replaced by $\widetilde{L}^{q}_{(\cdot,\,\beta)}(\Omega)$.

The weighted Hölder space $C^{l,\delta}_{(\mathfrak{X},\beta)}(\Omega)$, $l \geqslant 0$, $1 > \delta > 0$, consists of functions u, continuously differentiable up to the order l in Ω, for which the norm

$$\|u;\, C^{l,\delta}_{(\mathfrak{X},\beta)}(\Omega)\| = \|u;\, C^{l,\delta}(\Omega_{(k_0)})\| + \\ + \sum_{i=1}^{m}\sum_{|\alpha|\leqslant l}\sup_{x\in\Omega_i}\{g_i(x_n)^{\mathfrak{X}_i-l-\delta+|\alpha|}\exp(\beta_i\alpha_i(x_n))|D^\alpha u(x)|\} + \\ + \sum_{i=1}^{m}\sum_{|\alpha|=l}\sup_{x\in\Omega_i}\{g_i(x_n)^{\mathfrak{X}_i}\exp(\beta_i\alpha_i(x_n))\,[D^\alpha u]_\delta(x)\}$$

is finite.

The weighted function spaces in subdomains Ω' of Ω are defined by the same formulas with the only difference that either integrals or supremum are taken over Ω' instead of Ω. For example,

$$\|f;\, L^{q_i}_{(æ_i,\, \beta_i)}(\omega_{ik})\| = \left(\int_{\omega_{ik}} g_i(x_n)^{q_i æ_i} \exp\left(q_i \beta_i \alpha_i(x_n)\right) |f|^{q_i}\, dx \right)^{1/q_i},$$

$$i = 1, \ldots, m.$$

1.2. *Spaces of divergence-free vector fields.*

Let $\hat{H}^q_æ(\Omega)$ be the space of all solenoidal vector functions, having zero traces on $\partial\Omega$ and the finite norm $\|\nabla \boldsymbol{u};\, L^q_{(æ,\,0)}(\Omega)\|$ and $H^q_æ(\Omega)$ is the closure of $J^\infty_0(\Omega)$ in the norm $\|\cdot;\hat{H}^q_æ(\Omega)\|$. For simplicity in notations we put $H^q_0(\Omega) = H^q(\Omega)$, $\hat{H}^q_0(\Omega) = \hat{H}^q(\Omega)$, $H^q(\Omega) = H^q(\Omega)$, $\hat{H}^q(\Omega) = \hat{H}^q(\Omega)$ for $q_0 = q_1 = \ldots = q_m = q$ and $H^2(\Omega) = H(\Omega)$, $\hat{H}^2(\Omega) = \hat{H}(\Omega)$. It is obvious that

$$\hat{H}^q_æ(\Omega) \supset H^q_æ(\Omega).$$

THEOREM 1.1. *Assume that $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is the domain with m exits to infinity Ω_i, possessing the properties described above, and let for all $k > k_0$ the domain $\Omega_{(k)}$ has the Lipschitz boundary. If among the integrals*

$$\int_0^\infty g_i(t)^{-(n-1)(q_i-1) - q_i + q_i æ_i}\, dt, \qquad i = 1, \ldots, m,$$

there are exactly r which converge, then

$$\dim \hat{H}^q_æ(\Omega)/H^q_æ(\Omega) = r - 1.$$

The space $H^q_æ(\Omega)$ consists of those and only those $\boldsymbol{u} \in \hat{H}^q_æ(\Omega)$ which satisfy the condition

$$\int_{\sigma_i} \boldsymbol{u} \cdot \boldsymbol{n}\, ds = 0, \qquad i = 1, \ldots, m.$$

The proof of this theorem is completely analogous to that for the case $æ = \mathbf{0}$, $q_0 = q_1 = \ldots = q_m = q$ (see [18], [22]) and is based on the following lemma.

LEMMA 1.2 [29]. *Let Ω be a domain with $m > 1$ exits to infinity Ω_i*

of the form (1.1). *Then there exists a vector field* $\boldsymbol{A}$ *satisfying the following conditions*

$$\operatorname{div} \boldsymbol{A} = 0 \quad in\ \Omega,$$

$$\boldsymbol{A} = 0 \quad on\ \partial\Omega,$$

$$\int_{\sigma_i(t)} \boldsymbol{A} \cdot \boldsymbol{n}\, dS = F_i, \qquad i = 1, \ldots, m,$$

where $\boldsymbol{n}$ *is a unit vector of the normal to* σ_i *and* F_i *are given numbers with*

$$\sum_{i=1}^{m} F_i = 0.$$

Moreover, there holds the estimates

(1.8) $$|D_x^\alpha \boldsymbol{A}(x)| \leqslant C(|\boldsymbol{F}|)\, g_i(x_n)^{-n+1-|\alpha|},$$

$$i = 1, \ldots, m, \quad x \in \Omega_i, \quad |\alpha| \geqslant 0.$$

In (1.8) $|\boldsymbol{F}| = \left(\sum_{i=1}^{m} F_i^2\right)^{1/2}$.

1.3. *Weighted imbedding theorems.*

Below we use the following weighted imbedding theorems.

THEOREM 1.2. *Let* $u \in V^{l, q_i}_{(æ_i, 0)}(\omega_{ik})$.

(i) *If*

(1.9) $$q_i l \leqslant n, \qquad q_i \leqslant s_i \leqslant nq_i/(n - q_i l), \qquad n = 2, 3,$$

then $u \in L^{s_i}_{(æ_i - l + n/q_i - n/s_i, 0)}(\omega_{ik})$ *and*

(1.10) $$\|u; L^{s}_{(æ_i - l + n/q_i - n/s, 0)}(\omega_{ik})\| \leqslant c\|u; V^{l, q_i}_{(æ_i, 0)}(\omega_{ik})\|.$$

(ii) *If*

(1.11) $$q_i l > n, \qquad m + \delta \leqslant (q_i l - n)/q_i, \qquad \delta \in (0, 1),$$

then $u \in C^{m, \delta}_{(m + \delta - l + æ_i + n/q_i, 0)}(\omega_{ik})$ *and*

(1.12) $$\|u; C^{m, \delta}_{(m + \delta - l + æ_i + n/q_i, 0)}(\omega_{ik})\| \leqslant c\|u; V^{l, q_i}_{(æ_i, 0)}(\omega_{ik})\|.$$

The constants in (1.10), (1.12) *are independent of* k *and* u.

PROOF. After the transform of coordinates

$$y_j = 2M_i g_{ik}^{-1} x_j\,, \qquad j = 1, \ldots, n-1\,, \qquad y_n = 2M_i g_{ik}^{-1}(x_n - R_{ik-1})\,,$$

the domain ω_{ik} goes over to the standart domain $\hat{\omega}_{ik}$:

$$\hat{\omega}_{il} = \{y\colon |y'| < G_{il}(y_n),\, \gamma_{il}^{(1)} < y_n < 1 + \gamma_{il}^{(1)}\}\,,$$

$$\hat{\omega}_{il}^{*} = \{y\colon |y'| < G_{il}(y_n),\, 0 < y_n < 1 + \gamma_{il}^{(1)} + \gamma_{il}^{(2)}\}\,,$$

where

$$G_{il}(y_n) = g_{il}^{-1} 2M_i g_i (R_{il-1} + 2M_i^{-1} g_{il} y_n)\,,$$

$$\gamma_{il}^{(1)} = g_{il-1} g_{il}^{-1}\,, \quad \gamma_{il}^{(2)} = g_{il+1} g_{il}^{-1}\,.$$

The classical imbedding theorems in the domain $\hat{\omega}_{ik}$ yield either

$$\| \hat{u}; L^{s_i}(\hat{\omega}_{ik})\|^{s_i} \leqslant c\| \hat{u}; W^{l,\, q_i}(\hat{\omega}_{ik})\|^{s_i/q_i} \tag{1.13}$$

in the case (1.9), or

$$\| \hat{u}; C^{m,\, \delta}(\hat{\omega}_{ik})\| \leqslant c\| \hat{u}; W^{l,\, q_i}(\hat{\omega}_{ik})\| \tag{1.14}$$

in the case (1.11) (see e.g. R. A. Adams [2]). In (1.13), (1.14) $\hat{u}(y) = u(x(y))$ and the constants are independent of k and u. Returning in (1.13), (1.14) to coordinates $\{x\}$ and multiplying the obtained inequalities by $g_{ik}^{s_i(\varkappa_i - l + n/q_i)}$, we derive

$$\|u; L^{s_i}_{(\varkappa_i - l + n/q_i - n/s_i,\, 0)}(\omega_{ik})\|^{s_i} \leqslant c\|u; V^{l,\, q_i}_{(\varkappa_i,\, 0)}(\omega_{ik})\|^{s_i/q_i}\,, \tag{1.15}$$

$$\|u; C^{m,\, \delta}_{(m + \delta - l + \varkappa_i + n/q_i,\, 0)}(\omega_{ik})\| \leqslant c\|u; V^{l,\, q_i}_{(\varkappa_i,\, 0)}(\omega_{ik})\|\,. \tag{1.16}$$

The theorem is proved.

THEOREM 1.3. *Let* $u \in V^{l,\, q_i}_{(\varkappa_i,\, \beta_i)}(\Omega_i)$, $\beta \in \mathbb{R}^1$.

(i) *If* s_i *satisfies the conditions* (1.9), *then* $u \in L^{s_i}_{(\varkappa_i - l + n/q_i - n/s_i,\, \beta_i)}(\Omega_i)$ *and*

$$\|u; L^{s_i}_{(\varkappa_i - l + n/q_i - n/s_i,\, \beta_i)}(\Omega_i)\| \leqslant c\|u; V^{l,\, q_i}_{(\varkappa_i,\, \beta_i)}(\Omega_i)\|\,. \tag{1.17}$$

(ii) *If the conditions* (1.11) *are fullfild, then* $u \in C^{m,\, \delta}_{(m + \delta - l + \varkappa_i + n/q_i,\, \beta_i)}(\Omega_i)$ *and*

$$\|u; C^{m,\, \delta}_{(m + \delta - l + \varkappa_i + n/q_i,\, \beta_i)}(\Omega_i)\| \leqslant c\|u; V^{l,\, q_i}_{(\varkappa_i,\, \beta_i)}(\Omega_i)\|\,. \tag{1.18}$$

PROOF. Multiplying the estimates (1.15) by $\exp(s_i\beta_i\alpha_{ik})$ and taking the sum over all $k > k_0$, we derive

$$\|u;\, L^{s_i}_{(\varkappa_i - l + n/q_i - n/s_i,\, \beta_i)}(\Omega_i \setminus \Omega_{(k_0)})\|^{s_i} \leqslant c \sum_{k \geqslant k_0} \|u;\, V^{l,\,q_i}_{(\varkappa_i,\, \beta_i)}(\omega_{ik})\|^{s_i/q_i}.$$

Since $s_i/q_i > 1$, the right-hand side of the last inequality can be estimated by $c\|u;\, V^{l,\,q_i}_{(\varkappa_i,\,\beta_i)}(\Omega_i)\|^{s_i}$ and hence (1.17) is proved. Estimate (1.18) follows from (1.12) by multiplying it by $\exp(\beta_i\alpha_{ik})$ and taking the supremum over all $k > k_0$. The theorem is proved.

2. – Strong solvability of the Stokes problem.

Let us consider in the domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with $m > 1$ exits to infinity the Stokes problem (0.7), (0.8). We denote the problem (0.7)-(0.8) with zero fluxes, i.e. $F_i = 0$, $i = 1, \dots, m$, by (0.7)-$(0.8)_0$. Below we present (without proofs) the results from [27], [28].

2.1. *Weighted local estimates.*

THEOREM 2.1. (i) [27] *Let the function* $\boldsymbol{f}$ *has the representation* (1.7) *and* $\boldsymbol{f}^{(0)}, \boldsymbol{f}^{(j)} \in L^{q_i}_{\mathrm{loc}}(\Omega_i)$, $j = 1, \dots, n$, $q_i \geqslant 2$, $i = 0, 1, \dots, m$. *Then the weak solution* $\boldsymbol{u}$ *of the Stokes problem* (0. 7) *satisfies the local estimate*

$$(2.1)\quad \|\nabla \boldsymbol{u};\, L^{q_i}_{(\varkappa_i,\,\beta_i)}(\omega_{is})\| \leqslant c\Big(\|\boldsymbol{f}^{(0)};\, L^{q_i}_{(\varkappa_i+1,\,\beta_i)}(\omega^*_{is})\| + \sum_{j=1}^{n} \|\boldsymbol{f}^{(j)};\, L^{q_i}_{(\varkappa_i,\,\beta_i)}(\omega^*_{is})\| + \|\nabla \boldsymbol{u};\, L^{2}_{(\varkappa_i+\hat{\varkappa}_i,\,\beta_i)}(\omega^*_{is})\|\Big),$$

where $\hat{\varkappa}_i = n(2-q_i)/2q_i$, $\omega^*_{is} = \omega_{is-1} \cup \omega_{is} z \cup \omega_{is+1}$.

(ii) [27] *Assume that* $\partial\Omega \in C^{l+2}$, $\boldsymbol{f} \in W^{l,\,q_i}_{\mathrm{loc}}(\Omega_i)$, $q_i > 1$, $l \geqslant 0$. *Then the solution* $(\boldsymbol{u}, p)$ *of problem* (0.7) *satisfies the local estimates*

$$(2.2)\quad \|\boldsymbol{u};\, V^{l+2,\,q_i}_{(\varkappa_i,\,\beta_i)}(\omega_{is})\| + \|\nabla p;\, V^{l,\,q_i}_{(\varkappa_i,\,\beta_i)}(\omega_{is})\| \leqslant c(\|\boldsymbol{f};\, V^{l,\,q_i}_{(\varkappa_i,\,\beta_i)}(\omega^*_{is})\| + \|\nabla \boldsymbol{u};\, L^{q_i}_{(\varkappa_i-l-1,\,\beta_i)}(\omega^*_{is})\|),$$

$$(2.3)\quad \|\boldsymbol{u};\, V^{l+2,\,q_i}_{(\varkappa_i,\,\beta_i)}(\omega_{is})\| + \|\nabla p;\, V^{l,\,q_i}_{(\varkappa_i,\,\beta_i)}(\omega_{is})\| \leqslant c(\|\boldsymbol{f};\, V^{l,\,q_i}_{(\varkappa_i,\,\beta_i)}(\omega^*_{is})\| + \|\nabla \boldsymbol{u};\, L^{2}_{(\varkappa_i-l-n/2-1+n/q_i,\,\beta_i)}(\omega^*_{is})\|).$$

(iii) [28] *Assume that* $\partial\Omega \in C^{l+2,\delta}$ *and let* $\boldsymbol{f} \in C^{l,\delta}_{\mathrm{loc}}(\Omega_i)$, $l \geq 0$, $0 < \delta < 1$. *Then the solution* $(\boldsymbol{u}, p)$ *of* (0.7) *satisfies the local estimate*

$$(2.4)\qquad \|\boldsymbol{u};\, C^{l+2,\delta}_{(\ae_i,\beta_i)}(\omega_{is})\| + \|\nabla p;\, C^{l,\delta}_{(\ae_i,\beta_i)}(\omega_{is})\| \leq$$

$$\leq c(\|\boldsymbol{f};\, C^{l,\delta}_{(\ae_i,\beta_i)}(\omega^*_{is})\| + \|\nabla\boldsymbol{u};\, L^2_{(\ae_i - l - 1 - n/2 - \delta,\beta_i)}(\omega^*_{is})\|).$$

The constants in (2.1)-(2.4) *are independent of* s *and* $\boldsymbol{f}$.

2.2. The Stokes problem with zero fluxes.

THEOREM 2.2. (i) [27] *Assume that* $\partial\Omega \in C^{l+2}$, $\boldsymbol{f} \in \tilde{V}^{l,q}_{(\ae,\beta)}(\Omega)$, $l - 1$, $q_j > 1$, $|\beta_j| < \beta_*$, $\ae$ *is arbitrary. Then there exists a unique solution* $(\boldsymbol{u}, p)$(2) *of problem* (0.7)-$(0.8)_0$ *with* $\boldsymbol{u} \in V^{l+2,q}_{(\ae,\beta)}(\Omega)$ *and*

$$(2.5)\qquad \|\boldsymbol{u};\, V^{l+2,q}_{(\ae,\beta)}(\Omega)\| \leq c\|\boldsymbol{f};\tilde{V}^{l,q}_{(\ae,\beta)}(\Omega)\|.$$

Moreover, if $l \geq 0$, *then* $\nabla p \in V^{l,q}_{(\ae,\beta)}(\Omega)$ *and there holds the estimate*

$$(2.6)\qquad \|\boldsymbol{u};\, V^{l+2,q}_{(\ae,\beta)}(\Omega)\| + \|\nabla p;\, V^{l,q}_{(\ae,\beta)}(\Omega)\| \leq c\|\boldsymbol{f};\tilde{V}^{l,q}_{(\ae,\beta)}(\Omega)\|.$$

(ii) [28] *Assume that* $\partial\Omega \in C^{l+2,\delta}$ *and* $\boldsymbol{f} \in C^{l,\delta}_{(\ae,\beta)}(\Omega)$, *where* $l \geq 0$, $\delta \in (0, 1)$, $|\beta_i| < \beta_*$ *and* $\ae$ *is arbitrary. Then problem* (0.7)-$(0.8)_0$ *has a unique solution* $(\boldsymbol{u}, p)$ *with* $\boldsymbol{u} \in C^{l+2,\delta}_{(\ae,\beta)}(\Omega)$, $\nabla p \in C^{l,\delta}_{(\ae,\beta)}(\Omega)$. *Moreover, there holds the estimate*

$$(2.7)\qquad \|\boldsymbol{u};\, C^{l+2,\delta}_{(\ae,\beta)}(\Omega)\| + \|\nabla p;\, C^{l,\delta}_{(\ae,\beta)}(\Omega)\| \leq c\|\boldsymbol{f};\, C^{l,\delta}_{(\ae,\beta)}(\Omega)\|.$$

REMARK 2.1. *In particular, if* $\boldsymbol{f} \in C^{l,\delta}_{(\ae,\beta)}(\Omega)$ *with* $\beta_i > 0$ *(for example,* $\boldsymbol{f}$ *has a compact support), then from Theorem* 2.2 *follows the exponential decay estimates for the solution* $(\boldsymbol{u}, p)$ *of problem* (0.7)-$(0.8)_0$, *i.e. there hold the estimates*

$$|D^\alpha \boldsymbol{u}(x)| \leq c\|\boldsymbol{f};\, C^{l,\delta}_{(\ae,\beta)}(\Omega)\| \exp\left(-\beta_i \int_0^{x_n^{(i)}} g_i(t)^{-1}\, dt\right) g_i(x_n^{(i)})^{l+2+\delta-|\alpha|-\ae_i},$$

(2) Speaking about the uniqueness of the solution $(\boldsymbol{u}, p)$, we have in mind that the pressure p is unique up to an additive constant.

$$|D^\alpha \nabla p(x)| \leqslant c\|f; C^{l,\delta}_{(\varkappa,\beta)}(\Omega)\| \exp\left(-\beta_i \int\limits_0^{x_n^{(i)}} g_i(t)^{-1}\, dt\right) g(x_n^{(i)})^{l+\delta-|\alpha|-\varkappa_i},$$

$$x \in \Omega_i, \qquad 0 \leqslant |\alpha| \leqslant l+2.$$

2.3. *The Stokes problem with nonzero fluxes.*

Let us consider the Stokes problem with nonzero fluxes F_i, $i = 1, \dots, m$, i.e. the problem (0.7)-(0.8). We assume that for each $i \in \{1, \dots, m\}$ there exists a number q_i^* such that

$$\int\limits_0^\infty g_i^{-(n-1)(q_i^*-1)-q_i^*}(t)\, dt < \infty\,. \tag{2.8}$$

There hold the following results.

THEOREM 2.3. (i) [27] *Let condition* (2.8) *holds. Then for arbitrary* $f \in \widetilde{V}^{-1,q^*}_{(0,0)}(\Omega)$ *and* $F_i \in \mathbb{R}^1$, $i = 1, \dots, m$, *problem* (0.7)-(0.8) *has a unique solution* $u \in \widehat{H}^{q^*}(\Omega)$ *satisfying the estimate*

$$\|u; \widehat{H}^{q^*}(\Omega)\| \leqslant c\left(\|f; \widetilde{V}^{-1,q^*}_{(0,0)}(\Omega)\| + \sum_{i=1}^m |F_i|\right). \tag{2.9}$$

(ii) [27] *Let* $\partial\Omega \in C^{l+2}$, $F_i \in \mathbb{R}^1$, $i = 1, \dots, m$, $f \in \widetilde{V}^{l,q}_{(\varkappa^*,0)}(\Omega)$, *where* $l \geqslant -1$, $q_i > 1$ *and* $\varkappa^*$ *are defined by the formula*

$$\varkappa^* = (\varkappa_1^*, \dots, \varkappa_m^*), \quad \varkappa_i^* = n + 1 + l - nq_i^*/q_i\,, \quad i = 1, \dots, m\,, \tag{2.10}$$

Then there exists a unique solution (u, p) *of problem* (0.7)-(0.8) *with* $u \in V^{l+2,q}_{(\varkappa^*,0)}(\Omega)$, $\nabla p \in V^{l,q}_{(\varkappa^*,0)}(\Omega)$ *and*

$$\|u; V^{l+2,q}_{(\varkappa^*,0)}(\Omega)\| + \|\nabla p; V^{l,q}_{(\varkappa^*,0)}(\Omega)\| \leqslant \leqslant c\left(\sum_{i=1}^m |F_i| + \|f; \widetilde{V}^{l,q}_{(\varkappa^*,0)}(\Omega)\|\right). \tag{2.11}$$

In particular, if $f \in V^{-1,q}_{(\varkappa^*,0)}(\Omega)$, *then* $u \in \widehat{H}^{1,q}_{\varkappa^*}(\Omega)$.

(iii) [28] *Let* $\partial\Omega \in C^{l+2,\delta}$, $f \in C^{l,\delta}_{(\varkappa,0)}(\Omega)$, $l \geqslant 0$, $\delta \in (0, 1)$, *where* $\varkappa$ *is defined by the formula*

$$\varkappa = (\varkappa_1, \dots, \varkappa_m), \qquad \varkappa_i = n + 1 + l + \delta\,, \qquad i = 1, \dots, m\,. \tag{2.12}$$

Then there exists a unique solution (u, p) *of problem* (0.7)-(0.8) *such* *that* $\boldsymbol{u} \in C^{l+2,\delta}_{(\boldsymbol{\varkappa},0)}(\Omega)$, $\nabla p \in C^{l,\delta}_{(\boldsymbol{\varkappa},0)}(\Omega)$ *and there holds the estimate*

$$\|\boldsymbol{u}; C^{l+2,\delta}_{(\boldsymbol{\varkappa},0)}(\Omega)\| + \|\nabla p; C^{l,\delta}_{(\boldsymbol{\varkappa},0)}(\Omega)\| \leqslant \tag{2.13}$$
$$\leqslant c\left(\sum_{i=1}^{m} |F_i| + \|\boldsymbol{f}; C^{l,\delta}_{(\boldsymbol{\varkappa},0)}(\Omega)\|\right).$$

In particular, from (2.13) *follows that*

$$|D^{\alpha}\boldsymbol{u}(x)| \leqslant c\left(\sum_{i=1}^{m} |F_i| + \|\boldsymbol{f}; C^{l,\delta}_{(\boldsymbol{\varkappa},0)}(\Omega)\|\right) g_i(x_n)^{-n+1-|\alpha|}, \qquad x \in \Omega_i, \quad |\alpha| \geqslant 0, \tag{2.14}$$

$$|D^{\alpha}\nabla p(x)| \leqslant c\left(\sum_{i=1}^{m} |F_i| + \|\boldsymbol{f}; C^{l,\delta}_{(\boldsymbol{\varkappa},0)}(\Omega)\|\right) g_i(x_n)^{-n-1-|\alpha|}, \qquad x \in \Omega_i, \quad |\alpha| \geqslant 0. \tag{2.15}$$

Moreover,

$$|p(x)| \leqslant c\left(\sum_{i=1}^{m} |F_i| + \|\boldsymbol{f}; C^{l,\delta}_{(\boldsymbol{\varkappa},0)}(\Omega)\|\right)\left(\int_0^{x_n} g_i(t)^{-n-1}\,dt\right) + c_1, \qquad x \in \Omega_i. \tag{2.16}$$

3. – Weak solvability of the Navier-Stokes problem.

By a weak solution of the Navier-Stokes problem (0.2)-(0.3) we understand the vector function $\boldsymbol{u} \in W^{1,2}_{\mathrm{loc}}(\Omega)$ with

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{u} = 0 \quad \text{on } \partial\Omega,$$

satisfying the flux conditions (0.3) and the integral identity

$$\nu\int_{\Omega} \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{\eta}\,dx + \int_{\Omega}(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u}\cdot\boldsymbol{\eta}\,dx = \int_{\Omega}\boldsymbol{f}\cdot\boldsymbol{\eta}\,dx \tag{3.1}$$

for every test function $\boldsymbol{\eta} \in J_0^{\infty}(\Omega)$.

3.1. *Solutions with the finite Dirichlet integral.*

The theorem below has been proved by V. A. Solonnikov, K. Pileckas (1977) [29].

THEOREM 3.1. [29] *Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a domain with m exits to infinity of the form* (1.1). *Assume that the integrals*

$$\int_0^\infty g_i(t)^{-n-1}\, dt \tag{3.2}$$

are finite for $i = 1, \ldots, r_1$ and infinite for $i = r_1 + 1, \ldots, m$, and let $F_i = 0$ for $i = r_1 + 1, \ldots, m$. Then for arbitrary $\boldsymbol{f} \in D^{-1}(\Omega)$ and $F_i \in \mathbb{R}^1$, $i = 1, \ldots, r_1$ problem (0.2)-(0.3) *has at least one weak solution $\boldsymbol{u}$ such that $\boldsymbol{u} \in \widehat{H}(\Omega)$ and*

$$\|\nabla \boldsymbol{u}\|_{2,\Omega} \leqslant c\left(\|\boldsymbol{f}\|_{-1,2,\Omega} + \left(\sum_{i=1}^{r_1} |F_i|^2\right)^{1/2}\right). \tag{3.3}$$

3.2. *Solutions with the infinite Dirichlet integral.*

If the conditions (3.2) are violated, i.e. the integrals (3.2) are infinite for all $i = 1, \ldots, m$, one can not expect any more the existence of solutions with the finite Dirichlet integral. This is related to the fact that in such case there are no divergence free vector fields with the finite Dirichlet norm. However, as it were found by O. A. Ladyzhenskaya, V. A. Solonnikov (1980) [13], in this case there exist solutions, having an infinite dissipation of energy (i.e. infinite Dirichlet integral). In particular, there holds

THEOREM 3.2. [13]. *Let $\Omega \subset \mathbb{R}^3$ be a domain with m exits to infinity Ω_i. Suppose that the functions g_i satisfy conditions* (1.2), (1.3) *and, in addition,*

$$\int_0^\infty g_i(t)^{-4/3}\, dt = \infty\,, \qquad i = 1, \ldots, m\,, \tag{3.4}$$

$$|g_i'(t) g_i(t)^{1/3}| \leqslant \gamma \ll 1 \quad \text{for } t > k_0\,, \qquad i = 1, \ldots, m\,. \tag{3.5}$$

Let $\boldsymbol{f} = 0$. Then for arbitrary fluxes $F_i \in \mathbb{R}^1$, $i = 1, \ldots, m$, there exists at least one weak solution $\boldsymbol{u}$ to (0.2)-(0.3) *with an infinite Dirichlet integral. This solution admits the representation as a sum*

$$\boldsymbol{u} = \boldsymbol{A} + \boldsymbol{v}\,, \tag{3.6}$$

where $\boldsymbol{A}$ is a divergence free vector field satisfying the flux condition

(0.3) *and the estimates* (1.8). *The following estimates hold true*

$$\int_{\Omega_{(k)}} |\nabla \boldsymbol{v}|^2\, dx \leqslant c(|F|) \sum_{k=1}^{m} \int_0^{R_{k+1}} g_i(t)^{-4}\, dt, \tag{3.7}$$

$$\int_{\omega_{ik}} |\nabla \boldsymbol{v}|^2\, dx \leqslant c(|\boldsymbol{F}|) g_{ik}^{-8/3}, \qquad k > k_0, \qquad i = 1, \ldots, m, \tag{3.8}$$

where $|\boldsymbol{F}| = \left(\sum_{k=1}^{m} |F_i|^2 \right)^{1/2}$.

Moreover, if the fluxes F_i, $i = 1, \ldots, m$, *are sufficiently small, then any other* (*diffferent from* $\boldsymbol{u}$) *solulion* $\boldsymbol{u}'$ *of problem* (0.2)-(0.3) *satisfies the relation*

$$\liminf_{k \to \infty} k^{-3} \int_{\Omega_{(k)}} |\nabla(\boldsymbol{u}' - \boldsymbol{u})|^2\, dx > 0\,. \tag{3.9}$$

4. – Strong solvability of the Navier-Stokes problem.

In this section we consider the Navier-Stokes problem (0.2)-(0.3) in weighted Sobolev and Hölder spaces. Problem (0.2)-(0.3) with zero fluxes $F_i = 0$, $i = 1, \ldots, m$, we denote by $(0.2)\text{-}(0.3)_0$.

4.1. *Solvability of the problem* $(0.2)\text{-}(0.3)_0$.

Let us consider the problem $(0.2)\text{-}(0.3)_0$ in spaces $V^{l,q}_{(\varkappa, \beta)}(\Omega)$ and $C^{l,\delta}_{(\varkappa, \beta)}(\Omega)$, $\beta_i > 0$, $i = 1, \ldots, m$, of exponentially vanishing at infinity functions.

THEOREM 4.1. (i) *Let* $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, *be a domain with* $m \geqslant 1$ *exits to infinity,* $\partial\Omega \in C^{l+2}$ *and* $\boldsymbol{f} \in V^{l,q}_{(\varkappa, \beta)}(\Omega)$ *with*

$$l \geqslant 0\,, \qquad q_0 = q_1 = \ldots = q_m > 1\,, \qquad \beta_* > \beta_i 0\,, \qquad \varkappa_i \text{ are arbitrary}\,. \tag{4.1}$$

Then for sufficiently small $\|\boldsymbol{f}; V^{l,q}_{(\varkappa, \beta)}(\Omega)\|$ *problem* $(0.2)\text{-}(0.3)_0$ *has a unique solution* $(\boldsymbol{u}, p)$ *with* $\boldsymbol{u} \in V^{l+2,q}_{(\varkappa, \beta)}(\Omega)$, $\nabla p \in V^{l,q}_{(\varkappa, \beta)}(\Omega)$ *and the following estimate holds true*

$$\|\boldsymbol{u}; V^{l+2,q}_{(\varkappa, \beta)}(\Omega)\| + \|\nabla p; V^{l,q}_{(\varkappa, \beta)}(\Omega)\| \leqslant c\|\boldsymbol{f}; V^{l,q}_{(\varkappa, \beta)}(\Omega)\|\,. \tag{4.2}$$

(ii) *Let* $\partial\Omega \in C^{l+2,\delta}$ *and* $\boldsymbol{f} \in C^{l,\delta}_{(\varkappa, \beta)}(\Omega)$,

$$l \geqslant 0\,, \qquad 1 > \delta > 0\,, \qquad \beta_* > \beta_i 0\,, \qquad \varkappa_i \text{ are arbitrary}\,. \tag{4.3}$$

If the norm $\|\boldsymbol{f};\, C^{l,\delta}_{(\varkappa,\beta)}(\Omega)\|$ *is sufficiently small, then* (0.2)-(0.3)$_0$ *has a unique solution* $(\boldsymbol{u}, p)$ *with* $\boldsymbol{u} \in C^{l+2,\delta}_{(\varkappa,\beta)}(\Omega)$, $\nabla p \in C^{l,\delta}_{(\varkappa,\beta)}(\Omega)$ *and*

$$\|\boldsymbol{u};\, C^{l+2,\delta}_{(\varkappa,\beta)}(\Omega)\| + \|\nabla p;\, C^{l,\delta}_{(\varkappa,\beta)}(\Omega)\| \leq c\|\boldsymbol{f};\, C^{l,\delta}_{(\varkappa,\beta)}(\Omega)\|. \tag{4.4}$$

In particular, there hold the estimates from Remark 2.1.

PROOF. Let either $\boldsymbol{u} \in V^{l+2,q}_{(\varkappa,\beta)}(\Omega)$ or $\boldsymbol{u} \in C^{l+2,\delta}_{(\varkappa,\beta)}(\Omega)$. We define

$$M\boldsymbol{u} := (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}.$$

The direct computations, using the weighted imbedding Theorem 1.3 and the condition $\beta_i > 0$, imply $M\boldsymbol{u} \in V^{l,q}_{(\varkappa,\beta)}(\Omega)$ (or $M\boldsymbol{u} \in C^{l,\delta}_{(\varkappa,\beta)}(\Omega)$) and

$$\|M\boldsymbol{u};\, V^{l,q}_{(\varkappa,\beta)}(\Omega)\| \leq c_*\|\boldsymbol{u};\, V^{l+2,q}_{(\varkappa,\beta)}(\Omega)\|^2, \tag{4.5}$$

$$\|M\boldsymbol{u};\, C^{l,\delta}_{(\varkappa,\beta)}(\Omega)\| \leq c_*\|\boldsymbol{u};\, C^{l+2,\delta}_{(\varkappa,\beta)}(\Omega)\|^2. \tag{4.6}$$

Moreover,

$$\begin{aligned}\|M\boldsymbol{u} - M\boldsymbol{v};\, V^{l,q}_{(\varkappa,\beta)}(\Omega)\| &\leq \\ \leq c_{**}(\|\boldsymbol{u};\, V^{l+2,q}_{(\varkappa,\beta)}(\Omega)\| &+ \|\boldsymbol{v};\, V^{l+2,q}_{(\varkappa,\beta)}(\Omega)\|)\,\|\boldsymbol{u}-\boldsymbol{v};\, V^{l+2,q}_{(\varkappa,\beta)}(\Omega)\|,\end{aligned} \tag{4.7}$$

$$\begin{aligned}\|M\boldsymbol{u} - M\boldsymbol{v};\, C^{l,\delta}_{(\varkappa,\beta)}(\Omega)\| &\leq \\ \leq c_{**}(\|\boldsymbol{u};\, C^{l+2,\delta}_{(\varkappa,\beta)}(\Omega)\| &+ \|\boldsymbol{v};\, C^{l+2,\delta}_{(\varkappa,\beta)}(\Omega)\|)\,\|\boldsymbol{u}-\boldsymbol{v};\, C^{l+2,\delta}_{(\varkappa,\beta)}(\Omega)\|.\end{aligned} \tag{4.8}$$

Let $\mathcal{L}$ be the operator of the linear Stokes problem (0.7)-(0.8)$_0$. According to Theorem 2.2 there exist bounded inverse operators $\mathcal{L}_V^{-1}$ and $\mathcal{L}_C^{-1}$:

$$\mathcal{L}_V^{-1}\colon V^{l,q}_{(\varkappa,\beta)}(\Omega) \to V^{l+2,q}_{(\varkappa,\beta)}(\Omega),$$

$$\mathcal{L}_C^{-1}\colon C^{l,\delta}_{(\varkappa,\beta)}(\Omega) \to C^{l+2,\delta}_{(\varkappa,\beta)}(\Omega).$$

(We mention that because of the condition $q_0 = \ldots = q_m$ the spaces $V^{l+2,q}_{(\varkappa,\beta)}(\Omega)$ and $\tilde{V}^{l+2,q}_{(\varkappa,\beta)}(\Omega)$ coincide.) Hence, problem (0.2)-(0.3)$_0$ is equivalent to an operator equations either in the space $V^{l+2,q}_{(\varkappa,\beta)}(\Omega)$ or in the space $C^{l+2,\delta}_{(\varkappa,\beta)}(\Omega)$:

$$\boldsymbol{u} = \mathcal{L}_V^{-1}M\boldsymbol{u} + \mathcal{L}_V^{-1}\boldsymbol{f} \equiv \mathcal{A}_V\boldsymbol{u},$$

$$\boldsymbol{u} = \mathcal{L}_C^{-1}M\boldsymbol{u} + \mathcal{L}_C^{-1}\boldsymbol{f} \equiv \mathcal{A}_C\boldsymbol{u}.$$

Let

$$\mathcal{B}_V(r_0) = \{\boldsymbol{u} \in V^{l+2,q}_{(\boldsymbol{\varkappa},\boldsymbol{\beta})}(\Omega) \colon \|\boldsymbol{u}; V^{l+2,q}_{(\boldsymbol{\varkappa},\boldsymbol{\beta})}(\Omega)\| \leqslant r_0\}$$

and

$$\mathcal{B}_C(r_0) = \{\boldsymbol{u} \in C^{l+2,\delta}_{(\boldsymbol{\varkappa},\boldsymbol{\beta})}(\Omega) \colon \|\boldsymbol{u}; C^{l+2,\delta}_{(\boldsymbol{\varkappa},\boldsymbol{\beta})}(\Omega)\| \leqslant r_0\}$$

be the balls in $V^{l+2,q}_{(\boldsymbol{\varkappa},\boldsymbol{\beta})}(\Omega)$ and $C^{l+2,\delta}_{(\boldsymbol{\varkappa},\boldsymbol{\beta})}(\Omega)$. If

$$\|\mathcal{L}_V^{-1}\|(r_0 c_* + r_0^{-1}\|\boldsymbol{f}; V^{l,q}_{(\boldsymbol{\varkappa},\boldsymbol{\beta})}(\Omega)\|) < 1\,,$$

$$\|\mathcal{L}_C^{-1}\|(r_0 c_* + r_0^{-1}\|\boldsymbol{f}; C^{l,\delta}_{(\boldsymbol{\varkappa},\boldsymbol{\beta})}(\Omega)\|) < 1\,,$$

estimates (4.5)-(4.6) imply

$$\mathcal{A}_V\mathcal{B}_V(r_0) \subset \mathcal{B}_V(r_0)\,, \qquad \mathcal{A}_C\mathcal{B}_C(r_0) \subset \mathcal{B}_C(r_0)\,.$$

Moreover, from (4.7)-(4.8) follows that $\mathcal{A}_V$ and $\mathcal{A}_C$ define the contraction mappings in $\mathcal{B}_V(r_0)$ and $\mathcal{B}_C(r_0)$, provided that

$$\|\mathcal{L}_V^{-1}\| r_0 c_{**} < 1/2\,, \qquad \|\mathcal{L}_C^{-1}\| r_0 c_{**} < 1/2\,.$$

Thus, the existens of a unique fixed point of the above operator equations follows from the Banach contraction principle. The theorem is proved.

4.2. *Solvability of the problem* (0.2)-(0.3) *with nonzero fluxes*, $n = 3$.

In this subsection we prove the main result of the paper. Let Ω be a three-dimensional domain with exits to infinity. We consider the problem (0.2)-(0.3) with nonzero fluxes. We look for the solutions $\boldsymbol{u}$ of (0.2)-(0.3) in weighted Sobolev and Hölder spaces $V^{l+2,q}_{(\boldsymbol{\varkappa},\boldsymbol{\beta})}(\Omega)$ and $C^{l+2,\delta}_{(\boldsymbol{\varkappa},\boldsymbol{\beta})}(\Omega)$. Each element $\boldsymbol{u}$ of these spaces satisfies in certain sence the decay estimate

$$|D^\alpha \boldsymbol{u}(x)| \leqslant c g_i(x_3)^{\gamma - |\alpha|}\,, \qquad x \in \Omega_i\,, \qquad |\alpha| \geqslant 0\,. \tag{4.9}$$

If $\boldsymbol{u}$ is the weak solution of (0.2)-(0.3), by Theorem 3.2 we have for $\boldsymbol{u}$ the estimate (3.8). It easy to compute that (4.9) together with (3.8) imply, minimally, the decay rate

$$\boldsymbol{u} \sim g_i(x_3)^{-11/6 - |\alpha|}\,,$$

i.e. $\gamma = -11/6$. Then

$$\Delta\boldsymbol{u} \sim g_i^{-23/6}(x_3)\,, \qquad (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \sim g_i^{-28/6}(x_3)\,.$$

Thus, the nonlinear term $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ decays at infinity faster than the linear one $\Delta\boldsymbol{u}$ and by bootstrap argument we can improve the estimates for $\boldsymbol{u}$. Having this simple idea as background, by repeated application of weighted imbedding Theorem 1.2 and resuls on the linear Stokes problem (0.7)-(0.8)$_0$ (see Theorem 2.1, 2.2), we prove that the weak solution $\boldsymbol{u}$ of the problem (0.2)-(0.3) belongs for arbitrary data to certain weighted Sobolev or Hölder space.

THEOREM 4.2. *Let $\Omega\subset\mathbb{R}^3$ be a domain with $m>1$ exits to infinity Ω_i of the form* (1.1) *and let $\partial\Omega\in C^{l+2,\delta}$, $0<\delta<1$. Assume that the functions g_i satisfy conditions* (1.2), (1.3), (3.4), (3.5) *and that for each $i\in\{1,\ldots,m\}$ there exists a number q_i^* with*

$$q_i^*\geqslant 3/2\,,\qquad \int_0^\infty g_i(t)^{-3q_i^*+2}\,dt<\infty\,. \tag{4.10}$$

Let $\boldsymbol{f}=0$.

(i) *Then for arbitrary fluxes $F_i\in\mathbb{R}^1$, $i=1,\ldots,m$, the weak solution $\boldsymbol{u}$ of* (0.2)-(0.3) *with the infinite Dirichlet integral* (*see Theorem* 3.2) *belongs to the space $V^{l+2,q}_{(\boldsymbol{\ae}^*,0)}(\Omega)$ with*

$$\ae_i^*=4+l-3q_i^*/q_i\,,\qquad i=1,\ldots,m\,. \tag{4.11}$$

Moreover, there exists the pressure function $\nabla p\in V^{l,q}_{(\boldsymbol{\ae}^*,0)}(\Omega)$ and the following estimate holds true

$$\|\boldsymbol{u};V^{l+2,q}_{(\boldsymbol{\ae}^*,0)}(\Omega)\|+\|\nabla p;V^{l,q}_{(\boldsymbol{\ae}^*,0)}(\Omega)\|\leqslant C(|\boldsymbol{F}|)\,. \tag{4.12}$$

In particular, $\boldsymbol{u}\in\widehat{H}^{q}_{\boldsymbol{\ae}^*}(\Omega)$ ($\boldsymbol{u}\in\widehat{H}^{q^*}(\Omega)$).

(ii) *The solution $(\boldsymbol{u},p)$ of problem* (0.2)-(0.3) *admits the following pointwise estimates*

$$|D^\alpha\boldsymbol{u}(x)|\leqslant C(|\boldsymbol{F}|)\,g_i(x_3)^{-2-|\alpha|}\,,\qquad x\in\Omega_i\,,\qquad 0\leqslant|\alpha|\leqslant l\,, \tag{4.13}$$

$$|D^\alpha\nabla p(x)|\leqslant C(|\boldsymbol{F}|)\,g_i(x_3)^{-3-|\alpha|}\,,\qquad x\in\Omega_i\,,\qquad 1\leqslant|\alpha|\leqslant l\,, \tag{4.14}$$

$$|p(x)|\leqslant C(|\boldsymbol{F}|)\int_0^{x_3}g_i(t)^{-4}\,dt+c_1\,. \tag{4.15}$$

PROOF. First of all, we mention that the solution $(\boldsymbol{u},p)$ is locally smooth up to the boundary $\partial\Omega$ (see V. A. Solonnikov, V. E. Shchadilov

(1973) [34]). We represent the velocity field $\boldsymbol{u}$ as a sum

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{A},$$

where $\boldsymbol{A}$ is the divergence free vector field, satisfying the flux conditions (0.3) and the estimates (1.8) (see Lemma 1.2). Then for $(\boldsymbol{v}, p)$ we get the following problem

$$(4.16)\quad \begin{cases} -\nu\Delta\boldsymbol{v}+\nabla p = -(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}-(\boldsymbol{A}\cdot\nabla)\boldsymbol{v}-(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}+\nu\Delta\boldsymbol{A}-(\boldsymbol{A}\cdot\nabla)\boldsymbol{A} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{v} = 0 \quad \text{in } \Omega, & \\ \boldsymbol{v} = 0 \quad \text{on } \partial\Omega, \quad \displaystyle\int_{\sigma_i}\boldsymbol{v}\cdot\boldsymbol{n}\,ds = 0, \quad i = 1, \ldots, m. & \end{cases}$$

We consider now $(\boldsymbol{v}, p)$ as a solution of the linear Stokes problem (0.7)-$(0.8)_0$ with the right-hand side

$$\boldsymbol{f}_1 = -(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{A} + \nu\Delta\boldsymbol{A} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{A}$$

and we prove the statements of the theorem by the bootstrap argument.

From Theorem 1.2 (i) with $l = 1$, $q_i = 2$, $\varkappa_i = 0$, $s_i = 6$ it follows that $\boldsymbol{v} \in L^6(\omega_{ik})$ and

$$\|\boldsymbol{v}; L^6(\omega_{ik})\| \leqslant c\|\nabla\boldsymbol{v}\|_{2,\,\omega_{ik}},$$

Therefore, the Hölder inequality deliveres

$$\|(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}; L^{3/2}(\omega_{ik})\| \leqslant c\|\nabla\boldsymbol{v}\|^2_{2,\,\omega_{ik}}.$$

Moreover, estimates (1.8) for $\boldsymbol{A}$ and the Hölder inequality furnish

$$\|(\boldsymbol{A}\cdot\nabla)\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{A}; L^{3/2}(\omega_{ik})\| \leqslant c(|\boldsymbol{F}|)g_{ik}^{-3/2}\|\nabla\boldsymbol{v}\|_{2,\,\omega_{ik}},$$

$$\|\nu\Delta\boldsymbol{A} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{A}; L^{3/2}(\omega_{ik})\| \leqslant c(|\boldsymbol{F}|)g_{ik}^{-2}.$$

Thus, $\boldsymbol{f}_1 \in L^{3/2}(\omega_{ik})$ and

$$\|\boldsymbol{f}_1; L^{3/2}(\omega_{ik})\| \leqslant c(|\boldsymbol{F}|)(\|\nabla\boldsymbol{v}\|^2_{2,\,\omega_{ik}} + g_{ik}^{-3/2}\|\nabla\boldsymbol{v}\|_{2,\,\omega_{ik}}) + c(|\boldsymbol{F}|)g_{ik}^{-2}.$$

In virtue of (3.8) the last inequality yields

$$(4.17)\quad \|\boldsymbol{f}_1; L^{3/2}(\omega_{ik})\| \leqslant c(|\boldsymbol{F}|)(g_{ik}^{-8/3} + g_{ik}^{-3/2-8/6} + g_{ik}^{-2}) \leqslant c(|\boldsymbol{F}|)g_{ik}^{-2}.$$

Applying to the solution $\boldsymbol{v}$ the local estimate (2.3) with $q_i = 3/2$, $\varkappa_i = 1$, $\beta_i = 0$, following by (3.8), (4.17), we derive

$$(4.18)\quad \|\boldsymbol{v}; V^{2,\,3/2}_{(1,\,0)}(\omega_{ik})\| \leqslant c(g_{ik}\|\boldsymbol{f}_1; L^{3/2}(\omega^*_{ik})\| + g_{ik}^{-3/2+6/3}\|\nabla\boldsymbol{v}\|_{2,\,\omega^*_{ik}}) \leqslant$$
$$\leqslant c(|\boldsymbol{F}|)(g_{ik}^{-1} + g_{ik}^{-5/6}) \leqslant c(|\boldsymbol{F}|)g_{ik}^{-5/6}.$$

In virtue of Theorem 1.2 (i) from the condition $v \in V^{2,\,3/2}_{(1,\,0)}(\omega_{ik})$ it follows $v \in L^{t}_{(1-3/s,\,0)}(\omega_{ik})$ with $\forall t \geqslant 3/2$ and $\nabla v \in L^{3}_{(1,\,0)}(\omega_{ik})$. Let $3/2 \leqslant t < 3$. Then

$$\int_{\omega^*_{ik}} |v|^t\,|\nabla v|^t\,dx \leqslant \left(\int_{\omega^*_{ik}} |v|^{3t/(3-t)}\,dx\right)^{(3-t)/3}\left(\int_{\omega^*_{ik}} |\nabla v|^3\,dx\right)^{t/3} \leqslant$$

$$\leqslant c g_{ik}^{3-2t}\|v;\,L^{3t/(3-t)}_{(1-(3-t)/t,\,0)}(\omega^*_{ik})\|^t g_{ik}^{-t}\|\nabla v;\,L^3_{(1,\,0)}(\omega^*_{ik})\|^t \leqslant$$

$$\leqslant c g_{ik}^{3-3t}\|v;\,V^{2,\,3/2}_{(1,\,0)}(\omega^*_{ik})\|^{2t} \leqslant c(|F|)\,g_{ik}^{3-3t}g_{ik}^{-10t/6}$$

and

$$\|(A\cdot\nabla)v + (v\cdot\nabla)A;\,L^t(\omega^*_{ik})\|^t \leqslant c g_{ik}^{3-4t}\|\nabla v;\,L^3_{(1,\,0)}(\omega^*_{ik})\|^t \leqslant$$

$$\leqslant c g_{ik}^{3-4t}\|v;\,V^{2,\,3/2}_{(1,\,0)}(\omega^*_{ik})\|^t \leqslant c(|F|)\,g_{ik}^{3-3t}g_{ik}^{-11t/6},$$

$$\|\nu\Delta A - (A\cdot\nabla)A;\,L^t(\omega^*_{ik})\|^t \leqslant c(|F|)\,g_{ik}^{3-3t}g_{ik}^{-t}.$$

Thus,

$$\text{(4.19)}\qquad \|f_1;\,L^t_{(1,\,0)}(\omega^*_{ik})\|^t \leqslant$$

$$\leqslant c(|F|)\,g_{ik}^{t}g_{ik}^{3-3t}(g_{ik}^{-10t/6} + g_{ik}^{-11t/6} + g_{ik}^{-t}) \leqslant c(|F|)g_{ik}^{3-3t}.$$

Let $3/2q_i^* < 3$. Then we can take in (4.19) $t = q_i^*$ and we get in view of Lemma 1.1 (see the inequality (1.4))

$$\text{(4.20)}\qquad \|f_1;\,L^{q_i^*}_{(1,\,0)}(\Omega_i\setminus\Omega_{(k_0)})\|^{q_i^*} \leqslant c\sum_{k=k_0}^{\infty}\|f_1;\,L^{q_i^*}_{(1,\,0)}(\omega^*_{ik})\|^{q_i^*} \leqslant$$

$$\leqslant c(|F|)\sum_{k=k_0}^{\infty} g_{ik}^{3-3q_i^*} \leqslant c(|F|)\int_0^{\infty} g_i(t)^{-3q_i^*+2}\,dt < c(|F|).$$

Hence, $f_1 \in \widetilde{L}^{q_i^{**}}_{(1,\,0)}(\Omega)$ (3) and by Theorem 2.2 (i) the solution v of the Stokes problem (0.7)-(0.8)$_0$ belongs to the space $V^{2,\,q_i^*}_{(1,\,0)}(\Omega_i)$. Moreover, $\nabla p \in L^{q_i^*}_{(1,\,0)}(\Omega_i)$ and

$$\text{(4.21)}\qquad \|v;\,V^{2,\,q_i^*}_{(1,\,0)}(\Omega_i)\| + \|\nabla p;\,L^{q_i^*}_{(1,\,0)}(\Omega_i)\| \leqslant c\|f_1;\widetilde{L}^{q_i^*}_{(1,\,0)}(\Omega_i)\| \leqslant C(|F|).$$

(3) From (4.20) it follows that $f_1 \in L^{q_i^*}_{(1,\,0)}(\Omega\setminus\Omega_{(k_0)})$. However, we know that the solution v and the vector field A are locally smooth up to the boundary. Hence, $f_1 \in L^q(G)$ for $\forall q \geqslant 3/2$ and for arbitrary bounded subdomain $G \subset \Omega$.

Futher, for $3/2 < t < 3$ the local estimate (2.3) delivers

$$(4.22)\qquad \|v;\, V^{2,\,t}_{(1,\,0)}(\omega_{ik})\|^t \leq c(\|f_1;\, L^t_{(1,\,0)}(\omega^*_{ik})\|^t + g_{ik}^{3-3t/2}\|\nabla v\|^t_{2,\,\omega^*_{ik}}) \leq$$

$$\leq c(|F|)(g_{ik}^{3-3t} + g_{ik}^{3-17t/6}) \leq c(|F|)\, g_{ik}^{3-17t/6}\,.$$

We fix $t = 2$. Due to Theorem 1.3 the condition $v \in V^{2,\,2}_{(1,\,0)}(\omega^*_{ik})$ yields

$$(4.23)\qquad \begin{cases} v \in C^{0,\,\delta}_{(\delta+1/2,\,0)}(\omega^*_{ik}), & 0 < \delta \leq 1/2\,, \\ \nabla v \in L^s_{(3/2-3/s,\,0)}(\omega^*_{ik}), & 2 \leq s \leq 6\,. \end{cases}$$

By inclusion (4.23)

$$\sup_{\omega^*_{ik}}(|v|\, g_{ik}^{1/2}) < \infty$$

and, therefore, in virtue of (4.22) with $t = 2$, for arbitrary $2 \leq s \leq 6$ we have

$$g^s_{ik}\int_{\omega^*_{ik}} |v|^s\, |\nabla v|^s\, dx \leq c g_{ik}^{-s+3}\|v;\, C^{0,\,\delta}_{(\delta+1/2,\,0)}(\omega^*_{ik})\|^s\, \|\nabla v;\, L^s_{(3/2-3/s,0)}(\omega^*_{ik})\|^s \leq$$

$$\leq c g_{ik}^{-s+3}\|v;\, V^{2,\,2}_{(1,\,0)}(\omega^*_{ik})\|^{2s} \leq c(|F|)\, g_{ik}^{-s+3} g_{ik}^{-8s/3} \leq c(|F|) g_{ik}^{3-3s} g_{ik}^{-2s/3}\,,$$

$$g^s_{ik}\|(A\cdot\nabla)v + (v\cdot\nabla)A;\, L^s(\omega^*_{ik})\|^s \leq c g_{ik}^{-5s/2+3}\|\nabla v;\, L^s_{(1,\,0)}(\omega^*_{ik})\|^s \leq$$

$$\leq c g_{ik}^{3-5s/2}\|v;\, V^{2,\,2}_{(1,\,0)}(\omega^*_{ik})\|^s \leq c(|F|)\, g_{ik}^{3-3s} g_{ik}^{-5s/6}\,,$$

$$g^s_{ik}\|\nu\Delta A - (A\cdot\nabla)A;\, L^s(\omega^*_{ik})\|^s \leq c(|F|)\, g_{ik}^{3-3s}\,.$$

Thus,

$$(4.24)\qquad \|f_1;\, L^s_{(1,\,0)}(\omega^*_{ik})\|^s \leq$$

$$\leq c(|F|)(g_{ik}^{3-3s} g_{ik}^{-2s/3} + g_{ik}^{3-3s} g_{ik}^{-5s/6} + g_{ik}^{3-3s}) \leq c(|F|) g_{ik}^{3-3s}\,.$$

If $3 \leq q^*_i \leq 6$, we take $s = q^*_i$ and as in (4.20) we derive

$$\|f_1;\widetilde{L}^{q^*_i}_{(1,\,0)}(\Omega_i)\|^{q^*_i} \leq c(|F|)\int_0^\infty g_i(t)^{-3q^*_i+2}\, dt < c(|F|)\,.$$

Then by Theorem 2.2 (i) $v \in V^{2,\,q^*_i}_{(1,\,0)}(\Omega_i)$, and we get the estimate (4.21).

Let us take now $s = 6$ and repeat the above bootstrap arguments. We have $v \in V^{2,\,6}_{(1,\,0)}(\omega_{ik})$ and, in virtue of local estimate (2.3) followed by

(3.8), (4.24), we get

$$\|v;\, V^{2,6}_{(1,0)}(\omega_{ik})\| \leq c(\|f_1;\, L^{6}_{(1,0)}(\omega^*_{ik})\| + g_{ik}^{-3/2+1/2}\|\nabla v\|_{2,\omega^*_{ik}}) \leq \tag{4.25}$$

$$\leq c(|F|)(g_{ik}^{-5/2} + g_{ik}^{-7/3}) \leq c(|F|)\, g_{ik}^{-7/3}.$$

By Theorem 1.2 (ii)

$$V^{2,6}_{(1,0)}(\omega^*_{ik}) \subset C^{1,\delta}_{(\delta+1/2,0)}(\omega^*_{ik}), \qquad 0 < \delta \leq 1/2,$$

and it is easy to verify that $(v\cdot\nabla)v \in C^{0,\delta}_{(\delta,0)}(\omega^*_{ik})$ and

$$\|(v\cdot\nabla)v;\, C^{0,\delta}_{(\delta,0)}(\omega^*_{ik})\| \leq c\|v;\, C^{1,\delta}_{(\delta+1/2,0)}(\omega^*_{ik})\|^2 \leq \tag{4.26}$$

$$\leq c\|v;\, V^{2,6}_{(1,0)}(\omega^*_{ik})\|^2 \leq c(|F|)\, g_{ik}^{-14/3}.$$

Moreover,

$$\left\{\begin{aligned} &\|(A\cdot\nabla)v + (v\cdot\nabla)A;\, C^{0,\delta}_{(\delta,0)}(\omega^*_{ik})\| \leq \\ &\qquad \leq c(|F|)\, g_{ik}^{-5/2}\|v;\, C^{1,\delta}_{(\delta+1/2,0)}(\omega^*_{ik})\| \leq \\ &\qquad \leq c(|F|)\, g_{ik}^{-5/2} g_{ik}^{-7/3} \leq c(|F|)\, g_{ik}^{-29/6}, \\ &\|\nu\Delta A - (A\cdot\nabla)A;\, C^{0,\delta}_{(\delta,0)}(\omega^*_{ik})\| \leq c(|F|)\, g_{ik}^{-4}. \end{aligned}\right. \tag{4.27}$$

Hence,

$$\|f_1;\, C^{0,\delta}_{(\delta,0)}(\omega^*_{ik})\| \leq c(|F|)(g_{ik}^{-14/3} + g_{ik}^{-29/6} + g_{ik}^{-4}) \leq c(|F|)\, g_{ik}^{-4}. \tag{4.28}$$

Since the constant in (4.28) is independent of k, we obtain $f_1 \in C^{0,\delta}_{(4+\delta,0)}(\Omega_i)$ and by Theorem 2.2 (ii) $v \in C^{2,\delta}_{(4+\delta,0)}(\Omega_i)$, $\nabla p \in C^{0,\delta}_{(4+\delta,0)}(\Omega_i)$,

$$\|v;\, C^{2,\delta}_{(4+\delta,0)}(\Omega_i)\| + \|\nabla p;\, C^{0,\delta}_{(4+\delta,0)}(\Omega_i)\| \leq c\|f_1;\, C^{0,\delta}_{(4+\delta,0)}(\Omega_i)\| \leq c(|F|).$$

By induction we easily get $f_1 \in C^{l,\delta}_{(4+l+\delta,0)}(\Omega_i)$ and again by Theorem 2.2 (ii)

$$\|v;\, C^{l+2,\delta}_{(4+l+\delta,0)}(\Omega_i)\| + \|\nabla p;\, C^{l,\delta}_{(4+l+\delta,0)}(\Omega_i)\| \leq \tag{4.29}$$

$$\leq c\|f_1;\, C^{l,\delta}_{(4+l+\delta,0)}(\Omega_i)\| \leq c(|F|).$$

Hence, v and ∇p satisfy estimates (4.13), (4.14). By using (4.13), (4.14) it is easy to verify that $v \in V^{l+2,q_i}_{(\varkappa^*_i,0)}(\Omega_i)$, $\nabla p \in V^{l,q_i}_{(\varkappa^*_i,0)}(\Omega_i)$ with $\varkappa^*_i$ defined by (4.11) and that the estimate (4.12) holds true. Sinse A satisfies the same estimates, we derive (4.12)-(4.14) for u. Let us prove the last estimate (4.15) for $p(x)$. Let x be an arbitrary point in Ω_i and x_0 be a fixed point in ω_{ik_0}. Denote by γ a smooth counter connecting x_0 and x and lying in Ω_i. Assume that γ is given by the equations $x_j = \gamma_j(x_n)$, $j = 1, \ldots, n-1$, and that $(1 + \gamma_1'(x_n)^2 + \ldots + \gamma_{n-1}'(x_n)^2)^{1/2} \leqslant const.$ We represent the function $p(x)$ in the form

$$p(x) = p(x_0) + \int_{x_0}^{x} \frac{\partial p}{\partial \gamma}\, d\gamma .$$

Then,

$$(4.30) \qquad |p(x)| \leqslant |p(x_0)| + \left| \int_{x_0}^{x} \frac{\partial p}{\partial \gamma}\, d\gamma \right| \leqslant |p(x_0)| +$$

$$+ \left(\sup_{x \in \Omega_i} |\nabla p(x) g_i(x_n)^{n+1}| \right) \int_{x_{0n}}^{x_n} g_i(t)^{-n-1} \sqrt{1 + \gamma_1'(t)^2 + \ldots + \gamma_{n-1}'(t)^2}\, dt \leqslant$$

$$\leqslant |p(x_0)| + c \| \nabla p;\, C^{l,\delta}_{(\text{\ae},0)}(\Omega) \| \int_0^{x_n} g_i(t)^{-n-1}\, dt \leqslant$$

$$\leqslant c \| p;\, C^{l,\delta}(\omega_{ik_0}) \| + c \| \nabla p;\, C^{l,\delta}_{(\text{\ae},0)}(\Omega) \| \int_0^{x_n} g_i(t)^{-n-1}\, dt .$$

By local regularity results the norm $\|p;\, C^{l,\delta}(\omega_{ik_0})\|$ can be estimated by $c(|\boldsymbol{F}|) \int_0^{x_n} g_i(t)^{-n-1} dt + c_1$. Therefore, inequalities (4.29), (4.30) imply (4.15). The theorem is proved.

REMARK 4.1. The results of Theorem 4.2 remain valid also for nonzero right-hand sides $\boldsymbol{f}$ vanishing at infinity sufficiently rapidly.

REMARK 4.2. If the conditions (3.4), (3.5) are not valid, the results of Theorem 4.2 can be proved for sufficiently small data by using the Banach contraction principle and repeating the arguments of Theorem 4.1.

4.2. *Solvability of the problem* (0.2)-(0.3) *with nonzero fluxes,* $n = 2$.

For two dimentional domains Ω with exits to infinity the analogous result can not be proved by the same method. This is related to the fact that in this case the solenoidal vector field $\boldsymbol{A}$, satisfying the fluxconditons (0.3), has decay rate as $g_i(x_2)^{-1}$ and hence,

$$\Delta \boldsymbol{A} \sim g_i(x_2)^{-3}, \quad (\boldsymbol{A}\cdot\nabla)\boldsymbol{A} \sim g_i(x_2)^{-3} \quad \text{as } |x| \to \infty, \quad x \in \Omega_i .$$

If the solution $\boldsymbol{u}$ to (0.2), (0.3) also have such decay properties, then $-\Delta\boldsymbol{u}$ and the nonlinear term $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ have the same order $O(g_i(x_2)^{-3})$ as $|x| \to \infty$, $x \in \Omega_i$, and therefore, the bootstraps give no improvement. However, for small data we can reduce the problem (0.2)-(0.3) to a contraction operator equation and apply the Banach contraction principle. For example, we have the following

THEOREM 4.3. *Let $\Omega \subset \mathbb{R}^2$ be a domain with $m > 1$ exits to infinity Ω_i and let $\partial\Omega \in C^{l+2,\delta}$, $0 < \delta < 1$. Assume that $\boldsymbol{f} = 0$. Then for sufficiently small $|\boldsymbol{F}|$ problem* (0.2)-(0.3) *has a unique solution $(\boldsymbol{u}, p)$ with $\boldsymbol{u} \in C^{l+2,\delta}_{(\boldsymbol{\varkappa},0)}(\Omega)$, $\nabla p \in C^{l,\delta}_{(\boldsymbol{\varkappa},0)}(\Omega)$, where*

$$\varkappa_i = 3 + l + \delta, \quad i = 1, \dots, m. \tag{4.31}$$

Moreover, there holds the estimate

$$\|\boldsymbol{u}; C^{l+2,\delta}_{(\boldsymbol{\varkappa},0)}(\Omega)\| + \|\nabla p; C^{l,\delta}_{(\boldsymbol{\varkappa},0)}(\Omega)\| \leqslant C(|\boldsymbol{F}|). \tag{4.32}$$

In particular, the solution $(\boldsymbol{u}, p)$ of problem (0.2)-(0.3) *admits the following pointwise estimates*

$$\begin{cases} |D^\alpha \boldsymbol{u}(x)| \leqslant C(|\boldsymbol{F}|)\, g_i(x_2)^{-1-|\alpha|}, & x \in \Omega_i, \quad 0 \leqslant |\alpha| \leqslant l, \\ |D^\alpha \nabla p(x)| \leqslant C(|\boldsymbol{F}|)\, g_i(x_2)^{-2-|\alpha|}, & x \in \Omega_i, \quad 0 \leqslant |\alpha| \leqslant l, \\ |p(x)| \leqslant C(|\boldsymbol{F}|) \displaystyle\int_0^{x_2} g_i(t)^{-3}\, dt + c_1 . \end{cases} \tag{4.33}$$

PROOF. We look for the velocity field $\boldsymbol{u}$ in the form

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{A},$$

where the divergence free vector field A satisfies the flux condition (0.3) and the estimates (1.8). For $(\boldsymbol{v}, p)$ we obtain the problem (4.16). Let

$$M\boldsymbol{v} := -(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{A} + \nu\Delta\boldsymbol{A} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{A} .$$

If $\boldsymbol{v} \in C^{l+2,\delta}_{(\boldsymbol{\varkappa},0)}(\Omega)$ with $\boldsymbol{\varkappa}$ given by (4.31), we get in virtue of (1.8), that

$Mv \in C^{l,\delta}_{(\text{æ},0)}(\Omega)$ and

(4.34) $\|Mv; C^{l,\delta}_{(\text{æ},0)}(\Omega)\| \leq$

$$\leq c_* \|v; C^{l+2,\delta}_{(\text{æ},0)}(\Omega)\|^2 + C(|F|)(\|v; C^{l+2,\delta}_{(\text{æ},0)}(\Omega)\| + 1).$$

(4.35) $\|Mv - Mw; C^{l,\delta}_{(\text{æ},0)}(\Omega)\| \leq c_{**}(\|v; C^{l+2,\delta}_{(\text{æ},0)}(\Omega)\| +$

$$+\|w; C^{l+2,\delta}_{(\text{æ},0)}(\Omega)\|)\|v-w; C^{l+2,\delta}_{(\text{æ},0)}(\Omega)\| + C(|F|)\|v-w; C^{l+2,\delta}_{(\text{æ},0)}(\Omega)\|$$

with

$$C(|F|) \to 0, \qquad \text{as } |F| \to 0.$$

Problem (4.16) is equivalent to an operator equations in the space $C^{l+2,\delta}_{(\text{æ},0)}(\Omega)$:

(4.36) $$u = \mathcal{L}_C^{-1} Mv \equiv \mathcal{a}_C v.$$

($\mathcal{L}_C^{-1}$ is the inverse operators to the linear Stokes problem (0.7)-$(0.8)_0$, $\mathcal{L}_C^{-1}: C^{l,\delta}_{(\text{æ},0)}(\Omega) \to C^{l+2,\delta}_{(\text{æ},0)}(\Omega)$.) From (4.34), (4.35) it follows that for sufficiently small $|F|$ the operator $\mathcal{a}_C$ is a contraction in a small ball $\mathcal{B}_C(r_0) = \{v \in C^{l+2,\delta}_{(\text{æ},0)}(\Omega): \|v; C^{l+2,\delta}_{(\text{æ},0)}(\Omega)\| \leq r_0\}$ of the space $C^{l+2,\delta}_{(\text{æ},0)}(\Omega)$. Thus, the existens of a unique fixed point of (4.36) and the estimate (4.32) follow from the Banach contraction principle. Finally, the estimate $(4.33)_3$ can be proved just in the same way as (4.15) (see (4.30)). The theorem is proved.

5. – The Navier-Stokes problem in domains with cylindrical exits to infinity.

Let for some exit to infinity Ω_i we have $g_i(t) = g_0$, i.e. Ω_i coincides with a semicylinder $\{x \in \mathbb{R}^n: |x'| < g_0, x_n > 1\}$. Then the weights $g_i(x_n)^{\text{æ}_i}$ do not contribute in the norms of function spaces. Moreover,

$$\alpha_i(t) = \int_0^t g_i(\tau)^{-1}\, d\tau = g_0 t$$

and the norms of the spaces $V^{l,q}_{(\text{æ},\beta)}(\Omega)$ and $C^{l,\delta}_{(\text{æ},\beta)}(\Omega)$ can be simplified. For primality we suppose that all exits are cylindrical and, without any loss of generality, we take $g_0 = 1$. The function spaces $V^{l,q}_{(\text{æ},\beta)}(\Omega)$ and $C^{l,\delta}_{(\text{æ},\beta)}(\Omega)$ in this case we mention by $W^{l,q}_{\beta}(\Omega)$ and $\Lambda^{l,\delta}_{\beta}(\Omega)$. The corresponding norms are given by

$$\|u;\, W^{l,q}_{\beta}(\Omega)\| =$$

$$= \|u;\, W^{l,q_0}(\Omega_{(k_0+1)})\| + \sum_{i=1}^{m} \|u \exp(\beta_i x_n);\, W^{l,q_i}(\Omega_i \setminus \Omega_{(k_0)})\|\,;$$

and

$$\|u;\, \Lambda^{l,\delta}_{\beta}(\Omega)\| = \|u;\, C^{l,\delta}(\Omega_{(k_0)})\| + \sum_{i=1}^{m} \|u \exp(\beta_i x_n);\, C^{l,\delta}(\Omega_i)\|\,.$$

Analogously, from $\widetilde{V}^{l,q}_{(\varkappa,\beta)}(\Omega)$ we get the spase $\widetilde{W}^{l,q}_{\beta}(\Omega)$.

The Theorem 2.2 for domains with cylindrical exits to infinity specifies

THEOREM 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain with $m > 1$ cylindrical exits to infinity.*

(i) *Assume that $\partial\Omega \in C^{l+2}$, $\boldsymbol{f} \in W^{l,q}_{\beta}(\Omega)$, $l \geq 0$, $q_0 = \ldots = q_m > 1$, $\beta_* > \beta_i > 0$, and the norm $\|\boldsymbol{f};\, W^{l,q}_{\beta}(\Omega)\|$ is sufficiently small. Then there exists a unique solution $(\boldsymbol{u}, p)$ to problem (0.2)-(0.3)$_0$ with $\boldsymbol{u} \in W^{l+2,q}_{\beta}(\Omega)$, $\nabla p \in W^{l,q}_{\beta}(\Omega)$ and*

$$\|\boldsymbol{u};\, W^{l+2,q}_{\beta}(\Omega)\| + \|\nabla p;\, W^{l,q}_{\beta}(\Omega)\| \leq c\|\boldsymbol{f};\, W^{l,q}_{\beta}(\Omega)\|\,. \tag{5.1}$$

(ii) *Let $\partial\Omega \in C^{l+2,\delta}$, $\boldsymbol{f} \in \Lambda^{l,\delta}_{\beta}(\Omega)$, $l \geq 0$, $\delta \in (0,1)$, $0 < \beta_j < \beta_*$. Then for sufficiently small $\|\boldsymbol{f};\, \Lambda^{l,\delta}_{\beta}(\Omega)\|$ problem (0.2)-(0.3)$_0$ has a unique solution $(\boldsymbol{u}, p)$ with $\boldsymbol{u} \in \Lambda^{l+2,\delta}_{\beta}(\Omega)$, $\nabla p \in \Lambda^{l,\delta}_{\beta}(\Omega)$ and*

$$\|\boldsymbol{u};\, \Lambda^{l+2,\delta}_{\beta}(\Omega)\| + \|\nabla p;\, \Lambda^{l,\delta}_{\beta}(\Omega)\| \leq c\|\boldsymbol{f};\, \Lambda^{l,\delta}_{\beta}(\Omega)\|\,. \tag{5.2}$$

Let us consider problem (0.2)-(0.3) with nonzero fluxes. First, let Ω coincides with an infinite cylinder $Z = \{x \in \mathbb{R}^n\colon |x'| < 1\}$. It is well known that (0.2)-(0.3) has in Z an exact solution $(\boldsymbol{u}^0, p^0)$ with non-zero flux. This solution has been constructed by Poiseuille (e.g. [14]) and it has the form

$$\boldsymbol{u}^0(x) = (0, \ldots, 0, u^0_n(x'))\,, \qquad p^0(x) = \nu x_n + c\,, \tag{5.3}$$

where

$$-\Delta u^0_n(x') = 1 \quad \text{in } |x'| < 1\,,$$

$$u^0_n(x') = 0 \quad \text{on } |x'| = 1\,.$$

Moreover,

$$\kappa_0 \equiv \int\limits_{|x'|<1} u_n^0(x')\,dx = \int\limits_{|x'|<1} |\nabla_x u_n^0(x')|^2\,dx' > 0\,.$$

For the domain Ω with $m > 1$ cylindrical exits to infinity Ω_i we take in each exit Ω_i a corresponding exact solution $(\boldsymbol{u}_i^0, p_i^0)$, $i = 1, \ldots, m$, and we define

$$U^0 = \sum_{i=1}^m \zeta_i \frac{F_i}{\kappa_{i0}} \boldsymbol{u}_i^0 + \boldsymbol{w}\,, \qquad P^0 = \sum_{i=1}^m \zeta_i \frac{F_i}{\kappa_{i0}} p_i^0\,, \tag{5.4}$$

where ζ_i are smooth cut-off functions with $\zeta_i = 1$ in $\Omega_i \backslash \Omega_{ik_0+1}$, $\zeta_i = 0$ in Ω_{ik_0} and $\boldsymbol{w} \in W^{l+2,q_0}(\Omega_{(k_0+2)})$ (or $\boldsymbol{w} \in C^{l+2,\delta}(\Omega_{(k_0+2)})$) is the solution of the problem

$$\operatorname{div} \boldsymbol{w} = -\sum_{i=1}^m \frac{F_i}{\kappa_{i0}} \nabla\zeta_i \cdot \boldsymbol{u}_i^0 \quad \text{in } \Omega_{(k_0+2)}\,,$$

$$\operatorname{supp} \boldsymbol{w} \subset \overline{\Omega}_{(k_0+2)}\,, \qquad \boldsymbol{w}_i = 0 \quad \text{on } \partial\Omega$$

(see [9]). Then

$$\operatorname{div} U^0 = 0 \quad \text{in } \Omega\,, \qquad U^0 = 0 \quad \text{on } \partial\Omega$$

and

$$\int\limits_{\sigma_i} U^0 \cdot \boldsymbol{n}\,ds = F_i\,, \qquad i = 1, \ldots, m\,. \tag{5.5}$$

For small data we prove the existence of a unique solution $(\boldsymbol{u}, p)$ of (0.2)-(0.3) which tends exponentially in each exit to infinity to the corresponding Poiseuille solution.

THEOREM 5.2. *Assume that $\partial\Omega$ satisfies the condition of Theorem* 5.1.

(i) *Let $\boldsymbol{f} \in W_{\boldsymbol{\beta}}^{l,q}(\Omega)$, $l \geq 0$, $q_0 = \ldots = q_m > 1$, $0 < \beta_i < \beta_*$ and $\|\boldsymbol{f}; W_{\boldsymbol{\beta}}^{l,q}(\Omega)\|$, $|\boldsymbol{F}|$ be sufficiently small. Then there exists a unique solution $(\boldsymbol{u}, p)$ to problem* (0.2)-(0.3) *having the representation*

$$\boldsymbol{u} = U^0 + \boldsymbol{v}\,, \qquad p = P^0 + q\,, \tag{5.6}$$

where U^0, P^0 are functions (5.4). *Moreover,*

$$\|\boldsymbol{v}; W_{\boldsymbol{\beta}}^{l+2,q}(\Omega)\| + \|\nabla q; W_{\boldsymbol{\beta}}^{l,q}(\Omega)\| \leq c\left(\|\boldsymbol{f}; W_{\boldsymbol{\beta}}^{l,q}(\Omega)\| + \sum_{i=1}^m |F_i|\right). \tag{5.7}$$

(ii) *If* $f \in \Lambda_{\boldsymbol\beta}^{l,\delta}(\Omega)$, $l \geq 0$, $\delta \in (0,1)$, $0 < \beta_i < \beta_*$, *and* $\|f; \Lambda_{\boldsymbol\beta}^{l,\delta}(\Omega)\|$, $|F_i|$, $i = 1, \ldots, m$, *are sufficiently small, there exists a unique solution* $(\boldsymbol{u}, p)$ *to* (0.2)-(0.3) *having representation* (5.6) *with* $\boldsymbol{v} \in \Lambda_{\boldsymbol\beta}^{l+2,\delta}(\Omega)$, $\nabla q \in \Lambda_{\boldsymbol\beta}^{l,\delta}(\Omega)$ *and*

$$\|\boldsymbol{v}; \Lambda_{\boldsymbol\beta}^{l+2,\delta}(\Omega)\| + \|\nabla q; \Lambda_{\boldsymbol\beta}^{l,\delta}(\Omega)\| \leq c\left(\|\boldsymbol{f}; \Lambda_{\boldsymbol\beta}^{l,\delta}(\Omega)\| + \sum_{i=1}^{m} |F_i|\right). \tag{5.8}$$

PROOF. For $(\boldsymbol{v}, q)$ we get the following problem

$$-\nu\Delta\boldsymbol{v} + \nabla q = -(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{U}^0\cdot\nabla)\boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{U}^0 + \tilde{\boldsymbol{f}} \quad \text{in } \Omega,$$

$$\operatorname{div}\boldsymbol{v} = 0 \quad \text{in } \Omega,$$

$$\boldsymbol{v} = 0 \quad \text{on } \partial\Omega, \int_{\sigma_i} \boldsymbol{v}\cdot\boldsymbol{n}\, ds = 0, \quad i = 1, \ldots, m$$

with the right-hand side

$$\tilde{\boldsymbol{f}} = \boldsymbol{f} + \boldsymbol{f}_1, \quad \boldsymbol{f}_1 = \nu\Delta\boldsymbol{U}^0 - (\boldsymbol{U}^0\cdot\nabla)\boldsymbol{U}^0 - \nabla P^0.$$

Since $(\boldsymbol{U}^0\cdot\nabla)\boldsymbol{U}^0$ is equal to zero for large $|x|$ and (u_i^0, p_i^0) are the exact solutions to the Navier-Stokes system, $\boldsymbol{f}_1$ has a compact support. Therefore, for small $|F_i|$ and $\|\boldsymbol{f}; W_\beta^{l,q}(\Omega)\|$ (or $\|\boldsymbol{f}; \Lambda_{\boldsymbol\beta}^{l,\delta}(\Omega)\|$) the last problem is equivalent to an operator equation in the space $W_\beta^{l+2,q}(\Omega)$ (or $\Lambda_{\boldsymbol\beta}^{l+2,\delta}(\Omega)$):

$$\boldsymbol{v} = \mathcal{A}\boldsymbol{v}$$

with the contraction operator $\mathcal{A}$ and the statement of the theorem follows from Banach contraction principle.

REMARK 5.1. *For domains with cylindrical exits to infinity the Navier-Stokes problem with prescribed fluxes was considered by C. J. Amick* (1977, 1978) [3], [4], *O. A. Ladyzhenskaya, V. A. Solonnnikov* (1980) [13], *where a solution, approaching exponentially the Poiseuille flow, was constructed. We also mention the paper of S. A. Nazarov, K. Pileckas* (1983) [16], *where certain existence theorems for regular solutions are proved in domains with periodical exits to infinity.*

REFERENCES

[1] F. ABERGEL - J. L. BONA, *A mathematical theory for viscous free-surface flows over a perturbed plane*, Arch. Rational Mech. Anal., **118** (1992), pp. 71-93.

[2] R. A. ADAMS, *Sobolev Spaces*, Academic Press, New York, San Francisco, London (1975).

[3] C. J. AMICK, *Steady solutions of the Navier-Stokes equations in unbounded channels and pipes*, Ann. Scuola Norm. Sup. Pisa, 4 (1977), pp. 473-513.

[4] C. J. AMICK, *Properties of steady Navier-Stokes solutions for certain unbounded channels and pipes, Nonlinear analisys*, Theory, Meth., Appl., 2 (1978), pp. 689-720.

[5] C. J. AMICK - L. E. FRAENKEL, *Steady solutions of the Navier-Stokes equations representing plane flow in channels of various types*, Acta Math., **144** (1980), pp. 81-152.

[6] W. BORCHERS - K. PILECKAS, *Existence, uniqueness and asymptotics of steady jets*, Arch. Rational Mech. Anal., **120** (1992), pp. 1-49.

[7] W. BORCHERS - G. P. GALDI - K. PILECKAS, *On the uniqueness of Leray-Hopf solutions for the flow through an aperture*, Arch. Rational Mech. Anal., **122** (1993), pp. 19-33.

[8] G. P. GALDI, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations*, Springer Tracts in Natural Philosophy, **38**, **39** (1994).

[9] L. V. KAPITANSKII - K. PILECKAS, *On spaces of solenoidal vector fields and boundary value problems for the Navier-Stokes equations in domains with noncompact boundaries*, Trudy Mat. Inst. Steklov, **159** (1983), pp. 5-36; English transl.: Proc. Math. Inst. Steklov, **159**, issue 2 (1984), pp. 3-34.

[10] O. A. LADYZHENSKAYA, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, London, Paris (1969).

[11] O. A. LADYZHENSKAYA - V. A. SOLONNNIKOV, *On some problems of vector analysis and generalized formulations of boundary value problems for the Navier-Stokes equations*, Zapiski Nauchn. Sem. LOMI, **59** (1976), pp. 81-116; English transl.: J. Sov. Math., **10**, No. 2 (1978), pp. 257-285.

[12] O. A. LADYZHENSKAYA - V. A. SOLONNNIKOV, *On the solvability of boundary value problems for the Navier-Stokes equations in regions with noncompact boundaries*, Vestnik Leningrad. Univ. (Ser. Mat. Mekh. Astr. vyp. 3), **13** (1977), pp. 39-47; English transl.: Vestnik Leningrad Univ. Math., **10** (1982), pp. 271-280.

[13] O. A. LADYZHENSKAYA - V. A. SOLONNNIKOV, *Determination of the solutions of boundary value problems for stationary Stokes and Navier-Stokes equations having an unbounded Dirichlet integral*, Zapiski Nauchn. Sem. LOMI, **96** (1980), pp. 117-160; English transl.: J. Sov. Math., **21**, No. 5 (1983), pp. 728-761.

[14] L. D. LANDAU - E. M. LIFSCHITZ, *Hydromechanik*, Akademie-Verlag, Berlin (1971).

[15] V. N. MASLENNIKOVA - M. E. BOGOVSKII, *Sobolev spaces of solenoidal vector fields*, Sibirsk. Mat. Zh., **22** (1981), pp. 91-118; English transl.: Siberian Math. J., **22** (1982), pp. 399-420.

[16] S. A. NAZAROV - K. PILECKAS, *On the behaviour of solutions of the Stokes and Navier-Stokes systems in domains with periodically varying section*, Trudy Mat. Inst. Steklov, **159** (1983), pp. 137-149; English transl.: Proc. Math. Inst. Steklov, **159**, issue 2 (1984), pp. 141-154.

[17] S. A. NAZAROV - K. PILECKAS, *On noncompact free boundary problems for the plane stationary Navier-Stokes equations*, J. Reine Angewandte Math., **438** (1993), pp. 103-141.

[18] K. PILECKAS, *Three-dimensional solenoidal vectors*, Zapiski Nauchn. Sem. LOMI, **96** (1980), pp. 237-239; English transl.: J. Sov. Math., **21** (1983), pp. 821-823.
[19] K. PILECKAS, *On the solvability of certain problem of a plane motion of viscous incompressible liquid with a noncompact free boundary*, Zapiski Nauchn. Sem. LOMI, **110** (1981), pp. 174-179; English transl.: J. Sov. Math., **25**, No. 1 (1984), pp. 927-931.
[20] K. PILECKAS, *Existence of solutions for the Navier-Stokes equations having an infinite dissipation of energy, in a class of domains with noncompact boundaries*, Zapiski Nauchn. Sem. LOMI, **110** (1981), pp. 180-202; English transl.: J. Sov. Math., **25**, No. 1 (1984), pp. 932-947.
[21] K. PILECKAS, *Solvability of a problem of a plane motion of viscous fluid with noncompact boundary*, Differential Equations and their Application, Vilnius, **30** (1981), pp. 57-96 (in Russian).
[22] K. PILECKAS, *On spaces of solenoidal vectors*, Trudy Mat. Inst. Steklov, **159** (1983), pp. 137-149; English transl.: Proc. Math. Inst. Steklov, **159**, issue 2 (1984), pp. 141-154.
[23] K. PILECKAS, *On the problem of motion of heavy viscous incompressible fluid with noncompact free boundary*, Litovskii Mat. Sb., **28** (1988), pp. 315-333; English transl.: Lithuanian Math. J., **28**, No. 2 (1988).
[24] K. PILECKAS, *On plane motion of a viscous incompressible cappilary liquid with a noncompact free boundary*, Arch. Mech., **41**, No. 2-3 (1989), pp. 329-342.
[25] K. PILECKAS - V. A. SOLONNIKOV, *On stationary Stokes and Navier-Stokes systems in an open infinite channel*. I., Litovskii Mat. Sb., **29**, No. 1 (1989), pp. 90-108; English transl.: Lithuanian Math. J., **29**, No. 1 (1989); II., Litovskii Mat. Sb., **29**, No. 2 (1989), pp. 347-367. English transl.: Lithuanian Math. J., **29**, No. 2 (1989).
[26] K. PILECKAS - M. SPECOVIUS-NEUGEBAUER, *Solvability of a noncompact free boundary problem for the stationary Navier-Stokes system*. I., Litovskii Mat. Sb., **29**, No. 3 (1989), pp. 532-547. English transl.: Lithuanian Math. J., **29**, No. 3 (1989); II., Litovskii Mat. Sb., **29**, No. 4 (1989), pp. 773-784 (in Russian).
[27] K. PILECKAS, *Weighted L^q-solvability for the steady Stokes system in domains with noncompact boundaries*, Mathematical Models and Methods in Applied Sciences, **6**, No. 1 (1996), pp. 97-136.
[28] K. PILECKAS, *Classical solvability and uniform estimates for the steady Stokes system in domains with noncompact boundaries*, Mathematical Models and Methods in Applied Sciences, **6**, No. 2 (1996), pp. 149-167.
[29] V. A. SOLONNIKOV - K. PILECKAS, *Certain spaces of solenoidal vectors and the solvability of the boundary value problem for the Navier-Stokes system of equations in domains with noncompact boundaries*, Zapiski Nauchn. Sem. LOMI, **73** (1977), pp. 136-151; English transl.: J. Sov. Math., **34**, No. 6 (1986), pp. 2101-2111.
[30] V. A. SOLONNIKOV, *On the solvability of boundary and initial-boundary value problems for the Navier-Stokes system in domains with noncompact boundaries*, Pacific J. Math., **93**, No. 2 (1981), pp. 443-458.

[31] V. A. SOLONNIKOV, *On solutions of stationary Navier-Stokes equations with an infinite Dirichlet integral*, Zapiski Nauchn. Sem. LOMI, **115** (1982), pp. 257-263; English transl.: J. Sov. Math., 28, No. 5 (1985), pp. 792-799.
[32] V. A. SOLONNIKOV, *Stokes and Navier-Stokes equations in domains with noncompact boundaries*, College de France Seminar, 4 (1983), pp. 240-349.
[33] V. A. SOLONNIKOV, *Boundary and initial-boundary value problems for the Navier-Stokes equations in domains with noncompact boundaries*, Math. Topics in Fluid Mechanics, Pitman Research Notes in Mathematics Series, **274** (eds. J. F. Rodriques, A. Sequeira) (1991), pp. 117-162.
[34] V. A. SOLONNIKOV - V. E. SHCHADILOV, *On a boundary value problem for a stationary system of Navier-Stokes equations*, Trudy Mat. Inst. Steklov, **125** (1973), pp. 196-210; English transl.: Proc. Mat. Inst. Steklov, **125** (1973), pp. 186-199.
[35] R. TEMAM, *Navier-Stokes Equations*, North-Holland Pub. Co., Amsterdam, New York, Tokyo (1977).

Manoscritto pervenuto in redazione il 23 luglio 1995.