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A Note on *IA*-Endomorphisms of Two-Generated Metabelian Groups.

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1. - Introduction and preliminaries.

An endomorphism of a group G is called an *IA-endomorphism* if it induces the identity mapping on the factor group G/G' of G over its derived subgroup G' . We write $ia(G)$ for the monoid of the *IA*-endomorphisms of G and $IA(G)$ for the group of invertible elements of $ia(G)$, the *IA-automorphisms* of G . It is easily verified that, $\text{Inn}(G) \leq IA(G)$ where $\text{Inn}(G)$ denotes the group of inner automorphisms of G .

The monoid $ia(G)$ has been studied in the case that G is a metabelian two-generated group, in [3], [4], [6]. In these papers, the description of the *IA*-endomorphisms is based on the construction of either a certain semigroup or a module.

Our approach in this note is of ring-theoretical nature, using techniques introduced by H. Laue [7]. We feel that certain key results of the paper [1], [2], [3], [4] gain in clarity and elegance by deriving them from those ideas as a starting point. This conviction was the main motivation for this unpretentious note. We first recall some definitions and results of [7].

Let G be a group and A be an abelian normal subgroup of G . A *cocycle* of G into A is a mapping f of G into A such that $(xy)^f = x^f y^f$ for all $x, y \in G$. Let R denote the set of all cocycles of G into A . With respect to the usual addition of mappings into an abelian group and composition of mappings as multiplication, R is an associative ring.

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In [7], H. Laue establishes a close connection between $C_{\text{Aut}G}(G/A)$ and R . More, precisely, for any $h \in C_{\text{End}G}(G/A)$ let

$$f_h: G \rightarrow A, \quad x \mapsto x^{-1}x^h.$$

Then $f_h \in R$ and

$$\psi: C_{\text{End}G}(G/A) \rightarrow R, \quad h \mapsto f_h,$$

is an isomorphism of the monoid $C_{\text{End}G}(G/A)$ onto $(R, *)$, where $f * g = f + g + fg$ for all $f, g \in R$. Moreover, if the group of quasi regular elements of the ring R is denoted by $Q(R)$, then $(C_{\text{Aut}G}(G/A))^\psi = Q(R)$. If $f \in Q(R)$, then we write f^- for the inverse of f with respect to $*$.

Let $\text{End}_G A$ be the ring of G -endomorphisms of A . The restriction of any $f \in R$ to A is an element of $\text{End}_G A$, and the mapping

$$\varrho: R \rightarrow \text{End}_G A, \quad f \mapsto f|_A,$$

is a ring homomorphism such that $\ker \varrho = \text{Ann}_R(A) = \{f \mid f \in R, \forall x \in A \ x^f = 1\}$.

We remark that if $f \in R$, $\alpha \in \text{End}_G A$, then $f\alpha \in R$ and $(f\alpha)^e = f^e \alpha$. Thus R is a $\text{End}_G A$ -module and ϱ is an $\text{End}_G A$ -module homomorphism. Moreover we have

$$(1.1) \quad Q(R)^e = Q(R^e)$$

PROOF. Obviously $Q(R^e) \subseteq Q(R)^e$. Now, let $f \in R$ such that $f|_A \in \text{Ann}_R(A)$. Then there exists $g \in R$ such that $g|_A * f|_A = 0 = f|_A * g|_A$. Therefore we have $f * g, g * f \in \text{Ann}_R(A) \subseteq Q(R)$. Hence there exist $h, h' \in Q(R)$ such that

$$(f * g) * h = 0 = h' * (g * f).$$

Therefore $f \in Q(R)$, hence $f|_A \in Q(R)^e$. ■

We also observe

$$(1.2) \quad h \in C_{\text{Aut}G}(G/A) \Leftrightarrow h|_A \in \text{Aut } A.$$

2. - IA-endomorphisms of two-generated metabelian groups.

Let G be a two-generated metabelian group with generators a and b . Then $\text{End}_G G'$ is a finitely generated commutative ring, generated by automorphisms of G' induced by conjugation by a, a^{-1}, b, b^{-1} , and G'

is a cyclic $\text{End}_G G'$ -module generated by c , where $c = [a, b]$ (see, for example [3, 2]). These properties will be used freely in the sequel without further reference.

Now an application of our introductory general remarks leads to a short proof of the result that

$$(2.1) \quad IA(G) \quad \text{is a metabelian group} \quad ([4, 2.3], [2, \text{Cor. 1}]).$$

PROOF. The zero ideal $\text{Ann}_R(G')$ is contained in $Q(R)$ and therefore $\text{Ann}_R(G')$ is an abelian normal subgroup of $Q(R)$. Since $Q(R)/\text{Ann}_R(G') \cong Q(R)^e \subseteq \text{End}_G G'$, we have $Q(R)' \subseteq \text{Ann}_R(G')$. It follows that $Q(R)$ is a metabelian group and, hence $IA(G)$ is metabelian group. ■

If $h \in ia(G)$ then $a^h = au, b^h = bv$ for suitable elements $u, v \in G'$. Viceversa, we have the following

$$(2.2) \quad \text{For all } (u, v) \in G' \times G' \text{ there exists an endomorphism } h \text{ of } G \text{ such that } a^h = au, b^h = bv \text{ ([3, 3.1(i)]).}$$

PROOF. Let $(u, v) \in G' \times G'$ and $\alpha, \beta \in \text{End}_G G'$ such that $u = c^\alpha, v = c^\beta$. Then the application $h: G \rightarrow G, x \mapsto x[x, a]^{-\beta}[x, b]^\alpha$ is an element of $ia(G)$ and $a^h = au, b^h = bv$. ■

If z is any group element, we write \bar{z} for the inner automorphism induced by z . For later reference we remark as a consequence of the foregoing proof:

$$(1) \quad \text{For all } g \in R \text{ there are } \alpha, \gamma \in \text{End}_G G' \text{ such that } g = f_{\bar{a}}\gamma + f_{\bar{b}}\alpha.$$

Moreover, an application of (1.2) yields a criterion for h to be an automorphism ([3, 3.1(ii)]).

For all $f, g \in R$ we set $f \circ g =: fg - gf$. It is well known that $(R, +, \circ)$ is a Lie ring. As $\text{End}_G G'$ is commutative, we have $g * f * (f \circ g) = f * g$ for all $f, g \in R$. In particular, for all $f, g \in Q(R)$

$$(2) \quad f \circ g = [f, g]$$

where $[f, g] = f^- * g^- * f * g$.

It is readily verified that for arbitrary metabelian groups the Witt identity for group commutators reduces to the simple Jacobi-like equation $[x, y, z][y, z, x][z, x, y] = 1$. In terms of cocycles, this rule

may be expressed as follows:

$$(3) \quad f_{\bar{y}} \circ f_{\bar{z}} = f_{\overline{[y, z]}}$$

for all $y, z \in G$.

From these remarks we may deduce the following description of the descending central series of $IA(G)$:

$$(2.3) \quad \gamma_k(IA(G)) = \gamma_k(\text{In}(G)) \quad ([3, 4.1])$$

for all $k \geq 2$.

PROOF. We set $D := (\text{In}(G))^\psi$ and have to show that $\gamma_k(Q(R)) \subseteq \subseteq \gamma_k(D)$ for $k \geq 2$. We proceed by induction on k . If $g_1, g_2 \in Q(R)$ and $\alpha, \beta, \gamma, \delta \in \text{End}_G G'$ such that $g_1 = f_{\bar{a}}\alpha + f_{\bar{b}}\beta, g_2 = f_{\bar{a}}\gamma + f_{\bar{b}}\delta$, then (2) and (3) show that

$$[g_1, g_2] = f_{\overline{[(\bar{b}, \bar{a})]^\beta \gamma}} + f_{\overline{[(\bar{a}, \bar{b})]^\alpha \delta}} = f_{\overline{c^{\alpha\delta - \beta\gamma}}} \in ((\text{In}(G))')^\psi = \gamma_2(D)$$

which settles the case of $k = 2$.

Now let $k > 2$, $g_1 \in \gamma_{k-1}(Q(R)), g_2 \in Q(R)$. Inductively we assume that $g_1 = f_{\bar{z}}$ for an element $z \in \gamma_{k-1}(G)$, and we write $g_2 = f_{\bar{a}}\gamma + f_{\bar{b}}\delta$ for some $\gamma, \delta \in \text{End}_G G'$. Then

$$[g_1, g_2] = f_{\overline{[(z, \bar{a})]^\alpha}} + f_{\overline{[(z, \bar{b})]^\beta}} = f_{\overline{[z^\alpha, a][z^\beta, b]}} \in (\gamma_k(\text{In}(G)))^\psi = \gamma_k(D). \quad \blacksquare$$

The nilpotent case allows a neat characterization which has been useful in various places (see, for example, [8])

$$(2.4) \quad IA(G) \text{ nilpotent} \Leftrightarrow G \text{ nilpotent} \Leftrightarrow ia(G) = IA(G) \quad ([3, 3.1]).$$

PROOF. The first equivalence follows from (2.3). Furthermore, we know that $ia(G) = IA(G)$ if and only if $R = Q(R)$. As $\ker \varrho \subseteq Q(R)$, this is equivalent to saying that $R^e = Q(R^e)$ by (1.1), i.e. that R^e is a radical ring.

As R^e is an ideal of finitely generated commutative ring $\text{End}_G G'$, an application of [9, 4.1(i)] shows that R^e is a radical ring if and only if R^e is nilpotent. An easy induction shows that $\gamma_k(G) = c^{(R^e)^{k-2}}$ for $k \geq 2$, where $(R^e)^0 = \text{End}_G G'$. Hence the nilpotency of R^e is equivalent to the nilpotency of G . \blacksquare

Moreover, the nilpotent case allows the following description of the ascending central serie of $IA(G)$ (cf. [3, 4])

(2.5) *If G is nilpotent then, for all $k \in \mathbb{N}_0$*

$$\zeta_k(IA(G)) = C_{\text{Aut } G}(G/(G' \cap \zeta_k(G))).$$

PROOF. We remark that, for all $n \in N_0$, we have

$$(\zeta_k(IA(G)))^\psi = \zeta_k(Q(R))$$

and, by (2), $\zeta_k(Q(R)) = \zeta_k(R) = Z_k(R)$, where $Z_k(R)$ is the k -th term of the upper central series of the Lie ring $R(+, \circ)$. Moreover, if we put

$$R_k = (C_{\text{Aut } G}(G/(G' \cap \zeta_k(G))))^\psi$$

then it suffices to show that $Z_k(R) = R_k$ for all $k \in N_0$. As G is metabelian, one readily verifies that

$$(4) \quad x^{f \circ f_{\bar{v}}} = [y^f, x]$$

for any $f \in R$ and $x, y \in G$. The inclusion $Z_k(R) \subseteq R_k$ follows from (4) by an easy induction on k .

Now, we prove by induction that $R_k \subseteq Z_k(R)$ for all $k \in N_0$, the case $k = 0$ being trivial. Let $n \in N_0$ and assume that $R_k \subseteq Z_k(R)$. Let $f \in R_{n+1}$ and $g \in R$. By (1), there are $\alpha, \beta \in \text{End}_G G'$ such that $g = f_{\bar{a}}\alpha + f_{\bar{b}}\beta$. By (4) and by our inductive hypothesis it follows that $f \circ f_{\bar{a}}, f \circ f_{\bar{b}} \in Z_k(R)$. Hence

$$f \circ g = (f \circ f_{\bar{a}})\alpha + (f \circ f_{\bar{b}})\beta \in Z_k(R).$$

Therefore $f \in Z_{n+1}(R)$. ■

The study of $IA(G)$ has been of particular interest in the case that G is free metabelian of rank two. Then G' is a free abelian group with the conjugates of as a set of generators, and there is a canonical isomorphism of $\mathbf{Z}[G/G']$ onto $\text{End}_G G'$. We conclude this note by pointing out that the crucial step of the proof of the following well-known result is simplified considerably by a suitable application of (1):

$$(2.6) \quad \text{If } G \text{ is a free metabelian group of rank two, then } IA(G) = \text{Inn}(G) \text{ ([1, Theor.2], [4, 2.4], [2, Cor.3]).}$$

PROOF. It suffices to show that every $h \in IA(G)$ is an inner automorphism of G . The main step of the proof is to show this in the case that $h|_{G'} = \text{id}_{G'}$. Then, by (1), $f_{\bar{b}}^g \alpha = f_{\bar{a}}^g \beta$ for suitable elements $\alpha, \beta \in \text{End}_G G'$. We know that we may identify $\text{End}_G G'$ with the integral group ring of the free abelian group G/G' . By a well-known line of reasoning we obtain therefore an element $\gamma \in$

$\in \text{End}_G G'$ such that $\alpha = f_a^\alpha \gamma$, $\beta = f_b^\beta \gamma$. By (1) and (3) we now have

$$f_h = (f_b \circ f_a) \gamma = f_{c^{-1}} \gamma = f_{c^{-\gamma}}$$

hence $h = \overline{c^{-\gamma}} \in \text{In}(G)$.

The reduction of the case an arbitrary $h \in IA(G)$ to the case just settled is standard. The group of units of $\mathbf{Z}[G/G']$ is $\pm G/G'$ (see [5]) whence $h|_{G'}$ is induced by an inner automorphism \bar{g} of G , as h induces the identity automorphism on the non-trivial factor group $G'/\gamma_3(G)$. Therefore $h\bar{g}|_{G'}^{-1} = \text{id}_{G'}$, hence $h\bar{g}^{-1} \in \text{In}(G)$ by the part treated above. The claim follows. ■

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