RENDICONTI
*del*
SEMINARIO MATEMATICO
*della*
UNIVERSITÀ DI PADOVA

F. CATINO
M. M. MICCOLI

**A note on *IA*-endomorphisms of two-generated metabelian groups**

*Rendiconti del Seminario Matematico della Università di Padova*, tome 96 (1996), p. 99-104
<http://www.numdam.org/item?id=RSMUP_1996__96__99_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 96 (1996)

# A Note on $IA$-Endomorphisms of Two-Generated Metabelian Groups.

F. CATINO - M. M. MICCOLI (*)

## 1. - Introduction and preliminaries.

An endomorphism of a group $G$ is called an *IA-endomorphism* if it induces the identity mapping on the factor group $G/G'$ of $G$ over its derived subgroup $G'$. We write $ia(G)$ for the monoid of the $IA$-endomorphisms of $G$ and $IA(G)$ for the group of invertible elements of $ia(G)$, the *IA-automorphisms* of $G$. It is easily verified that, $\mathrm{Inn}(G) \leqslant IA(G)$ where $\mathrm{Inn}(G)$ denotes the group of inner automorphisms of $G$.

The monoid $ia(G)$ has been studied in the case that $G$ is a metabelian two-generated group, in [3], [4], [6]. In these papers, the description of the $IA$-endomorphisms is based on the construction of either a certain semigroup or a module.

Our approach in this note is of ring-theoretical nature, using techniques introduced by H. Laue [7]. We feel that certain key results of the paper [1], [2], [3], [4] gain in clarity and elegance by deriving them from those ideas as a starting point. This conviction was the main motivation for this unpretentious note. We first recall some definitions and results of [7].

Let $G$ be a group and $A$ be an abelian normal subgroup of $G$. A *cocycle* of $G$ into $A$ is a mapping $f$ of $G$ into $A$ such that $(xy)^f = x^{fy} y^f$ for all $x, y \in G$. Let $R$ denote the set of all cocycles of $G$ into $A$. With respect to the usual addition of mappings into an abelian group and composition of mappings as multiplication, $R$ is an associative ring.

(*) Indirizzo degli AA.: Dipartimento di Matematica, Via Provinciale Lecce-Arnesano, P.O. Box 193, I-73100 Lecce; e-mail: catino@ingle01.unile.it.

The authors acknowledge support from the Italian MURST (40%).

In [7], H. Laue establishes a close connection between $C_{\mathrm{Aut}G}(G/A)$ and $R$. More, precisely, for any $h \in C_{\mathrm{End}G}(G/A)$ let

$$f_h\colon G \to A, \qquad x \mapsto x^{-1}x^h .$$

Then $f_h \in R$ and

$$\psi\colon C_{\mathrm{End}G}(G/A) \to R, \qquad h \mapsto f_h,$$

is an isomorphism of the monoid $C_{\mathrm{End}G}(G/A)$ onto $(R, *)$, where $f*g = f+g+fg$ for all $f, g \in R$. Moreover, if the group of quasi regular elements of the ring $R$ is denoted by $Q(R)$, then $(C_{\mathrm{Aut}G}(G/A))^{\psi} = Q(R)$. If $f \in Q(R)$, then we write $f^-$ for the inverse of $f$ with respect to $*$.

Let $\mathrm{End}_G A$ be the ring of $G$-endomorphisms of $A$. The restriction of any $f \in R$ to $A$ is an element of $\mathrm{End}_G A$, and the mapping

$$\varrho\colon R \to \mathrm{End}_G A, \qquad f \mapsto f_{|A},$$

is a ring homomorphism such that $\ker \varrho = \mathrm{Ann}_R(A) = \{f \mid f \in R, \forall x \in A \; x^f = 1\}$.

We remark that if $f \in R$, $\alpha \in \mathrm{End}_G A$, then $f\alpha \in R$ and $(f\alpha)^{\varrho} = f^{\varrho}\alpha$. Thus $R$ is a $\mathrm{End}_G A$-module and $\varrho$ is an $\mathrm{End}_G A$-module homomorphism. Moreover we have

$$Q(R)^{\varrho} = Q(R^{\varrho}) \tag{1.1}$$

PROOF. Obviously $Q(R^{\varrho}) \subseteq Q(R^{\varrho})$. Now, let $f \in R$ such that $f_{|A} \in Q(R^{\varrho})$. Then there exists $g \in R$ such that $g_{|A} * f_{|A} = 0 = f_{|A} * g_{|A}$. Therefore we have $f*g$, $g*f \in \mathrm{Ann}_R(A) \subseteq Q(R)$. Hence there exist $h, h' \in Q(R)$ such that

$$(f*g)*h = 0 = h' * (g*f).$$

Therefore $f \in Q(R)$, hence $f_{|A} \in Q(R)^{\varrho}$. ■

We also observe

$$h \in C_{\mathrm{Aut}G}(G/A) \Leftrightarrow h_{|A} \in \mathrm{Aut}\, A. \tag{1.2}$$

## 2. – *IA*-endomorphisms of two-generated metabelian groups.

Let $G$ be a two-generated metabelian group with generators $a$ and $b$. Then $\mathrm{End}_G G'$ is a finitely generated commutative ring, generated by automorphisms of $G'$ induced by conjugation by $a$, $a^{-1}$, $b$, $b^{-1}$, and $G'$

is a cyclic $\mathrm{End}_G G'$-module generated by $c$, where $c = [a, b]$ (see, for example [3, 2]). These properties will be used freely in the sequel without further reference.

Now an application of our introductory general remarks leads to a short proof of the result that

(2.1) $IA(G)$ *is a metabelian group* ([4, 2.3], [2, Cor. 1]).

PROOF. The zero ideal $\mathrm{Ann}_R(G')$ is contained in $Q(R)$ and therefore $\mathrm{Ann}_R(G')$ is an abelian normal subgroup of $Q(R)$. Since $Q(R)/\mathrm{Ann}_R(G') \cong Q(R)^{\varrho} \subsetneq \mathrm{End}_G G'$, we have $Q(R)' \subsetneq \mathrm{Ann}_R(G')$. It follows that $Q(R)$ is a metabelian group and, hence $IA(G)$ is metabelian group. ■

If $h \in ia(G)$ then $a^h = au$, $b^h = bv$ for suitable elements $u, v \in G'$. Viceversa, we have the following

(2.2) *For all $(u, v) \in G' \times G'$ there exists an endomorphism $h$ of $G$ such that $a^h = au$, $b^h = bv$* ([3, 3.1(i)]).

PROOF. Let $(u, v) \in G' \times G'$ and $\alpha, \beta \in \mathrm{End}_G G'$ such that $u = c^{\alpha}$, $v = c^{\beta}$. Then the application $h$: $G \to G$, $x \mapsto x[x, a]^{-\beta}[x, b]^{\alpha}$ is an element of $ia(G)$ and $a^h = au$, $b^h = bv$. ■

If $z$ is any group element, we write $\bar{z}$ for the inner automorphism induced by $z$. For later reference we remark as a consequence of the foregoing proof:

(1) *For all $g \in R$ there are $\alpha, \gamma \in \mathrm{End}_G G'$ such that* $g = f_{\bar{a}}\gamma + f_{\bar{b}}\alpha$.

Moreover, an application of (1.2) yields a criterion for $h$ to be an automorphism ([3, 3.1(ii)]).

For all $f, g \in R$ we set $f \circ g =: fg - gf$. It is well known that $(R, +, \circ)$ is a Lie ring. As $\mathrm{End}_G G'$ is commutative, we have $g * f * (f \circ g) = f * g$ for all $f, g \in R$. In particular, for all $f, g \in Q(R)$

(2) $$f \circ g = [f, g]$$

where $[f, g] = f^- * g^- * f * g$.

It is readily verified that for arbitrary metabelian groups the Witt identity for group commutators reduces to the simple Jacobi-like equation $[x, y, z][y, z, x][z, x, y] = 1$. In terms of cocycles, this rule may be expressed as follows:

(3) $$f_{\bar{y}} \circ f_{\bar{z}} = f_{\overline{[y,z]}}$$

for all $y, z \in G$.

From these remarks we may deduce the following description of the descending central series of $IA(G)$:

(2.3) $$\gamma_k(IA(G)) = \gamma_k(\mathrm{In}(G)) \qquad ([3,\ 4.1])$$

for all $k \geq 2$.

PROOF. We set $D := (\mathrm{In}(G))^{\psi}$ and have to show that $\gamma_k(Q(R)) \subseteq \gamma_k(D)$ for $k \geq 2$. We proceed by induction on $k$. If $g_1, g_2 \in Q(R)$ and $\alpha, \beta, \gamma, \delta \in \mathrm{End}_G G'$ such that $g_1 = f_{\bar{a}}\alpha + f_{\bar{b}}\beta$, $g_2 = f_{\bar{a}}\gamma + f_{\bar{b}}\delta$, then (2) and (3) show that

$$[g_1, g_2] = f_{\overline{([b,a])^{\beta\gamma}}} + f_{\overline{([a,b])^{\alpha\delta}}} = f_{\overline{c^{\alpha\delta-\beta\gamma}}} \in ((\mathrm{In}(G))')^{\psi} = \gamma_2(D)$$

which settles the case of $k = 2$.

Now let $k > 2$, $g_1 \in \gamma_{k-1}(Q(R))$, $g_2 \in Q(R)$. Inductively we assume that $g_1 = f_{\bar{z}}$ for an element $z \in \gamma_{k-1}(G)$, and we write $g_2 = f_{\bar{a}}\gamma + f_{\bar{b}}\delta$ for some $\gamma, \delta \in \mathrm{End}_G G'$. Then

$$[g_1, g_2] = f_{\overline{([z,a])^{\alpha}}} + f_{\overline{([z,b])^{\beta}}} = f_{\overline{[z^{\alpha},a][z^{\beta},b]}} \in (\gamma_k(\mathrm{In}(G)))^{\psi} = \gamma_k(D). \qquad \blacksquare$$

The nilpotent case allows a neat characterization which has been useful in various places (see, for example, [8])

(2.4) *$IA(G)$ nilpotent $\Leftrightarrow G$ nilpotent $\Leftrightarrow ia(G) = IA(G)$* ([3, 3.1]).

PROOF. The first equivalence follows from (2.3). Furthermore, we know that $ia(G) = IA(G)$ if and only if $R = Q(R)$. As $\ker \varrho \subseteq Q(R)$, this is equivalent to saying that $R^{\varrho} = Q(R^{\varrho})$ by (1.1), i.e. that $R^{\varrho}$ is a radical ring.
As $R^{\varrho}$ is an ideal of finitely generated commutative ring $\mathrm{End}_G G'$, an application of [9, 4.1(i)] shows that $R^{\varrho}$ is a radical ring if and only if $R^{\varrho}$ is nilpotent. An easy induction shows that $\gamma_k(G) = c^{(R^{\varrho})^{k-2}}$ for $k \geq 2$, where $(R^{\varrho})^0 = \mathrm{End}_G G'$. Hence the nilpotency of $R^{\varrho}$ is equivalent to the nilpotency of $G$. $\blacksquare$

Moreover, the nilpotent case allows the following description of the ascending central serie of $IA(G)$ (cf. [3, 4])

(2.5) *If G is nilpotent then, for all $k \in N_0$*

$$\zeta_k(IA(G)) = C_{\mathrm{Aut}\,G}(G/(G' \cap \zeta_k(G))).$$

PROOF. We remark that, for all $n \in \boldsymbol{N}_0$, we have

$$(\zeta_k(IA(G)))^{\psi} = \zeta_k(Q(R))$$

and, by (2), $\zeta_k(Q(R)) = \zeta_k(R) = Z_k(R)$, where $Z_k(R)$ is the $k$-th term of the upper central series of the Lie ring $R(+, \circ)$. Moreover, if we put

$$R_k = (C_{\mathrm{Aut}G}(G/(G' \cap \zeta_k(G)))^{\psi}$$

then it suffices to show that $Z_k(R) = R_k$ for all $k \in \boldsymbol{N}_0$. As $G$ is metabelian, one readily verifies that

$$x^{f \circ f_{\bar{y}}} = [y^f, x] \tag{4}$$

for any $f \in R$ and $x, y \in G$. The inclusion $Z_k(R) \subseteq R_k$ follows from (4) by an easy induction on $k$.
Now, we prove by induction that $R_k \subseteq Z_k(R)$ for all $k \in \boldsymbol{N}_0$, the case $k = 0$ being trivial. Let $n \in \boldsymbol{N}_0$ and assume that $R_k \subseteq Z_k(R)$. Let $f \in R_{n+1}$ and $g \in R$. By (1), there are $\alpha, \beta \in \mathrm{End}_G G'$ such that $g = f_{\bar{a}}\alpha + f_{\bar{b}}\beta$. By (4) and by our inductive hypothesis it follows that $f \circ f_{\bar{a}}$, $f \circ f_{\bar{b}} \in Z_k(R)$. Hence

$$f \circ g = (f \circ f_{\bar{a}})\,\alpha + (f \circ f_{\bar{b}})\,\beta \in Z_k(R).$$

Therefore $f \in Z_{n+1}(R)$. ■

The study of $IA(G)$ has been of particular interest in the case that $G$ is *free* metabelian of rank two. Then $G'$ is a free abelian group with the coniugates of as a set of generators, and there is a canonical isomorphism of $\boldsymbol{Z}[G/G']$ onto $\mathrm{End}_G G'$. We conclude this note by pointing out that the crucial step of the proof of the following well-known result is simplified considerably by a suitable application of (1):

(2.6) *If $G$ is a free metabelian group of rank two, then* $IA(G) = \mathrm{Inn}(G)$ ([1, Theor.2], [4, 2.4], [2, Cor.3]).

PROOF. It suffices to show that every $h \in IA(G)$ is an inner automorphism of $G$. The main step of the proof is to show this in the case that $h_{|G'} = \mathrm{id}_{G'}$. Then, by (1), $f_{\bar{b}}^{\varrho}\alpha = f_{\bar{a}}^{\varrho}\beta$ for suitable elements $\alpha, \beta \in \mathrm{End}_G G'$. We know that we may identify $\mathrm{End}_G G'$ with the integral group ring of the free abelian group $G/G'$. By a well-known line of reasoning we obtain therefore an element $\gamma \in$

$\in \mathrm{End}_G G'$ such that $\alpha = f_{\bar{a}}^{\varrho}\gamma$, $\beta = f_{\bar{b}}^{\varrho}\gamma$. By (1) and (3) we now have

$$f_h = (f_{\bar{b}} \circ f_{\bar{a}})\gamma = f_{\overline{c^{-1}}}\gamma = f_{\overline{c^{-\gamma}}}$$

hence $h = \overline{c^{-\gamma}} \in \mathrm{In}(G)$.
The reduction of the case an arbitrary $h \in IA(G)$ to the case just settled is standard. The group of units of $\mathbb{Z}[G/G']$ is $\pm G/G'$ (see [5]) whence $h_{|G'}$ is induced by an inner automorphism $\bar{g}$ of $G$, as $h$ induces the identity automorphism on the non-trivial factor group $G'/\gamma_3(G)$. Therefore $h\bar{g}_{|G'}^{-1} = \mathrm{id}_{G'}$, hence $h\bar{g}^{-1} \in \mathrm{In}(G)$ by the part treated above. The claim follows. ■

## REFERENCES


[1] S. BACHMUTH, *Automorphisms of free metabelian groups*, Trans. Amer. Math. Soc., **118** (1965), pp. 93-104.

[2] S. BACHMUTH - G. BAUMSLAG - J. DYER - H. Y. MOCHIZUKI, *Automorphism groups of two generator metabelian groups*, J. London Math. Soc., **36** (1987), pp. 393-406.

[3] A. CARANTI - C. M. SCOPPOLA, *Endomorphisms of two-generated metabelian groups that induce the identity modulo the derived subgroup*, Arch. Math. (Basel), **56** (1991), pp. 218-227.

[4] C. K. GUPTA, *IA-automorphisms of two generator metabelian groups*, Arch. Math., **37** (1981), pp. 106-112.

[5] G. KARPILOVSKY, *Unit Groups of Group Rings*, Longman Sci. Techn. (1989).

[6] YU. V. KUZ'MIN, *Inner endomorphisms of abelian groups* (Russian), Sibirsk. Mat. Zh., **16** (1975), pp. 736-744; translation in Siberian Math. J., **16** (1975), pp. 563-568.

[7] H. LAUE, *On group automorphisms which centralize the factor group by an abelian normal subgroup*, J. Algebra, **96** (1985), pp. 532-543.

[8] F. MENEGAZZO, *Automorphisms of p-groups with cyclic commutator subgroup*, Rend. Sem. Mat. Univ. Padova, **90** (1993), pp. 81-101.

[9] B. A. F. WEHRFRITZ, *Infinite Linear Group*, Springer-Verlag, Berlin (1973).


Manoscritto pervenuto in redazione il 29 novembre 1994
e, in forma revisionata, il 15 maggio 1995.