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A capacity method for the study of Dirichlet problems for elliptic systems in varying domains

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ABSTRACT - The asymptotic behaviour of solutions of second order linear elliptic systems with Dirichlet boundary conditions on varying domains is studied by means of a suitable notion of capacity.

Introduction.

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) and let \( \mathcal{A} : H^1_0(\Omega, \mathbb{R}^m) \to H^{-1}(\Omega, \mathbb{R}^m) \) be an elliptic operator of the form

\[
\langle \mathcal{A}u, v \rangle = \int_{\Omega} (ADu, Dv) \, dx,
\]

where \( A(x) \) is a fourth order tensor and \( (\cdot, \cdot) \) denotes the scalar product between matrices. Given a sequence \( (\Omega_j) \) of open subsets of \( \Omega \), we consider for every \( f \in H^{-1}(\Omega, \mathbb{R}^m) \) the sequence \( (u_j) \) of the solutions of the Dirichlet problems

\[
\begin{cases}
  u_j \in H^1_0(\Omega_j, \mathbb{R}^m), \\
  \mathcal{A}u_j = f \quad \text{in } \Omega_j,
\end{cases}
\]

(0.1)

extended to \( \Omega \) by setting \( u_j = 0 \) on \( \Omega \setminus \Omega_j \). We want to describe the asymptotic behaviour of \( (u_j) \) as \( j \to \infty \). As in the scalar case, a relaxation phenomenon may occur. Namely, if \( (u_j) \) converges weakly in \( H^1_0(\Omega, \mathbb{R}^m) \) to some function \( u \), then there exist an \( m \times m \) matrix \( B(x) \),

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with $|B(x)| = 1$, and a measure $\mu$, not charging polar sets, such that $u$ is the solution of the relaxed Dirichlet problem

$$\begin{cases}
u \in H^1_0(\Omega, \mathbb{R}^m) \cap L^2_\mu(\Omega, \mathbb{R}^m), \\
\int_\Omega (ADu, Dv) \, dx + \int_\Omega (Bu, v) \, d\mu = (f, v), \\
\forall v \in H^1_0(\Omega, \mathbb{R}^m) \cap L^2_\mu(\Omega, \mathbb{R}^m),
\end{cases}$$

(0.2)

where, in the second integral, $(\cdot, \cdot)$ denotes the scalar product in $\mathbb{R}^m$, while $(\cdot, \cdot)$ is the duality pairing between $H^{-1}(\Omega, \mathbb{R}^m)$ and $H^1_0(\Omega, \mathbb{R}^m)$. Compactness and localization results for the relaxed Dirichlet problems are established in [8] for symmetric $A$ and $B$, and in [4] in the general case.

The problem we consider in this paper is the identification of the pair $(B, \mu)$ which appears in the limit problem (0.2). To this aim we introduce a suitable notion of capacity. If $K$ is a compact subset of $\Omega$ and $\xi, \eta \in \mathbb{R}^m$, then the $\alpha$-capacity of $K$ in $\Omega$ relative to $\xi$ and $\eta$ is defined as

$$C_{\alpha}(K, \xi, \eta) = \int_{\Omega \setminus K} (ADu^\xi, Du^\eta) \, dx,$$

where, for every $\xi \in \mathbb{R}^m$, $u^\xi$ is the weak solution in $\Omega \setminus K$ of the Dirichlet problem

$$\begin{cases}
u \in H^1(\Omega \setminus K, \mathbb{R}^m), \\
u^\xi = \xi \quad \text{on } \partial K, \\
u^\xi = 0 \quad \text{on } \partial \Omega, \\
\int_{\Omega \setminus K} (ADu^\xi, Dv) \, dx = 0, \\
\forall v \in H^1_0(\Omega \setminus K, \mathbb{R}^m).
\end{cases}$$

For every $x \in \mathbb{R}^n$ let $D_\varrho(x)$ be the closed ball with centre $x$ and radius $\varrho$. Assume that the limit

$$\lim_{j \to +\infty} C_{\alpha}(D_\varrho(x) \setminus \Omega_j, \xi, \eta) = \alpha(D_\varrho(x), \xi, \eta)$$

exists for every $x \in \Omega$ and for almost every $\varrho > 0$ such that $D_\varrho(x) \subset \Omega$. Our main result, Theorem 3.7, shows that, if $\alpha$ can be majorized by a Kato measure $\lambda$ (Definition 1.1), then for $\lambda$-almost every $x \in \Omega$ there exists an $m \times m$ matrix $G(x)$ such that

$$\operatorname{ess \, lim}_{\varrho \to 0} \frac{\alpha(D_\varrho(x), \xi, \eta)}{\lambda(D_\varrho(x))} = (G(x) \xi, \eta), \quad \forall \xi, \eta \in \mathbb{R}^m.$$
Moreover, for every $f \in H^{-1}(\Omega, \mathbb{R}^m)$, the sequence $(u_\varepsilon)$ of the solutions of (0.1) converges weakly in $H^1_0(\Omega, \mathbb{R}^m)$ to the solution $u$ of (0.2) with $B(x) = (G(x))/|G(x)|$ and $\mu(E) = \int_E |G(x)| d\lambda$. If $\varepsilon$ is symmetric, the same result (Theorem 4.3) holds whenever $\lambda$ is a bounded measure.

1. – Notation and preliminaries.

Let $\mathbb{M}^{m \times n}$ be the space of all real $m \times n$ matrices $\xi = (\xi_{ij})$ endowed with the scalar product

$$ (\xi, \xi) = \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij}^a \xi_{ij}^a $$

and with the corresponding norm $|\xi|^2 = (\xi, \xi)$. As usual, $\mathbb{R}^m$ is identified with $\mathbb{M}^{m \times 1}$. Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$, $n \geq 3$. The case $n = 2$ can be treated in a similar way by using the logarithmic potentials. We assume that the boundary $\partial \Omega$ of $\Omega$ is of class $C^1$. The Sobolev space $H^1(\Omega, \mathbb{R}^m)$ is defined as the space of all functions $u$ in $L^2(\Omega, \mathbb{R}^m)$ whose first order distribution derivatives $D_j u$ belong to $L^2(\Omega, \mathbb{R}^m)$, endowed with the norm

$$ \|u\|_{H^1(\Omega, \mathbb{R}^m)} = \int_{\Omega} |Du|^2 dx + \int_{\Omega} |u|^2 dx, $$

where $Du = (D_j u)$ is the Jacobian matrix of $u$. The space $H^1_0(\Omega, \mathbb{R}^m)$ is the closure of $C^0_0(\Omega, \mathbb{R}^m)$ in $H^1(\Omega, \mathbb{R}^m)$, and $H^{-1}(\Omega, \mathbb{R}^m)$ is the dual of $H^1_0(\Omega, \mathbb{R}^m)$. The symbol $\mathbb{R}^m$ will be omitted when $m = 1$.

For every subset $E$ of $\Omega$ the (harmonic) capacity of $E$ with respect to $\Omega$ is defined by $\text{cap}(E) = \inf \int_{\Omega} |Du|^2 dx$, where the infimum is taken over all functions $u \in H^1_0(\Omega)$ such that $u \geq 1$ almost everywhere in a neighbourhood of $E$, with the usual convention $\inf \varnothing = +\infty$.

A function $u: \Omega \to \mathbb{R}^m$ is said to be quasicontinuous if for every $\varepsilon > 0$ there exists a set $E \subset \Omega$, with $\text{cap}(E) \leq \varepsilon$, such that the restriction of $u$ to $\Omega \setminus E$ is continuous. We recall that for every $u \in H^1_0(\Omega, \mathbb{R}^m)$ there exists a quasicontinuous function $\tilde{u}$, unique up to sets of capacity zero, such that $u = \tilde{u}$ almost everywhere in $\Omega$. We shall always identify $u$ with $\tilde{u}$.

By a Borel measure on $\Omega$ we mean a positive, countably additive set function with values in $[0, +\infty]$ defined on the $\sigma$-field of all Borel subsets of $\Omega$; by a Radon measure on $\Omega$ we mean a Borel measure which is
finite on every compact subset of \( \Omega \). By \( \mathcal{M}_0(\Omega) \) we denote the set of all positive Borel measures \( \mu \) on \( \Omega \) such that \( \mu(E) = 0 \) for every Borel set \( E \subset \Omega \) with \( \text{cap}(E) = 0 \). If \( E \) is \( \mu \)-measurable in \( \Omega \), we define the Borel measure \( \mu \upharpoonright E \) by \( (\mu \upharpoonright E)(B) = \mu(E \cap B) \) for every Borel set \( B \subset \Omega \), while \( \mu \upharpoonright E \) is the measure on \( E \) given by \( \mu \upharpoonright E(B) = \mu(B) \) for every Borel subset \( B \) of \( E \).

For every \( x \in \mathbb{R}^n \) and \( \varrho > 0 \) we set \( U_\varrho(x) = \{ y \in \mathbb{R}^n : |x - y| < \varrho \} \) and \( D_\varrho(x) = U_\varrho(x) \). A special class of measures we shall frequently use is the Kato space.

**Definition 1.1.** The Kato space \( K^+(\Omega) \) is the cone of all positive Radon measures \( \mu \) on \( \Omega \) such that

\[
\lim_{\varrho \to 0^+} \sup_{x \in \Omega} \int_{U_\varrho(x) \cap \Omega} |y - x|^{2-n} d\mu(y) = 0.
\]

We recall that every measure in \( K^+(\Omega) \) is bounded and belongs to \( H^{-1}(\Omega) \). For more details about Kato measures we refer to [10] and [6].

Let \( A(x) = (a_{ij}^\alpha(x)) \), with \( 1 \leq i, j \leq n \) and \( 1 \leq \alpha, \beta \leq m \), be a family of functions in \( C(\overline{\Omega}) \) satisfying the following conditions: there exist two constants \( c_1 > 0 \) and \( c_2 > 0 \) such that

\[
\begin{align*}
&\left\{ \begin{array}{l}
  c_1 |\xi|^2 \leq \sum_{i,j} \sum_{\alpha,\beta} a_{ij}^{\alpha\beta}(x) \xi_i^\alpha \xi_j^\beta, & \forall x \in \Omega, \forall \xi \in M^{m \times n}, \\
  \sum_{i,j} \sum_{\alpha,\beta} |a_{ij}^{\alpha\beta}(x)| \leq c_2, & \forall x \in \Omega,
\end{array} \right.
\end{align*}
\]

and let \( \mathcal{A} : H_0^1(\Omega, \mathbb{R}^m) \to H^{-1}(\Omega, \mathbb{R}^m) \) be the elliptic operator defined by

\[
\langle \mathcal{A}u, v \rangle = \int_{\Omega} (ADu, Dv) \, dx,
\]

where \( ADu \) is the \( m \times n \) matrix defined by

\[
(ADu)^{\alpha}_i = \sum_j a_{ij}^{\alpha\beta} D_j u^\beta.
\]

For fixed \( x \in \Omega \) the Green's function \( G(x, y) = G_x(y) \) is the solution
of the problem

$$
\begin{cases}
\mathcal{A}^* G^x = \delta_x I & \text{in } \Omega, \\
G^x \in H^{1,p}_0(\Omega, \mathbb{M}^{m \times m}), & 1 < p < \frac{n}{n-1},
\end{cases}
$$

where $\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$, $\delta_x$ is the Dirac distribution at $x$, and $I$ is the $m \times m$ identity matrix. Since the coefficients are continuous, the existence of the Green’s function can be obtained by a classical duality argument. It is well-known that, as the boundary of $\Omega$ is of class $C^1$, there exists a constant $c_3 > 0$ such that

$$
|G(x, y)| \leq c_3 |x - y|^{2-n}, \quad \forall x, y \in \Omega.
$$

This estimate can be proved by using classical regularity results, as in [1]. For any $\mathbb{R}^m$-valued bounded Radon measure $\mu$, the solution $u$ of the problem

$$
\begin{cases}
\mathcal{A} u = \mu & \text{in } \Omega, \\
u \in H^{1,p}_0(\Omega, \mathbb{R}^m), & 1 < p < \frac{n}{n-1},
\end{cases}
$$

can be represented for almost every $x \in \Omega$ as

$$
u(x) = \int_{\Omega} G(x, y) \, d\mu(y).
$$

If, in addition, $\mu \in H^{-1}(\Omega, \mathbb{R}^m)$, then this formula provides the quasi-continuous representative of the solution $u$.

2. Definition and properties of the $\mu$-capacity.

We introduce now two notions of capacity associated with the operator $\mathcal{A}$.

**Definition 2.1.** Let $\xi, \eta \in \mathbb{R}^m$ and let $K$ be a compact subset of $\Omega$. The capacity of $K$ in $\Omega$ relative to the operator $\mathcal{A}$ and to the vectors $\xi$ and $\eta$ is defined by

$$
C_\mathcal{A}(K, \xi, \eta) = \int_{\Omega \setminus K} (ADu^\xi, Du^\eta) \, dx,
$$

where, for every $\xi \in \mathbb{R}^m$, $u^\xi$ is the weak solution in $\Omega \setminus K$ of the Dirichlet
We extend \( u^\xi \) to \( \Omega \) by setting \( u^\xi = \xi \) in \( K \). In (2.2) the boundary conditions are understood in the following sense: for every \( \varphi \in C_0^\infty (\Omega, \mathbb{R}^m) \) with \( \varphi = \xi \) on \( K \) we have \( u^\xi - \varphi \in H_0^1 (\Omega \setminus K, \mathbb{R}^m) \).

**Remark 2.2.** For every \( \psi \in C_0^\infty (\Omega, \mathbb{R}^m) \) with \( \psi = \eta \) on \( K \) we have

\[
C_\alpha (K, \xi, \eta) = \int_\Omega (ADu^\xi, D\psi) \, dx.
\]

This can be easily seen by taking \( u^\eta - \psi \), which belongs to \( H_0^1 (\Omega \setminus K, \mathbb{R}^m) \), as test function in the equation (2.2) satisfied by \( u^\xi \).

**Remark 2.3.** The function \( C_\alpha (K, \xi, \eta) \) is bilinear with respect to \( \xi \) and \( \eta \). Moreover there exist two constants \( c_4 > 0 \) and \( c_5 > 0 \), depending on \( n, m \), and on the constants \( c_1 \) and \( c_2 \) which appear in (1.1), such that

\[
C_\alpha (K, \xi, \xi) \geq c_4 \text{cap}(K) |\xi|^2 \quad \text{and} \quad |C_\alpha (K, \xi, \eta)| \leq c_5 \text{cap}(K) |\xi| |\eta|,
\]

for every compact set \( K \subset \Omega \) and for every \( \xi, \eta \in \mathbb{R}^m \). For the proof see Proposition 2.7.

Let \( \mu \in M_0 (\Omega) \) and let \( B = (b_{\alpha \beta}) \) be an \( m \times m \) matrix of Borel functions satisfying the following conditions: there exist two constants \( c_6 > 0 \) and \( c_7 > 0 \) such that

\[
c_6 |\xi|^2 \leq \sum_{\alpha, \beta} b_{\alpha \beta} (x) \xi^\alpha \xi^\beta, \quad \sum_{\alpha, \beta} |b_{\alpha \beta} (x)| \leq c_7,
\]

for \( \mu \)-almost every \( x \in \Omega \) and every \( \xi \in \mathbb{R}^m \).

**Definition 2.4.** Let \( \xi, \eta \in \mathbb{R}^m \). For every Borel set \( E \subset \Omega \) the \((B, \mu)\)-capacity of \( E \) in \( \Omega \) relative to \( c_\alpha, \xi, \) and \( \eta \) is defined by

\[
C_{c_\alpha}^{B, \mu} (E, \xi, \eta) = \int_{\Omega} (ADu^\xi, Du^\eta) \, dx + \int_E (B(u^\xi - \xi), (u^\eta - \eta)) \, d\mu,
\]
where, for every $\zeta \in \mathbb{R}^m$, $u^\zeta$ is the solution of

\[
\begin{align*}
&u^\zeta \in H^1_0(\Omega, \mathbb{R}^m), \\
&u^\zeta - \zeta \in L^2_\mu(E, \mathbb{R}^m), \\
&(\mathbb{A}D(u^\zeta), D\xi)dx + \int_{\mathbb{E}} (B(u^\zeta - \zeta), \psi)d\mu = 0, \\
&\forall \psi \in H^1_0(\Omega, \mathbb{R}^m) \cap L^2_\mu(E, \mathbb{R}^m).
\end{align*}
\] (2.4)

The existence and the uniqueness of the solution $u^\zeta$ of problem (2.4) follow from the Lax-Milgram Lemma.

**Remark 2.5.** For any $\psi \in H^1_0(\Omega, \mathbb{R}^m)$ with $\psi - \eta \in L^2_\mu(E, \mathbb{R}^m)$, we have

\[
C^B_{\alpha, \mu}(E, \xi, \eta) = \int_{\mathbb{Q}} (\mathbb{A}D(u^\xi), D\psi)dx + \int_{\mathbb{E}} (B(u^\xi - \xi), (\psi - \eta))d\mu .
\] (2.5)

To prove this fact it is enough to take $u^\eta - \psi$, which belongs to $H^1_0(\Omega, \mathbb{R}^m) \cap L^2_\mu(E, \mathbb{R}^m)$, as test function in the equation (2.4) satisfied by $u^\zeta$. In particular (2.5) gives

\[
C^B_{\alpha, \mu}(E, \xi, \eta) = \int_{\mathbb{Q}} (\mathbb{A}D(u^\zeta), D\psi)dx ,
\]

if $\psi = \eta \mu$-almost everywhere on $E$.

**Remark 2.6.** If $\mu$ is bounded, then $u^\eta \in L^2_\mu(E, \mathbb{R}^m)$, thus we may take $u^\eta$ as test function in the equation satisfied by $u^\zeta$ and we obtain

\[
C^B_{\alpha, \mu}(E, \xi, \eta) = -\int_{\mathbb{E}} (B(u^\xi - \xi), \eta)d\mu .
\]

We shall compare now the capacity $C^B_{\alpha, \mu}$ with the $\mu$-capacity $C^\mu$ relative to the Laplacian, introduced in [7], Definition 5.1.

**Proposition 2.7.** There exist two constants $c_8 > 0$ and $c_9 > 0$, depending on $n, m$, and on $c_1, c_2, c_6, c_7$, such that for every Borel set $E \subset \subset \Omega$

\[
c_8 C^\mu(E)|\xi|^2 \leq C^B_{\alpha, \mu}(E, \xi, \xi), \quad \forall \xi \in \mathbb{R}^m ,
\]

(2.6)

\[
|C^B_{\alpha, \mu}(E, \xi, \eta)| \leq c_9 C^\mu(E)|\xi| |\eta| , \quad \forall \xi, \eta \in \mathbb{R}^m .
\]

(2.7)

**Proof.** To prove (2.6), let $v^\alpha = (u^\xi)^\alpha/\xi^\alpha$, if $\xi^\alpha \neq 0$, and $v^\alpha = 0$ otherwise. Then, using the ellipticity of $A$ and $B$, for every Borel subset
\( E \subset \Omega \) and for every \( \xi \in \mathbb{R}^m \) we obtain

\[
\int_{\Omega} (ADu^\xi, Du^\xi) \, dx + \int_{\mathbb{R}} (B(u^\xi - \xi), u^\xi - \xi) \, d\mu \geq
\]

\[
\geq k \left( \int_{\Omega} |Du^\xi|^2 \, dx + \int_{\mathbb{R}} |u^\xi - \xi|^2 \, d\mu \right) \geq
\]

\[
\geq k |\xi|^2 \sum_{a=1}^{m} \left( \int_{\Omega} |Dv^a|^2 \, dx + \int_{\mathbb{R}} |v^a - 1|^2 \, d\mu \right),
\]

where \( k = \min\{c_1, c_6\} \). This implies that

\[ C_{\alpha}^{B,\mu}(E, \xi, \eta) \geq mkC^\mu(E) |\xi|^2. \]

Using Hölder Inequality it can be easily proved that

\[ |C_{\alpha}^{B,\mu}(E, \xi, \eta)| \leq (C_{\alpha}^{B,\mu}(E, \xi, \xi))^{1/2} (C_{\alpha}^{B,\mu}(E, \eta, \eta))^{1/2}. \]

Hence it suffices to prove (2.7) for \( \xi = \eta \). Let \( v_E \) be the \( C^\mu \)-capacitary potential of \( E \) in \( \Omega \) (see [6], Definition 3.1). Define \( \psi^a = (1 - v_E)\xi^a \). By (2.5), using the boundedness of \( A \) and \( B \), Young Inequality, and then Poincaré Inequality we get

\[ C_{\alpha}^{B,\mu}(E, \xi, \xi) \leq M \left( \int_{\Omega} |Du^\xi| |D\psi| \, dx + \int_{\mathbb{R}} |u^\xi - \xi| |\psi - \xi| \, d\mu \right) \leq
\]

\[
\leq \frac{M}{2} \left( \varepsilon \int_{\Omega} |Du^\xi|^2 \, dx + \frac{1}{\varepsilon} \int_{\Omega} |D\psi|^2 \, dx +
\right.
\]

\[
+ \varepsilon \int_{\mathbb{R}} |u^\xi - \xi|^2 \, d\mu + \frac{1}{\varepsilon} \int_{\mathbb{R}} |\psi - \xi|^2 \, d\mu \right).
\]

For a suitable choice of \( \varepsilon \) the sum of the terms containing \( u^\xi \) can be majorized by \( (1/M)C_{\alpha}^{B,\mu}(E, \xi, \xi) \), hence there exists a constant \( K \) such that

\[ C_{\alpha}^{B,\mu}(E, \xi, \xi) \leq K \left( \int_{\Omega} |D\psi|^2 \, dx + \int_{\mathbb{R}} |\psi - \xi|^2 \, d\mu \right) \leq
\]

\[
\leq K |\xi|^2 \left( \int_{\Omega} |Dv_E|^2 \, dx + \int_{\mathbb{R}} |v_E|^2 \, d\mu \right) = K |\xi|^2 C^\mu(E). \]
PROPOSITION 2.8. For every Kato measure \( \mu \), the solution \( u^\xi \) of (2.4) corresponding to a Borel subset \( E \) of \( \Omega \) of sufficiently small diameter belongs to \( L^\infty (\Omega, \mathbb{R}^m) \) and tends to 0 in \( L^\infty (\Omega, \mathbb{R}^m) \) as the diameter of \( E \) tends to zero.

PROOF. Let \( E \) be a Borel subset of \( \Omega \) and let \( u^\xi \) be the solution of (2.4). If \( u^\xi \in L^\infty (\Omega, \mathbb{R}^m) \), then the representation formula (1.3) for the solution of a linear system of second order partial differential equations gives

\[
(2.8) \quad u^\xi (x) = - \int_E G(x, y) B(y)(u^\xi (y) - \zeta) d\mu(y) \quad \text{for a.e. } x \in \Omega,
\]

where \( G(x, y) \) is the Green’s function associated with the operator \( A \) and with the domain \( \Omega \). In this case the measure \( B(u^\xi - \zeta) \mu \ll E \) belongs to \( H^{-1}(\Omega, \mathbb{R}^m) \) and (2.8) provides the quasicontinuous representative of \( u^\xi \).

Let us consider the operator \( T : L^\infty (\Omega, \mathbb{R}^m) \to L^\infty (\Omega, \mathbb{R}^m) \) defined by

\[
Tf(x) = - \int_E G(x, y) B(y)(f(y) - \zeta) d\mu(y).
\]

Since the functions \( b_{\alpha\beta} \) are bounded, we may apply estimate (1.2) for the Green’s function and we obtain

\[
\| Tf_1 - Tf_2 \|_{L^\infty (\Omega, \mathbb{R}^m)} \leq c_3 c_7 \| f_1 - f_2 \|_{L^\infty (\Omega, \mathbb{R}^m)} \sup_{x \in \Omega} \int_E |x - y|^{2-n} d\mu(y).
\]

As \( \mu \in K^+ (\Omega) \), the integral in the above formula tends to zero as \( \text{diam}(E) \) tends to zero, so that for sets \( E \) of sufficiently small diameter the operator \( T \) is a contraction, hence it has a unique fixed point \( w \) in \( L^\infty (\Omega, \mathbb{R}^m) \). By (1.3), for \( f \in L^\infty (\Omega, \mathbb{R}^m) \) the function \( w_f = Tf \) is the solution of the Dirichlet problem

\[
\begin{cases}
 w_f \in H^1_0 (\Omega, \mathbb{R}^m), \\
 Aw_f = - B(f - \zeta) \mu \ll E \quad \text{in } \Omega,
\end{cases}
\]

so that the fixed point \( w \) belongs to \( H^1_0 (\Omega, \mathbb{R}^m) \) and is a solution in the sense of distributions of \( Aw = - B(w - \zeta) \mu \ll E \), and hence a solution of
(2.4). Therefore \( u^\xi = w \) and we conclude that for sets \( E \) of sufficiently small diameter \( u^\xi \in L^\infty_w (\Omega, R^m) \). Then, from (2.8), for the quasicontinuous representative of \( u^\xi \) we have

\[
| u^\xi (x) | = \left| \int_E G(x, y) B(y) (u^\xi (y) - \xi) \, d\mu(y) \right| \leq \\
\leq \int_E | G(x, y) | | B(y) | | u^\xi (y) - \xi | d\mu(y) \leq \\
\leq c_3 c_7 \| u^\xi - \xi \|_{L^\infty_w (\Omega, R^m)} \sup_{x \in \Omega} \int_E | x - y |^{2-n} \, d\mu(y),
\]

which implies that \( \| u^\xi \|_{L^\infty_w (\Omega, R^m)} \leq c_3 \| u^\xi \|_{L^\infty_w (\Omega, R^m)} + c_7 \| \xi \| \), where the coefficient \( c_E \) is given by \( c_3 c_7 \sup_{x \in \Omega} \int_E | x - y |^{2-n} \, d\mu \) and tends to zero as the diameter of \( E \) tends to zero. As \( u^\xi \in H^1_0 (\Omega, R^m) \) and \( \mu \) vanishes on sets of capacity zero, \( \| u^\xi \|_{L^\infty_w (\Omega, R^m)} \leq \| u^\xi \|_{L^\infty (\Omega, R^m)} \) and from the previous inequality we obtain that \( \| u^\xi \|_{L^\infty (\Omega, R^m)} \) tends to zero as the diameter of \( E \) tends to zero.

**THEOREM 2.9.** If \( \mu \) is a Kato measure then

\[
\lim_{e \to 0^+} \frac{C_{\alpha}^{B, \mu} (D_e (x), \xi, \eta)}{\mu (D_e (x))} = (B(x) \xi, \eta)
\]

for \( \mu \)-almost every \( x \in \Omega \) and for every \( \xi, \eta \in R^m \).

**PROOF.** Let \( x \in \Omega \). Since every \( \mu \in K^+ (\Omega) \) is bounded, by Remark 2.6 we have

\[
(2.9) \quad C_{\alpha}^{B, \mu} (D_e (x), \xi, \eta) = - \int_{D_e (x)} (B(y)(u^\xi (y) - \xi), \eta) \, d\mu(y).
\]

By the Besicovitch Differentiation Theorem (see, e.g., [9], 1.6.2),

\[
(2.10) \quad \lim_{e \to 0^+} \frac{1}{\mu (D_e (x))} \int_{D_e (x)} (B(y), \eta) \, d\mu(y) = (B(x) \xi, \eta)
\]

for \( \mu \)-almost every \( x \in \Omega \) and for every \( \xi, \eta \in R^m \). The conclusion follows now from (2.9), (2.10), and Proposition 2.8. □
3. \( \gamma^\alpha \)-convergence.

In order to study the asymptotic behaviour of sequences of solutions of Dirichlet problems in varying domains we introduce the notion of \( \gamma^\alpha \)-convergence and show that under certain hypotheses the \( \gamma^\alpha \)-limit can be identified.

**DEFINITION 3.1.** Let \((\Omega_j)\) be a sequence of open subsets of \(\Omega\), let \(\mu \in \mathcal{M}_0(\Omega)\), and let \(B\) be an \(m \times m\) matrix of Borel functions satisfying (2.3). We say that \((\Omega_j)\) \(\gamma^\alpha_{\Omega}\)-converges to \((B, \mu)\), and we use the notation \(\Omega_j \xrightarrow{\gamma^\alpha_{\Omega}} (B, \mu)\), if for every \(f \in H^{-1}(\Omega, R^m)\) the sequence \((u_j)\) of the solutions of the problems

\[
\begin{aligned}
\int_{\Omega_j} (Au_j, Dv) \, dx &= \langle f, v \rangle, \\
\forall v &\in H_0^1(\Omega, R^m),
\end{aligned}
\]

extended by zero on \(\Omega \setminus \Omega_j\), converges weakly in \(H_0^1(\Omega, R^m)\) to the solution of the relaxed Dirichlet problem

\[
\begin{aligned}
\int_{\Omega} (Au, Dv) \, dx + \int_{\Omega} (Bu, v) \, d\mu &= \langle f, v \rangle, \\
\forall v &\in H_0^1(\Omega, R^m) \cap L^2(\Omega, R^m).
\end{aligned}
\]  

**REMARK 3.2.** Let \(\mu \in \mathcal{M}_0(\Omega)\), let \(B\) be an \(m \times m\) matrix of Borel functions satisfying (2.3), and let \(v\) and \(C\) be defined by

\[
v(E) = \int_E |B| \, d\mu, \quad C(x) = \frac{B(x)}{|B(x)|}.
\]

Then the measure \(v\) belongs to \(\mathcal{M}_0(\Omega)\) and the matrix \(C\) satisfies (2.3). Moreover \(\Omega_j \xrightarrow{\gamma^\alpha_{\Omega}} (B, \mu)\) if and only if \(\Omega_j \xrightarrow{\gamma^\alpha_{\Omega}} (C, v)\). This shows that, in Definition 3.1, it is not restrictive to assume \(|B(x)| = 1\) for every \(x \in \Omega\). However, it is sometimes useful to consider also matrices \(B\) which do not satisfy this condition.

If \(m = 1\) and \(\alpha = -A\), we shall always assume that \(B(x) = 1\) for every \(x \in \Omega\). In this case we use the notation \(\Omega_j \xrightarrow{\gamma^\alpha} \mu\).

The following compactness result is proved in [4].
THEOREM 3.3. For every sequence \((\Omega_j)\) of open subsets of \(\Omega\) there exist a subsequence \((\Omega'_{j_k})\), a measure \(\mu \in \mathcal{P}_0(\Omega)\), and an \(m \times m\) matrix of Borel functions satisfying (2.3), such that \(\Omega_{j_k} \xrightarrow{\gamma_\alpha} \mu\) and \(\Omega_{j_k} \xrightarrow{\gamma_\beta}(B, \mu)\).

The localization property of the \(\gamma_\alpha\)-convergence is also proved in [4].

THEOREM 3.4. If \(\Omega_j \xrightarrow{\gamma_\beta}(B, \mu)\) then \(\Omega_j \cap U \xrightarrow{\gamma_\beta}(B_{1_U}, \mu_{1_U})\) for every open subset \(U\) of \(\Omega\).

PROPOSITION 3.5. Suppose that \(\Omega_j \xrightarrow{\gamma_\beta}(B, \mu)\) and \(\Omega_j \xrightarrow{\gamma_\beta}(B, \bar{\mu})\). If \(\mu = \bar{\mu}\) and \(\mu(\Omega) < +\infty\), then \(B(x) = \bar{B}(x)\) for \(\mu\)-almost every \(x \in \Omega\).

PROOF. Let \(f \in H^{-1}(\Omega, \mathbb{R}^m)\) and let \(u\) be the solution of the relaxed Dirichlet problem (3.1). Then we have

\[
\int_\Omega ((B - \bar{B})u, v) \, d\mu = 0, \quad \forall v \in H^1_0(\Omega, \mathbb{R}^m) \cap L^2_\mu(\Omega, \mathbb{R}^m).
\]

In particular, since \(\mu(\Omega) < +\infty\), this equality holds true for every \(v \in C_c^\infty(\Omega, \mathbb{R}^m)\). So, varying \(v\), we obtain that \((B - \bar{B})u = 0\) \(\mu\)-almost everywhere in \(\Omega\). Since \(\mu(\Omega) < +\infty\), the set of all solutions \(u\) of (3.1) corresponding to different data \(f \in H^{-1}(\Omega, \mathbb{R}^m)\) is dense in \(H^1_0(\Omega, \mathbb{R}^m)\). This implies that \(B = \bar{B}\) \(\mu\)-almost everywhere in \(\Omega\). □

For every \(x \in \Omega\) let \(d_\Omega(x) = \text{dist}(x, \partial \Omega)\).

THEOREM 3.6. If \(\Omega_j \xrightarrow{\gamma_\alpha}\mu\), with \(\mu(\Omega) < +\infty\), and \(\Omega_j \xrightarrow{\gamma_\beta}(B, \mu)\), then for every \(x \in \Omega\) there exists a countable set \(N(x) \subset \mathbb{R}\) such that

\[
C_\alpha(D_\varphi(x)\setminus \Omega_j, \xi, \eta) \rightarrow C^{B, \mu}_\alpha(D_\varphi(x), \xi, \eta)
\]

for every \(\varphi \in (0, d_\Omega(x))\setminus N(x)\).

PROOF. Let us fix \(x \in \Omega\). It is proved in [7] that there exists a countable set \(N_1(x) \subset \mathbb{R}\) such that for all \(\varphi \in (0, d_\Omega(x))\setminus N_1(x)\)

\[
\Omega \setminus (D_\varphi(x) \setminus \Omega_j) \xrightarrow{\gamma_\alpha}\mu \setminus D_\varphi(x).
\]

Then, applying Theorem 3.3 to the sequence \(\tilde{\Omega}_j = \Omega \setminus (D_\varphi(x) \setminus \Omega_j)\), we obtain that there exist a subsequence, still denoted by the same index \(j\),
and an \( m \times m \) matrix \( \tilde{B} \) of Borel functions satisfying (2.3) such that
\[
\tilde{Q}_j \xrightarrow{\gamma^B_{\partial}} (\tilde{B}, \mu \llcorner D_\varrho(x)).
\]
Now we apply the localization result (Theorem 3.4) to the sequence \((Q_j)\) and we obtain
\[
Q_j \cap U_\varrho(x) \xrightarrow{\gamma_{U_\varrho(x)}} \mu \llcorner_{U_\varrho(x)} \quad \text{and} \quad Q_j \cap U_\varrho(x) \xrightarrow{\gamma^B_{U_\varrho(x)}} (B \llcorner_{U_\varrho(x)}, \mu \llcorner_{U_\varrho(x)}).
\]
The same localization result applied to the sequence \( \tilde{Q}_j \) gives
\[
Q_j \cap U_\varrho(x) = \tilde{Q}_j \cap U_\varrho(x) \xrightarrow{\gamma^B_{U_\varrho(x)}} (\tilde{B} \llcorner_{U_\varrho(x)}, \mu \llcorner_{U_\varrho(x)}),
\]
hence \( B = \tilde{B} \) \( \mu \)-almost everywhere in \( U_\varrho(x) \) by Proposition 3.5. On the other hand, since \( \mu(\Omega) < +\infty \), for every \( x \in \Omega \) there exists a countable set \( N_2(x) \subset \mathbb{R} \) such that \( \mu(\partial D_\varrho(x)) = 0 \) for all \( \varrho \in (0, d_\varrho(x)) \backslash N_2(x) \). Together with the previous results this implies that
\[
\tilde{Q}_j \xrightarrow{\gamma^B_{\partial}} (B, \mu \llcorner D_\varrho(x)), \quad \forall \varrho \in (0, d_\varrho(x)) \backslash (N_1(x) \cup N_2(x)).
\]
Let \( K_j = D_\varrho(x) \backslash Q_j = \Omega \backslash \tilde{Q}_j \) and let \( u_j \) be the weak solution in \( \tilde{Q}_j \) of the problem
\[
\begin{cases}
\begin{align*}
u_j &\in H^1(\tilde{Q}_j), \\
u_j &\equiv \xi \quad \text{on } \partial K_j, \\
u_j &\equiv 0 \quad \text{on } \partial \Omega,
\end{align*}
\end{cases}
\int (A D u_j, D v) \, dx = 0, \quad \forall v \in H^1_0(\tilde{Q}_j, \mathbb{R}^m).
\]
As usual we extend \( u_j \) to \( \Omega \) by setting \( u_j = \xi \) on \( K_j \). Let \( \varphi \in C^\infty_0(\Omega, \mathbb{R}^m) \) with \( \varphi = \xi \) on \( D_\varrho(x) \), and let \( z_j = u_j - \varphi \). Then \( z_j \) is the solution of the problem
\[
\begin{cases}
z_j &\in H^1_0(\tilde{Q}_j, \mathbb{R}^m), \\
\int (A D z_j, D v) \, dx = \langle f, v \rangle, \quad \forall v \in H^1_0(\tilde{Q}_j, \mathbb{R}^m),
\end{cases}
\]
where \( f \) is the element of \( H^{-1}(\Omega, \mathbb{R}^m) \) defined by \( \langle f, v \rangle = -\int \Omega (A D \varphi, D v) \, dx \). By Definition 3.1 the sequence \( (z_j) \) converges weakly
in $H^1_0(\Omega, \mathbb{R}^m)$ to the solution $z$ of the problem

$$
\begin{cases}
z \in H^1_0(\Omega, \mathbb{R}^m) \cap L^2(D_\varphi(x), \mathbb{R}^m), \\
\int (ADz, Dv) \, dx + \int_{D_\varphi(x)} (Bz, v) \, d\mu = (f, v), \\
\forall v \in H^1_0(\Omega, \mathbb{R}^m) \cap L^2_\mu(D_\varphi(x), \mathbb{R}^m).
\end{cases}
$$

This implies that $u^\xi$ converges weakly in $H^1_0(\Omega, \mathbb{R}^m)$ to the solution $u^\xi$ of (2.4) corresponding to $\xi = \xi$ and $E = D_\varphi(x)$. Consequently $(ADu_j)$ converges to $ADu^\xi$ weakly in $L^2(\Omega, M^{m \times m})$. Let us fix now $\psi \in C^\infty_0(\Omega, \mathbb{R}^m)$ with $\psi = \eta$ on $D_\varphi(x)$. Then, by Remarks 2.2 and 2.5,

$$
C^1_\alpha(D_\varphi(x) \setminus \Omega_j, \xi, \eta) = \int_{\Omega} (ADu_j, D\psi) \, dx,
$$

and the conclusion follows from the weak convergence of $(ADu_j)$. □

Given a family $(f_\varphi)_\varphi > 0$ of real numbers, we say that $\text{ess lim}_{\varphi \to 0} f_\varphi = a$ if for every neighbourhood $V$ of $a$ there exists a neighbourhood $U$ of 0 such that $f_\varphi \in V$ for almost every $\varphi \in U$. Let $(\Omega_j)$ be a sequence of open subsets of $\Omega$. For every closed ball $D_\varphi(x) \subset \Omega$ and for every $\xi, \eta \in \mathbb{R}^m$ we define

$$
\begin{cases}
\alpha'(D_\varphi(x), \xi, \eta) = \liminf_{j \to \infty} C_{\alpha_\varphi}(D_\varphi(x) \setminus \Omega_j, \xi, \eta), \\
\alpha''(D_\varphi(x), \xi, \eta) = \limsup_{j \to \infty} C_{\alpha_\varphi}(D_\varphi(x) \setminus \Omega_j, \xi, \eta).
\end{cases}
$$

We are now in a position to prove the main result of the paper.

**Theorem 3.7.** Assume that there exists a measure $\lambda \in K^+(\Omega)$ such that

$$
\alpha''(D_\varphi(x), \xi, \xi) \leq \lambda(D_\varphi(x)) \, |\xi|^2
$$

for every closed ball $D_\varphi(x) \subset \Omega$ and for every $\xi \in \mathbb{R}^m$. Assume, in addition, that for every $x \in \Omega$

$$
\alpha'(D_\varphi(x), \xi, \eta) = \alpha''(D_\varphi(x), \xi, \eta) \quad \text{for a.e. } \varphi \in (0, d_\Omega(x)).
$$
Then there exists an $m \times m$ matrix $G(x)$ of bounded Borel functions such that

\[
(3.5) \quad \lim_{\varepsilon \to 0} \, \frac{\alpha'(D_\varepsilon(x), \xi, \eta)}{\lambda(D_\varepsilon(x))} = \lim_{\varepsilon \to 0} \, \frac{\alpha''(D_\varepsilon(x), \xi, \eta)}{\lambda(D_\varepsilon(x))} = (G(x) \xi, \eta)
\]

for $\lambda$-almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^m$. Let $B$ and $\mu$ be defined by

\[
B(x) = \frac{G(x)}{|G(x)|} \quad \text{for } \lambda\text{-a.e. } x \in \Omega,
\]

\[
\mu(E) = \int_E |G| \, d\lambda \quad \text{for every Borel set } E \subset \Omega,
\]

with the convention that $0/0$ is the $m \times m$ identity matrix $I$. Then $B$ satisfies (2.3) and $\Omega_j \xrightarrow{\gamma^B_{\mu}} (B, \mu)$.

**Remark 3.8.** Theorems 3.3 and 3.6 imply that every sequence $(\Omega_j)$ has a subsequence which satisfies (3.4). Therefore condition (3.3) is the only non-trivial hypothesis of Theorem 3.7.

**Remark 3.9.** For every closed ball $D_\varepsilon(x) \subset \Omega$ let

\[
\beta''(D_\varepsilon(x)) = \limsup_{j \to +\infty} \text{cap}(D_\varepsilon(x) \setminus \Omega_j).
\]

If there exists a measure $\lambda \in K^+(\Omega)$ such that $\beta''(D_\varepsilon(x)) \leq \lambda(D_\varepsilon(x))$ then the estimates in Remark 2.3 imply that (3.3) is satisfied with $\lambda$ replaced by $c_\varepsilon \lambda$. This condition is satisfied, for instance, in the periodic case with a critical size of the holes (see [5]) and for the sequences of domains considered in [11] and [12].

**Proof of Theorem 3.7.** Let us fix $x \in \Omega$. From the compactness result (Theorem 3.3) we obtain that there exist a subsequence, still denoted by $(\Omega_j)$, and a pair $(\tilde{B}, \tilde{\mu})$, with $\tilde{B}$ satisfying (2.3) and $\tilde{\mu} \in \mathcal{M}_0(\Omega)$, such that $\Omega_j \xrightarrow{\gamma^\Omega} \tilde{\mu}$ and $\Omega_j \xrightarrow{\gamma^\mu_{\tilde{B}}} (\tilde{B}, \tilde{\mu})$. Let us fix $x \in \Omega$. By Theorem 5.15 in [7] for almost every $q \in (0, d_\Omega(x))$ we have $\text{cap}(D_\varepsilon(x) \setminus \Omega_j) \to \rightarrow C^\mu(D_\varepsilon(x))$. The first estimate in Remark 2.3 gives

\[
c_4|\xi|^2 \text{cap}(D_\varepsilon(x) \setminus \Omega_j) \leq C_\alpha(D_\varepsilon(x) \setminus \Omega_j, \xi, \xi),
\]
and passing to the limit we get

\[ c_4 \tilde{\mu}(D_\theta(x))|\xi|^2 \leq \limsup_{\theta \to 0} C_\alpha(D_\theta(x) \setminus \Omega_j, \xi, \tilde{\xi}) = \alpha''(D_\theta(x), \xi, \tilde{\xi}) \leq \lambda(D_\theta(x))|\xi|^2. \]

Applying now Theorem 2.3 in [3] we get that \( \tilde{\mu} \) is absolutely continuous with respect to \( \lambda \) and that the density \((d\tilde{\mu}/d\lambda)(x)\) is bounded, hence \( \tilde{\mu} \in K^+(\Omega) \). Let

\[ G(x) = \tilde{B}(x) \frac{d\tilde{\mu}}{d\lambda}(x), \quad B(x) = \frac{G(x)}{|G(x)|}, \]

\[ \mu(E) = \int_E |G| d\lambda = \int \tilde{B} d\tilde{\mu}, \]

with the convention that \(0/0\) is the \(m \times m\) identity matrix \(I\). Then

\[ B(x) = \frac{\tilde{B}(x)}{|\tilde{B}(x)|}, \quad \text{if } \frac{d\tilde{\mu}}{d\lambda}(x) > 0, \quad \text{and } B(x) = I, \quad \text{if } \frac{d\tilde{\mu}}{d\lambda}(x) = 0. \]

As \( \tilde{B} \) satisfies (2.3) \( \tilde{\mu} \)-almost everywhere, \( B \) satisfies (2.3) \( \mu \)-almost everywhere. Since \( \Omega_j \overset{\gamma_2}{\longrightarrow} (\tilde{B}, \tilde{\mu}) \) and \( B(x) = \tilde{B}(x)/|\tilde{B}(x)| \tilde{\mu} \)-almost everywhere in \( \Omega \), by Remark 3.2 we have also \( \Omega_j \overset{\gamma_2}{\longrightarrow} (B, \mu) \).

Let us prove now (3.5). Applying Theorem 3.6 we obtain that

\[ C_{\alpha}(D_\theta(x) \setminus \Omega_j, \xi, \eta) \rightarrow C_{\alpha}^{B, \tilde{\mu}}(D_\theta(x), \xi, \eta) \]

for almost every \( \theta \in (0, d_\Omega(x)) \). Thus

\[ \alpha'(D_\theta(x), \xi, \eta) = \alpha''(D_\theta(x), \xi, \eta) = C_{\alpha}^{B, \tilde{\mu}}(D_\theta(x), \xi, \eta) \]

for almost every \( \theta \in (0, d_\Omega(x)) \) and for every \( \xi, \eta \in \mathbb{R}^m \). We may now apply Theorem 2.9 and the Besicovitch Differentiation Theorem to obtain

\[ \text{ess lim}_{\theta \to 0} \frac{\alpha'(D_\theta(x), \xi, \eta)}{\lambda(D_\theta(x))} = \text{ess lim}_{\theta \to 0} \frac{C_{\alpha}^{B, \tilde{\mu}}(D_\theta(x), \xi, \eta)}{\tilde{\mu}(D_\theta(x))} = \text{ess lim}_{\theta \to 0} \frac{\tilde{\mu}(D_\theta(x))}{\lambda(D_\theta(x))} = \]

\[ \left( \tilde{B}(x), \xi, \eta \right) \frac{d\tilde{\mu}}{d\lambda}(x) = (G(x), \xi, \eta) \]

for every \( \xi, \eta \in \mathbb{R}^m \) and for \( \lambda \)-almost every \( x \in \Omega \) such that \((d\tilde{\mu}/d\lambda)(x) > 0\). Since \( C_{\alpha}^{B, \tilde{\mu}}(D_\theta(x), \xi, \eta) \leq c_5 C_{\mu}(D_\theta(x)) |\xi| |\eta| \leq \]
I we obtain that for $A$-almost every $x \in \Omega$ such that $(dJ_i/d\lambda)(x) = 0$. This concludes the proof of (3.5).

4. The symmetric case.

If the operator $c_i$ is symmetric, then the $c_i$-capacity can be obtained by solving a minimum problem. If $\Omega \xrightarrow{\gamma_B} (B, \mu)$, with $\mu(\Omega) < +\infty$, then the matrix $B$ is symmetric (see [8], Corollary 5.4). In this case we have

$$C_{c_i}^{B,\mu}(E, \xi, \xi) = \min_{u \in H_0^1(\Omega, \mathbb{R}^m)} \left\{ \int_{\Omega} (Du^\xi, Du^\xi) \, dx + \int_{E} (B(u^\xi - \xi), (u^\xi - \xi)) \, d\mu \right\}$$

for every measure $\mu \in \mathcal{M}_0(\Omega)$, for every $\xi \in \mathbb{R}^m$, and for every Borel set $E \subset \subset \Omega$.

**Remark 4.1.** Assume that $c_i$ and $B$ are symmetric. If $\mu_1 \leq \mu_2$, then $C_{c_i}^{B,\mu_1}(E, \xi, \xi) \leq C_{c_i}^{B,\mu_2}(E, \xi, \xi)$ for every Borel set $E \subset \subset \Omega$ and every $\xi \in \mathbb{R}^m$.

This monotonicity property of the capacity with respect to the measure allows us to extend the derivation theorem to any bounded measure in $\mathcal{M}_0(\Omega)$.

**Theorem 4.2.** Assume that $c_i$ is symmetric. Let $\mu, \nu \in \mathcal{M}_0(\Omega)$, with $\nu(\Omega) < +\infty$, and let $B$ be an $m \times m$ symmetric matrix of Borel functions satisfying (2.3). For every $x \in \Omega$ and for every $\xi \in \mathbb{R}^m$ let

$$f(x, \xi) =$$

$$\liminf_{\varepsilon \to 0} \frac{C_{c_i}^{B,\mu}(D_{\varepsilon}(x), \xi, \xi)}{\nu(D_{\varepsilon}(x))} \quad (\text{with the convention that } 0/0 = 1).$$
Assume that there exists $\xi \in \mathbb{R}^m \backslash \{0\}$ such that

\begin{equation}
(4.2) \quad f(x, \xi) < + \infty \quad \forall x \in \Omega \quad \text{and} \quad \int_{\Omega} f(x, \xi) \, d\nu < + \infty.
\end{equation}

Then $\mu(\Omega) < + \infty$, $\mu$ is absolutely continuous with respect to $\nu$, and

\[ f(x, \xi) = (B(x) \xi, \xi) \frac{d\mu}{d\nu}(x) \quad \text{for } \nu - \text{a.e. } x \in \Omega \quad \text{and} \quad \forall \xi \in \mathbb{R}^m. \]

Moreover, the lim inf in the definition of $f$ is a limit for $\nu$-almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^m$.

**Proof.** For every $x \in \Omega$ let

\[ f_1(x) = \liminf_{\varepsilon \to 0} \frac{C^\mu(D_\varepsilon(x))}{\nu(D_\varepsilon(x)).} \]

The estimates in Proposition 2.7 give

\begin{equation}
(4.3) \quad c_8 \sqrt{\frac{1}{2}} f_1(x) \leq f(x, \xi) \leq c_9 \sqrt{\frac{1}{2}} f_1(x), \quad \forall x \in \Omega, \, \forall \xi \in \mathbb{R}^m,
\end{equation}

thus $f_1 \in L^1(\Omega)$ and $f_1(x) < + \infty$ for every $x \in \Omega$. Then from Proposition 2.3 in [3] we deduce that $\mu(\Omega) < + \infty$ and that $\mu = f_1 \nu$, i.e., $\mu(E) = \int_E f_1 \, d\nu$ for every Borel set $E \subset \Omega$. By Proposition 2.5 of [2] there exist a measure $\lambda \in K^+(\Omega)$ and a Borel function $g: \Omega \to [0, + \infty]$ such that $\mu = g\lambda$. For every $k \in \mathbb{N}$ let $g_k(x) = \min\{g(x), k\}$. Since $g_k \lambda$ belongs to $K^+(\Omega)$, Theorem 2.9 implies the existence of a subset $E_1$ of $\Omega$ such that

\[ \int_{E_1} g_k \lambda \, d\lambda = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{C^\lambda_{B,g_k \lambda}(D_\varepsilon(x), \xi, \xi)}{(g_k \lambda)(D_\varepsilon(x))} = (B(x) \xi, \xi), \quad \forall x \in \Omega \backslash E_1, \, \forall \xi \in \mathbb{R}^m, \, \forall k \in \mathbb{N}. \]

Since $\lambda + \nu$ is a bounded measure on $\Omega$, by the Besicovitch Differentiation Theorem there exists a set $E_2 \subset \Omega$ such that $(\lambda + \nu)(E_2) = 0$ and

\[ \lim_{\varepsilon \to 0} \frac{(g_k \lambda)(D_\varepsilon(x))}{(\lambda + \nu)(D_\varepsilon(x))} = g_k(x) \frac{d\lambda}{d(\lambda + \nu)}(x) < + \infty, \quad \forall x \in \Omega \backslash E_2, \, \forall k \in \mathbb{N}, \]

\[ \lim_{\varepsilon \to 0} \frac{\nu(D_\varepsilon(x))}{(\lambda + \nu)(D_\varepsilon(x))} = \frac{d\nu}{d(\lambda + \nu)}(x) \leq 1, \quad \forall x \in \Omega \backslash E_2. \]
By (4.2) and (4.3) we have \( f_1(x) < +\infty \) and \( f(x, \xi) < +\infty \) for every \( x \in \Omega \) and for every \( \xi \in \mathbb{R}^m \). Let \( E = E_1 \cup E_2 \). For \( x \in \Omega \setminus E \) and \( \xi \in \mathbb{R}^m \) we have

\[
g_k(x)(B(x) \xi, \xi) \frac{d\lambda}{d(\lambda + v)}(x) =
\]

\[
= \lim_{\epsilon \to 0} \frac{(g_k \lambda)(D_\epsilon(x))}{(\lambda + v)(D_\epsilon(x))} \lim_{\epsilon \to 0} \frac{C^{B, g_k \lambda}(D_\epsilon(x), \xi, \xi)}{(g_k \lambda)(D_\epsilon(x))} =
\]

\[
= \lim_{\epsilon \to 0} \frac{C^{B, g_k \lambda}(D_\epsilon(x), \xi, \xi)}{(\lambda + v)(D_\epsilon(x))} \leq
\]

\[
\leq \liminf_{\epsilon \to 0} \frac{C^{B, g_k \lambda}(D_\epsilon(x), \xi, \xi)}{v(D_\epsilon(x))} \lim_{\epsilon \to 0} \frac{v(D_\epsilon(x))}{(\lambda + v)(D_\epsilon(x))} = f(x, \xi) \frac{dv}{d(\lambda + v)}(x).
\]

So, for every Borel set \( F \subset \Omega \setminus E \) and for every \( \xi \in \mathbb{R}^m \) we have

\[
\int_F \left[ g_k(x)(B(x) \xi, \xi) \frac{d\lambda}{d(\lambda + v)}(x) \right] d(\lambda + v) \leq
\]

\[
\leq \int_F f(x, \xi) \frac{dv}{d(\lambda + v)}(x) d(\lambda + v),
\]

hence

\[
\int_F g_k(x)(B(x) \xi, \xi) d\lambda \leq \int_F f(x, \xi) dv
\]

for every Borel set \( F \subset \Omega \). Passing now to the limit as \( k \to +\infty \), by the monotone convergence theorem we have

\[
\int_F (B(x) \xi, \xi) d\mu = \int_F g(x)(B(x) \xi, \xi) d\lambda \leq \int_F f(x, \xi) dv
\]

for every Borel set \( F \subset \Omega \) and every \( \xi \in \mathbb{R}^m \). Thus, \( f_1(x)(B(x) \xi, \xi) \leq f(x, \xi) \) for \( \nu \)-almost every \( x \in \Omega \) and for every \( \xi \in \mathbb{R}^m \). Since

\[
C^{B, \mu}(D_\epsilon(x), \xi, \xi) \leq \int_{D_\epsilon(x)} (B(y) \xi, \xi) f_1(y) dv(y),
\]

by the Besicovitch Differentiation Theorem we obtain \( f(x, \xi) \leq f_1(x)(B(x) \xi, \xi) \) for \( \nu \)-almost every \( x \in \Omega \) and for every \( \xi \in \mathbb{R}^m \). So we proved that \( f(x, \xi) = f_1(x)(B(x) \xi, \xi) \) for every \( \xi \in \mathbb{R}^m \) and \( \nu \)-almost
every $x \in \Omega$. Moreover, by the Besicovitch Differentiation Theorem for $\nu$-almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^m$ we have

$$f(x, \xi) = \liminf_{\epsilon \to 0} \frac{C_{\alpha}^{B, \nu}(D_\epsilon(x), \xi, \xi)}{\nu(D_\epsilon(x))} \leq \limsup_{\epsilon \to 0} \frac{C_{\alpha}^{B, \nu}(D_\epsilon(x), \xi, \xi)}{\nu(D_\epsilon(x))} \leq$$

$$\leq \limsup_{\epsilon \to 0} \frac{1}{\nu(D_\epsilon(x))} \int_{D_\epsilon(x)} (B(y) \xi, \xi) f_1(y) d\nu(y) = f_1(x)(B(x) \xi, \xi),$$

and this completes the proof. ■

The hypotheses in Theorem 3.7 can be weakened by using the monotonicity of the $\alpha$-capacity and the previous result.

**Theorem 4.3.** Assume that $\alpha$ is symmetric and that there exists a bounded Radon measure $\lambda$ on $\Omega$ such that

$$\alpha''(D_\epsilon(x), \xi, \xi) \leq \lambda(D_\epsilon(x)) |\xi|^2$$

for every closed ball $D_\epsilon(x) \subset \Omega$ and for every $\xi \in \mathbb{R}^m$. Assume, in addition, that for every $x \in \Omega$ there exists a dense set $D \subset (0, d_\Omega(x))$ such that

$$\alpha'(D_\epsilon(x), \xi, \xi) = \alpha''(D_\epsilon(x), \xi, \xi), \quad \forall \epsilon \in D, \forall \xi \in \mathbb{R}^m.$$ (4.4)

Then there exists an $m \times m$ symmetric matrix $G(x)$ of bounded Borel functions such that

$$\text{esslim}_{\epsilon \to 0} \frac{\alpha'(D_\epsilon(x), \xi, \xi)}{\lambda(D_\epsilon(x))} = \text{esslim}_{\epsilon \to 0} \frac{\alpha''(D_\epsilon(x), \xi, \xi)}{\lambda(D_\epsilon(x))} = (G(x) \xi, \xi)$$

for $\lambda$-almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^m$. Let $B$ and $\mu$ be defined by

$$B(x) = \frac{G(x)}{|G(x)|} \quad \text{for } \lambda - \text{a.e. } x \in \Omega,$$

$$\mu(E) = \int_E |G| d\lambda \quad \text{for every Borel set } E \subset \Omega,$$

with the convention that $0/0$ is the $m \times m$ identity matrix $I$. Then $\mu \in \mathcal{M}_0(\Omega)$, $B$ satisfies (2.3), and $\Omega \overset{\gamma^B}{\longrightarrow} (B, \mu)$.

**Proof.** Since $C_{\alpha}^B(\cdot, \xi, \xi)$ is an increasing set function, $\alpha'(D_\epsilon(x), \xi, \xi)$ and $\alpha''(D_\epsilon(x), \xi, \xi)$ are increasing functions of $\epsilon$, hence
(4.4) implies that \( \alpha'(D_\omega(x), \xi, \xi) = \alpha''(D_\omega(x), \xi, \xi) \) for almost every \( q \in (0, d_\Omega(x)) \). As in the proof of Theorem 3.7, we obtain that \( \Omega_j \stackrel{\gamma}{\rightarrow} (B, \bar{\mu}) \), with \( \bar{\mu} \) absolutely continuous with respect to \( \lambda \). Since \( (d\bar{\mu}/d\lambda)(x) \) is bounded, we have \( \bar{\mu}(\Omega) < +\infty \). Let \( G(x) = \tilde{B}(x)(d\bar{\mu}/d\lambda)(x) \). Since \( \mu(E) = \int |G|d\lambda = \int |\tilde{B}|d\bar{\mu} \), and \( \bar{\mu} \in \mathcal{M}_0(\Omega) \), we have \( \mu \in \mathcal{M}_0(\Omega) \). The conclusion follows now by repeating the same arguments as in Theorem 3.7, the only difference being that now we apply Theorem 4.2 instead of Theorem 2.9.

**REFERENCES**


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