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## On an Elliptic Equation with Exponential Growth.

J. A. AGUILAR CRESPO - I. PERAL ALONSO (\*)(\*\*)

ABSTRACT - In this paper we deal with the following nonlinear degenerate elliptic problem

$$(P) \quad \begin{cases} -\Delta_p u \equiv -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = \lambda V(x) e^u & \text{in } \Omega \subset \mathbf{R}^N, \quad N \geq 2, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where  $p > 1$ ,  $\lambda > 0$ ,  $\Omega$  is a bounded domain in  $\mathbf{R}^N$  and  $V(x)$  is a given function in  $L^q(\Omega)$  ( $q$  depending on the relationship between  $N$  and  $p$ ). In particular, we study the existence of solutions in  $W_0^{1,p}(\Omega)$ , considering the cases: 1) Existence of solution for  $\lambda$  small and  $V$  possibly changing sign in  $\Omega$ . 2) Conditions for positivity of solutions, with  $V$  changing sign in  $\Omega$ . 3) Existence and behavior of the minimal solution for  $V(x) \geq 0$  in  $\Omega$  and  $p < N$ . 4) Existence of solution for  $V$  possibly changing sign in  $\Omega$  and  $p \geq N$ . 5) Full analysis of the radial solutions for  $V = r^{-\alpha}$ ,  $\alpha < p$ ,  $|x| = r$ . It has to be remarked that these results are new even for the semilinear case,  $p = 2$ .

### Introduction.

We study the following problem,

$$(P) \quad \begin{cases} -\Delta_p u \equiv -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = \lambda V(x) e^u & \text{in } \Omega \subset \mathbf{R}^N, \quad N \geq 2, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where  $p > 1$ ,  $\lambda > 0$ ,  $x \in \Omega$  a bounded domain, and  $V(x)$  is a given function which may change sign in  $\Omega$ .

The case  $V \equiv \text{constant}$  has been studied in [GP] and [GPP] for gen-

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eral  $p$ . The semilinear case  $p = 2$ , with constant  $V$  too, has been extensively studied; see for instance the papers [Bd], [F], [Ge], [GMP], [JL], [MP1] and [MP2]. Motivation for the model in this case  $p = 2$  can be found in [Ch], [FK] and [KW].

On the other hand, the case  $p = 2$  and  $V$  a given function has a few precedents, see [BM] and [KW] for  $N = 2$ ; the general case is new at all.

The aim of this paper is to show the following results for  $V \in L^q(\Omega)$ , where  $q \geq 1$  for  $p > N$ ,  $q > 1$  for  $p = N$  or  $q > N/p > 1$  for  $1 < p < N$ .

(1) If  $\lambda$  is small enough, then there exists at least one solution to problem (P) for  $q > N/p > 1$ .

(2) If  $\lambda$  is small enough, then there exist at least one solution to problem (P) for  $p \geq N$ .

(3) If  $\lambda$  is large enough and  $V \geq 0$ , the problem (P) has no solution.

(Obviously, if  $V < 0$  then the maximum principle implies that problem (P) has one negative solution).

In addition, we find a sufficient condition related to the existence of positive solutions to (P) with  $V$  changing sign in  $\Omega$ , a result new even for the semilinear case  $p = 2$ .

We also prove that the first eigenvalue for the  $p$ -Laplacian with weight  $V \in L^q(\Omega)$ ,  $V(x) \geq 0$ , is simple and isolated. This result is an extension to our context of those by [An], [B] and [L]; it will be used in the study of the nonexistence of solutions to (P).

The paper is organized as follows: first, we show the existence of solution to (P) for  $V \in L^q(\Omega)$ , in case  $V$  may change sign in  $\Omega$  and a sufficient condition about the positivity of the solution when  $V$  changes sign is obtained. Next section is devoted to study the behavior of the minimal solution for  $1 < p < N$ ,  $V \geq 0$ . Variational methods are used for the case  $p \geq N$ ,  $V$  possibly with non constant sign. Finally, after dealing with the nonexistence of solutions, we analyze the radial solutions on the unit ball for  $V(r) = r^{-\alpha}$ ,  $\alpha < p$ ,  $r = |x|$ .

Before studying the existence of solutions, we give some definitions for solutions to the problem (P).

**DEFINITION.** We say that  $u \in W_0^{1,p}(\Omega)$  is a regular solution (P) if and only if  $e^u \in L^\infty(\Omega)$ , and the equation holds in the sense of  $W^{-1,p'}(\Omega)$ . If  $V(x)e^u \in L^1(\Omega)$ , we say that  $u$  is a singular solution of (P), and the equation holds in the sense of  $\mathcal{O}'(\Omega)$ .

Obviously, if  $u \in W_0^{1,p}(\Omega)$  with  $p > N$ , Morrey's Theorem [GT, Ch. 7] implies that  $e^u \in L^\infty(\Omega)$ . So, we just need  $V(x) \in L^1(\Omega)$ . On the other hand, by using the Stampacchia's lemma [S] and Trudinger's inequality [GT, p. 162], we get that a solution of (P) for  $p = N$ ,  $u \in W_0^{1,N}(\Omega)$ , verifies  $u \in L^\infty(\Omega)$  whenever the function  $V$  belongs to  $L^q(\Omega)$ ,  $q > 1$ . In other words

**PROPOSITION.** *If  $V \in L^q(\Omega)$ , with either  $q \geq 1, p > N$  or  $q > 1, p = N$ , then any singular solution of (P)  $u \in W_0^{1,p}(\Omega)$  is a regular solution.*

**1. - Existence of solution for  $\lambda$  small.**

We show in this section the existence of solution to the problem

$$(P) \quad \begin{cases} -\Delta_p u = \lambda V(x) e^u & \text{in } \Omega \subset \mathbf{R}^N, \quad N \geq 2, \\ u|_{\partial\Omega} = 0, \end{cases}$$

by means of a fixed point argument, where  $\lambda > 0$ ,  $x \in \Omega$ , a bounded domain, and  $V(x)$  is a given function in  $L^q(\Omega)$ ,  $q \geq 1$  if  $p > N$ ,  $q > 1$  if  $p = N$  and  $q > N/p$  otherwise.

It has to be noted that  $V$  may change sign in  $\Omega$ .

**LEMMA 1.1.** *Let  $B_\delta = \{\varphi \in C(\Omega): |\varphi| < \delta, \varphi|_{\partial\Omega} = 0\}$ ,  $\delta > 0$ . Let  $F_\lambda: B_\delta \rightarrow L^\infty(\Omega)$  defined by  $\varphi \rightarrow F_\lambda(\varphi) = \psi$ , where  $\psi$  verifies the following problem*

$$\begin{cases} -\Delta_p \psi = \lambda V(x) e^\varphi & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

Then,

$$\|\psi\|_\infty \leq C(\lambda e^\delta \|V\|_q)^\gamma$$

where  $C = C(p, N, \Omega)$  and  $\gamma > 0$ .

**PROOF.** *Case  $p > N$ .* In this case,  $W_0^{1,p}(\Omega) \subset L^\infty(\Omega)$  and by Sobolev inequality we obtain the following estimate

$$\|\psi\|_\infty \leq C(p, N, \Omega)(\lambda e^\delta \|V\|_q)^{1/(p-1)}$$

Case 1  $1 < p < N$ . Since  $\varphi$  is bounded, we get that  $\lambda V(x)e^\varphi$  belongs to  $L^q(\Omega)$ ,  $q > N/p$ . Hence,  $\lambda V(x)e^\varphi$  belongs to  $W^{-1,r}(\Omega)$  for  $r > N/(p-1)$ . Then, there exist  $f_1, f_2, \dots, f_N$  in  $L^r(\Omega)$  such that,  $\nabla \eta \in \mathcal{W}_0^{1,p}(\Omega)$ ,

$$\int_{\Omega} |\dot{\nabla} \psi|^{p-2} \langle \nabla \psi, \nabla \eta \rangle dx = \int_{\Omega} \langle f, \nabla \eta \rangle dx$$

where  $f = (f_1, f_2, \dots, f_N)$  and  $\lambda V(x)e^\varphi = -\operatorname{div} f$  (see [Br, Prop. IX.20]). For  $k > 0$ , if we take as test

$$\eta = \operatorname{sign}(\psi)(|\psi| - k)^+ = \begin{cases} \psi - k & \psi \geq k, \\ \psi + k & \psi \leq -k, \\ 0 & \text{otherwise,} \end{cases}$$

then  $\nabla \eta = \nabla \psi$  in  $A(k) = \{x \in \Omega : |\psi(x)| > k\}$  and  $\eta = 0$  in  $\Omega \setminus A(k)$ . Then

$$\int_{A(k)} |\nabla \psi|^p dx = \int_{A(k)} \langle f, \nabla \eta \rangle dx \leq \left( \int_{A(k)} |\nabla \psi|^p dx \right)^{1/p} \|f\|_r |A(k)|^{1-1/p-1/r}.$$

That is

$$\left( \int_{A(k)} |\nabla \psi|^p dx \right)^{(p-1)/p} \leq \|f\|_r |A(k)|^{1-1/p-1/r}.$$

For  $p < N$ , by the Sobolev's inequality ( $S^{1/p}$  being the best constant for this inequality,  $p^* = Np/(N-p)$ )

$$S^{1/p} \left( \int_{A(k)} |\psi|^{p^*} dx \right)^{p/p^*} \leq \left( \int_{A(k)} |\nabla \psi|^p dx \right)^{p/p}.$$

The case  $p = N$  is reduced to the case  $p < N$  because of the embedding  $W_0^{1,N}(\Omega) \subset \mathcal{W}_0^{1,p}(\Omega)$  for  $1 < p < N$  ( $\Omega$  is bounded). Therefore

$$S^{(p-1)/p} \left( \int_{A(k)} |\psi|^{p^*} dx \right)^{(p-1)/p^*} \leq \|f\|_r |A(k)|^{1-1/p-1/r}.$$

If  $0 < k < h$ ,  $A(h) \subset A(k)$ . Then

$$|A(k)|^{1/p^*} (h - k) = \left( \int_{A(k)} (h - k)^{p^*} dx \right)^{1/p^*} \leq \left( \int_{A(h)} |\psi|^{p^*} dx \right)^{1/p^*} \leq \left( \int_{A(k)} |\psi|^{p^*} dx \right)^{1/p^*}.$$

Finally

$$|A(k)|^{1/p^*} \leq \frac{1}{S^{1/p}} \frac{1}{h - k} \|f\|_r^{1/(p-1)} (|A(k)|^{1-1/p-1/r})^{1/(p-1)}.$$

In other words

$$|A(h)| \leq \frac{1}{S^{p^*/p}} \frac{1}{(h - k)^{p^*}} \|f\|_r^{p^*/(p-1)} |A(k)|^{p^*(1/p-1/(r(p-1)))}.$$

Since  $r > N/(p - 1)$ , the exponent for  $|A(k)|$  is greater than 1. So, we can apply Stampacchia's Lemma, [S], to conclude that there exists some  $h$  for which  $|A(h)| = 0$ , that is,  $\psi \in L^\infty(\Omega)$  and

$$\|\psi\|_\infty \leq C(p, r, N, \Omega)(\lambda e^\delta \|V\|_q)^{p'/p}. \quad \blacksquare$$

In addition, the inequalities in  $\mathbf{R}^N$

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y \rangle \geq \begin{cases} C_p |x - y|^p & \text{if } p \geq 2, \\ C_p \frac{|x - y|^2}{(|x| + |y|)^{2-p}} & \text{if } p \leq 2, \end{cases}$$

imply the following result

**LEMMA 1.2.** *Given  $f_1, f_2 \in W^{-1, p'}(\Omega)$ , consider  $u_1, u_2 \in W_0^{1, p}(\Omega)$  such that  $-\Delta_p u_i = f_i, i = 1, 2$ . Then:*

$$\int_{\Omega} (f_1 - f_2)(u_1 - u_2) dx = \int_{\Omega} \langle |\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2, \nabla u_1 - \nabla u_2 \rangle dx \geq \begin{cases} C_p \int_{\Omega} |\nabla(u_1 - u_2)|^p dx & \text{if } p \geq 2, \\ C_p \int_{\Omega} \frac{|\nabla(u_1 - u_2)|^2}{(|\nabla u_1| + |\nabla u_2|)^{2-p}} dx & \text{if } p \leq 2. \end{cases}$$

As a consequence

$$C_p \int_{\Omega} |\nabla(u_1 - u_2)|^p dx \leq \|u_1 - u_2\|_{W_0^{1,p}(\Omega)} \|f_1 - f_2\|_{W^{-1,p'}(\Omega)} \quad \text{if } p \geq 2,$$

$$\|u_1 - u_2\|_{W_0^{1,p}(\Omega)} \leq$$

$$\leq C_p (\|\nabla u_1\|_{W_0^{1,p}(\Omega)}^p + \|\nabla u_2\|_{W_0^{1,p}(\Omega)}^p)^{(2-p)/p} \|f_1 - f_2\|_{W^{-1,p'}(\Omega)} \quad \text{if } p < 2.$$

Now we can state the existence theorem

**THEOREM 1.3.** *If  $\lambda$  is small enough, then there exists one solution to the problem (P).*

**PROOF.** Lemma 1.1 implies that, for  $\lambda$  small enough,  $F_\lambda$  applies the ball of radius  $\delta$  in  $L^\infty(\Omega)$  to itself. On the other hand, if  $\psi_1 = F_\lambda \varphi_1$ ,  $\psi_2 = F_\lambda \varphi_2$ , where  $\varphi_1, \varphi_2 \in B_\delta$ , Lemma 1.2 implies

$$\gamma_0 \int_{\Omega} |\nabla \psi_1 - \nabla \psi_2|^p dx \leq$$

$$\leq \lambda e^\delta \|\varphi_1 - \varphi_2\|_\infty \|\psi_1 - \psi_2\|_\infty \|V\|_q |\Omega|^{1/q'} \quad \gamma_0(p) > 0, \quad p \geq 2,$$

$$\gamma_1 \int_{\Omega} |\nabla \psi_1 - \nabla \psi_2|^p dx \leq \lambda e^\delta \|\varphi_1 - \varphi_2\|_\infty^{p/2} \|\psi_1 - \psi_2\|_\infty^{p/2}.$$

$$\cdot (\|\psi_1\|_\infty + \|\psi_2\|_\infty)^{(2-p)/2} \|V\|_q |\Omega|^{1/q'} \quad \gamma_1(p) > 0, \quad p < 2,$$

by means of the mean value theorem. Therefore by either Sobolev inclusion in the case  $p > N$ , or by Stampacchia method in the general case,

$$\|F_\lambda \varphi_1 - F_\lambda \varphi_2\|_\infty = \|\psi_1 - \psi_2\|_\infty \leq C(p, N, \Omega, \|V\|_q) (\lambda e^\delta)^{\gamma(p)} \|\varphi_1 - \varphi_2\|_\infty$$

where  $\gamma(p) > 0$ . So we have proved that  $F_\lambda$  is contractive if  $\lambda$  is small enough; therefore, the classical *Banach-Picard fixed point theorem* allows us to conclude the proof. ■

**REMARK.** *In the case  $p = N$ , the potential can be considered with less regularity. Precisely  $V \in L^1(\log L)^\beta(\Omega)$ , the usual Zygmund space with  $\beta > N - 1$ , gives that each iteration in the proof of Theorem 1.3 verifies  $u \in L^\infty(\Omega)$ . (See [BPV]).*

**2. – A sufficient condition of existence of positive solution.**

When the sign of the potential  $V$  is constant, it is easy to know the sign of the corresponding solutions. In this section we will give a sufficient condition related to positive solutions with  $V$  changing sign. This result is new even for the semilinear case,  $p = 2$ , which is treated following.

Let us consider the problem,

$$(P') \quad \begin{cases} -\Delta w = V(x) & \text{in } \Omega \subset \mathbf{R}^N, \quad N \geq 1, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where  $x \in \Omega$ , a bounded domain, and  $V(x)$  is a given function in  $L^q(\Omega)$  changing sign in  $\Omega$  ( $q \geq 1$  if  $N > 2$ ,  $q > N/2$  otherwise). We assume that  $\Omega$  verifies the classical *interior ball condition*.

Let us assume that  $w > 0$  in  $\Omega$ , that is, if  $G(x, \xi)$  is the Green's function for  $\Omega$ ,

$$w(x) = \int_{\Omega} G(x, \xi) V(\xi) d\xi > 0.$$

Since  $V = V^+ - V^-$ , we get ( $V^+ = \max(V, 0)$ ,  $V^- = \max(-V, 0)$ )

$$w_1(x) = \int_{\Omega} G(x, \xi) V^+(\xi) d\xi > \int_{\Omega} G(x, \xi) V^-(\xi) d\xi = w_2(x)$$

where  $w_1, w_2$  are the solutions of the problems

$$\begin{cases} -\Delta w_1 = V^+(x) & \text{in } \Omega, \\ w_1|_{\partial\Omega} = 0, \end{cases} \quad \begin{cases} -\Delta w_2 = V^-(x) & \text{in } \Omega, \\ w_2|_{\partial\Omega} = 0. \end{cases}$$

Let  $y(x)$  be the function defined in  $\bar{\Omega}$  by

$$y(x) = \begin{cases} \frac{w_1(x)}{w_2(x)} & \text{if } x \in \Omega, \\ \frac{\partial_\nu w_1(x)}{\partial_\nu w_2(x)} & \text{if } x \in \partial\Omega, \end{cases}$$

where  $\nu$  is the exterior unit normal to  $\partial\Omega$ , This function is well defined, positive and continuous in  $\bar{\Omega}$  by the Hopf's Lemma. Let us suppose that

$\min_{x \in \Omega} y(x) = 1 + m$  for some  $m > 0$ . That implies  $w_1 > (1 + m)w_2$ . Let  $M$  be such that  $M = \|y\|_\infty$ .

Now, if we take a function  $\varphi$  such that  $0 \leq \varphi \leq \delta \log(y(x))$ , with  $\delta > 0$  to be determined, we can define for  $\lambda > 0$  the application  $T_\lambda$  as follows

$$\psi = T_\lambda \varphi = \lambda \int_\Omega G(x, \xi) V(\xi) e^{\varphi(\xi)} d\xi.$$

Thus

$$\begin{aligned} T_\lambda \varphi &\geq \lambda \int_\Omega G(x, \xi) V^+(\xi) d\xi - \\ &\quad - \lambda \int_\Omega G(x, \xi) V^-(\xi) (y(\xi))^\delta d\xi \geq \lambda(w_1(x) - M^\delta w_2(x)), \end{aligned}$$

then we can take  $\delta$  small enough to get

$$T_\lambda \varphi \geq \lambda(w_1(x) - M^\delta w_2(x)) > \lambda(w_1(x) - (1 + m)w_2(x)) > 0.$$

Then,  $T_\lambda \varphi$  is positive if  $\delta$  is small enough. By fixing a  $\delta$  in these hypothesis,  $T_\lambda$  sends the ball of radius  $M^\delta$  in  $L^\infty(\Omega)$  to itself and is contractive for  $\lambda$  small enough, by Theorem 1.3. In this way the existence of one positive solution to the problem

$$(P'') \quad \begin{cases} -\Delta u = \lambda V(x) e^u & \text{in } \Omega \subset \mathbf{R}^N, \\ u|_{\partial\Omega} = 0, \end{cases}$$

can be shown by means of a fixed point argument ( $u = T_\lambda u$ ), as a consequence of the positivity of the solution to (P').

Then we conclude with the following result.

**PROPOSITION 2.1.** *Let  $p = 2$ ,  $w_1, w_2$  as above. If  $w_1(x) = (1 + \rho(x))w_2(x)$  bounded function and  $\rho(x) \geq m > 0$  in  $\bar{\Omega}$ , then problem (P'') has at least one positive solution for  $\lambda$  small enough.*

Now we take  $p$  general. The corresponding result is the following

**THEOREM 2.2.** *Let  $w$  be the solution of*

$$\begin{cases} -\Delta_p w = V^+(x) - (1 + \mu(x))V^-(x) & \text{in } \Omega \subset \mathbf{R}^N, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where  $V \in L^q(\Omega)$  changes sign in  $\Omega$  ( $q \geq 1$  if  $p > N$ ,  $q > 1$  if  $p = N$ ,  $q > N/p$  otherwise) and  $\mu(x)$  verifies that there exists a positive constant  $m$  such that  $\mu(x) \geq m > 0$ . Suppose that  $w > 0$  in  $\Omega$ ; then, if  $\lambda$  is small enough, there exists one positive solution to (P).

**PROOF.** Let  $\delta = \log(1 + m)^{1/2}$  (hence,  $e^{2\delta} = 1 + m$  and  $e^{-\delta} = (1 + m)^{-1/2}$ ). Theorem 1.3 now implies that there exists one solution to (P) belonging to the ball  $B_\delta$ . Let  $\psi$  such a solution. then

$$\begin{aligned} -\Delta_p \psi &= \lambda(V^+(x)e^\psi - V^-(x)e^\psi) \geq \\ &\geq \lambda(V^+(x)e^{-\delta} - V^-(x)e^\delta) = \lambda e^{-\delta}(V^+(x) - V^-(x)e^{2\delta}). \end{aligned}$$

That is

$$-\Delta_p \psi \geq \lambda(1 + m)^{-1/2}(V^+(x) - (1 + m)V^-(x)) \geq \lambda(1 + m)^{-1/2}(-\Delta_p w).$$

The weak comparison principle allows us to conclude that

$$\psi \geq C(\lambda, m, N)w > 0 \quad \text{in } \Omega \quad \blacksquare$$

### 3. - $V \geq 0$ and $1 < p < N$ . Minimal solution.

We show in this section the existence of a solution to the problem (P) for the case  $1 < p < N$ ,  $V \geq 0$  by comparison arguments. The following results are extensions to the variable coefficients case of those in [GPP]. We give the proofs of the results that need some changes.

**DEFINITION.** We say that  $u \in W_0^{1,p}(\Omega) (W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$  for  $1 < p \leq N$ ) is a regular supersolution of problem (P) if

$$\begin{cases} -\Delta_p u \geq \lambda V(x)e^u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

We say that  $u_m$ , a solution of (P) is minimal if, for each supersolution  $u$  of (P), we have  $u_m \leq u$ .

With this definition at hand, we study the existence and behavior of minimal solution of (P).

LEMMA 3.1. *Let  $u_0$  be a regular supersolution of (P). Then, there exists  $0 \leq u_m \leq u_0$ ,  $u_m$  being a minimal regular solution of (P).*

COROLLARY 3.2. *If there exists regular solution of (P) for  $\lambda_0 > 0$ , then there exists regular solution for all  $\lambda \leq \lambda_0$ .*

THEOREM 3.3. *If  $V \in L^q(\Omega)$ ,  $q > N/p > 1$ , there exists a constant  $\lambda^*$  such that if  $\lambda < \lambda^*$ , problem (P) has one positive solution.*

THEOREM 3.4. *If  $u_0 \in W_0^{1,p}(\Omega)$  is a singular solution of*

$$(P_{\lambda^*}) \quad \begin{cases} -\Delta_p u_0 = \lambda^* V(x) e^{u_0} & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where  $V \in L^q(\Omega)$ ,  $q > N/p > 1$ , then, for all  $\lambda \in (0, \lambda^*)$  the problem

$$(P_\lambda) \quad \begin{cases} -\Delta_p u = \lambda V(x) e^u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

has one positive minimal regular solution  $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ .

PROOF. If  $u_0$  is a singular solution, then  $V(x) e^{u_0} \in L^1(\Omega)$  and we can consider  $V(x) e^{u_0} \in W^{-1,p'}(\Omega)$ . The function

$$u_1 = \left( \frac{\lambda}{\lambda^*} \right)^{1/(p-1)} u_0$$

is a solution of the problem

$$\begin{cases} -\Delta_p u_1 = \lambda V(x) e^{u_1} & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

and it verifies  $V(x)^{(\lambda/\lambda^*)^{1/(p-1)}} e^{u_1} \in L^{(\lambda^*/\lambda)^{1/(p-1)}}$ ,  $0 < u_1 < u_0$ , and  $V(x) e^{u_1} \in W^{-1,p'}(\Omega)$ ; moreover

$$\int_{\Omega} V(x) u_1^r dx < \int_{\Omega} V(x) u_0^r dx < \int_{\Omega} V(x) e^{u_0} dx < \infty \quad \forall r \in (1, \infty).$$

If we consider the problem

$$\begin{cases} -\Delta_p u_2 = \lambda V(x) e^{u_1} & \text{in } \Omega, \\ u_2|_{\partial\Omega} = 0; \end{cases}$$

then  $u_2 \in W_0^{1,p}(\Omega)$ ; by using the weak comparison principle, we have  $0 < u_2 \leq u_1 < u_0$  and

$$\int_{\Omega} V(x) u_2^r dx < \infty.$$

By the convexity of  $f(t) = e^{tx_0}$  for  $0 < t < 1$ , we get

$$e^{tx_0} + (1-t)x_0 e^{tx_0} \leq e^{x_0}.$$

Then, if  $t = (\lambda/\lambda^*)^{1/(p-1)}$  and  $x_0 = u_0$ ,

$$e^{u_1} + (1-t)u_0 e^{u_1} \leq e^{u_0}.$$

Since  $u_2 \leq u_1 < u_0$ ,

$$(*) \quad \lambda e^{u_1} \leq e^{u_0} - \lambda(1-t)u_2 e^{u_1}.$$

In addition,

$$-\Delta_p \left( \frac{p-1}{p} v^{p/(p-1)} \right) = -v \Delta_p v - |\nabla v|^p.$$

By replacing  $v$  for  $u_2$  in the last equality, we arrive to

$$-\Delta_p \left( \frac{p-1}{p} u_2^{p/(p-1)} \right) = -u_2 \Delta_p u_2 - |\nabla u_2|^p \leq \lambda u_2 V(x) e^{u_1}.$$

Now, by the homogeneity and (\*)

$$\begin{aligned} -\Delta_p \left( (1-t)^{1/(p-1)} \frac{p-1}{p} u_2^{p/(p-1)} \right) &\leq \lambda(1-t)u_2 V(x) e^{u_1} \leq \\ &\leq \lambda V(x) e^{u_1} + \lambda(1-t)u_2 V(x) e^{u_1} \leq \lambda V(x) e^{u_0} = -\Delta_p u_1. \end{aligned}$$

If we assume that  $u_2^{p/(p-1)} \in W_0^{1,p}(\Omega)$ , by applying the weak comparison principle,

$$(1-t)^{1/(p-1)} \frac{p-1}{p} u_2^{p/(p-1)} \leq u_1$$

and

$$V(x) e^{(1-t)^{1/(p-1)}((p-1/p)u_2^{p/(p-1)})} \in L^1(\Omega).$$

Therefore, if  $V(x) \in L^q(\Omega)$ , with  $q > N/p > 1$ , then  $V(x) e^{u_2} \in L^r(\Omega)$ , for  $r > N/p$  (by the Hölder inequality). So, the Stampacchia's lemma [S] implies that  $u_3 \in L^\infty(\Omega)$ ,  $u_3$  being the third iteration. In this way, we have obtained a regular supersolution of  $(P_\lambda)$ . Lemma 3.1 states that there exists one positive minimal regular solution of  $(P_\lambda)$ .

It remains to show that  $u_2^{p/(p-1)} \in W_0^{1,p}(\Omega)$ . We observe that

$$(**) \quad -\Delta_p \left( \frac{p-1}{p} u_2^{p/(p-1)} \right) = \lambda u_2 V(x) e^{u_1} - |\nabla u_2|^p \in L^1(\Omega)$$

since  $V(x) e^{u_1} \in W^{-1,p'}(\Omega)$  and

$$\lambda \int_{\Omega} u_2 V(x) e^{u_1} dx = \int_{\Omega} |\nabla u_2|^p dx = \|\nabla u_2\|_p^p < \infty.$$

It we define  $w_k$  as

$$w_k(x) \equiv \begin{cases} \frac{p-1}{p} u_2^{p/(p-1)} & \text{if } \frac{p-1}{p} u_2^{p/(p-1)} \leq k, \\ k & \text{if } \frac{p-1}{p} u_2^{p/(p-1)} \geq k, \end{cases}$$

By multiplying  $(**)$  by  $w_k$ , the Hölder inequality gives

$$\begin{aligned} \int_{\Omega} |\nabla w_k|^p dx &\leq \lambda \int_{\Omega} w_k u_2 V(x) e^{u_1} dx \leq \lambda \frac{p-1}{p} \int_{\Omega} u_2^{p/(p-1)+1} V(x) e^{u_1} dx = \\ &= \lambda \frac{p-1}{p} \int_{\Omega} [V(x)^{1-(\lambda/\lambda^*)^{1/(p-1)}} u_2^{p/(p-1)+1}] [V(x)^{(\lambda/\lambda^*)^{1/(p-1)}} e^{u_1}] dx \leq \\ &\leq \lambda \frac{p-1}{p} \left( \int_{\Omega} V(x) u_2^r dx \right)^{1-(\lambda/\lambda^*)^{1/(p-1)}} \left( \int_{\Omega} V(x) e^{u_0} dx \right)^{(\lambda/\lambda^*)^{1/(p-1)}} < \infty \end{aligned}$$

Then  $\{w_k\}$  is uniformly bounded in  $W_0^{1,p}(\Omega)$  and therefore the limit  $(p-1)/p u_2^{p/(p-1)} \in W_0^{1,p}(\Omega)$ . ■

We need the following result.

LEMMA 3.5. *Let  $\underline{u} = \underline{u}(\lambda)$  be a minimal regular solution of (P). If*

we define the set  $\mathcal{X}$  as

$$\mathcal{X} = \{v \in W_0^{1,p}(\Omega) \mid 0 \leq v \leq \underline{u}\}$$

the following functional

$$J(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \lambda \int_{\Omega} V(x) e^u dx$$

is well defined on  $\mathcal{X}$ . Then, the minimizer  $u \in \mathcal{X}$  for  $J$  is the minimal solution  $\underline{u}$ . Moreover,  $\underline{u}$  satisfies the estimate

$$\lambda \int_{\Omega} V(x) e^{\underline{u}} w^2 dx \leq (p-1) \int_{\Omega} |\nabla \underline{u}|^{p-2} |\nabla w|^2 dx \quad \forall w \in W_0^{1,p}(\Omega).$$

(See [GPP] for a proof).

**THEOREM 3.6.** *Let  $\{\lambda_n\}_{n \in \mathbb{N}}$  be an increasing sequence such that*

$$\lambda_n \rightarrow \lambda^* \equiv \sup \{\lambda \mid (P_{\lambda}) \text{ has solution}\}$$

*If  $V \in L^q(\Omega)$ ,  $q > N/p > 1$ , and  $\underline{u}_n = \underline{u}_n(\lambda)$  is the corresponding minimal solution of  $(P_{\lambda_n})$ , then  $\underline{u}_n \rightarrow u^*$  strongly in  $W_0^{1,p}(\Omega)$ ,  $V(x) e^{\underline{u}_n} \rightarrow V(x) e^{u^*}$  in  $L^{p^*/(p^*-1)}(\Omega)$ , and  $u^*$  is a singular solution of  $(P_{\lambda^*})$ .*

**PROOF.** If  $\underline{u}_n$  is the minimal solution of  $(P_{\lambda_n})$  we get, taking  $w = \underline{u}_n$  and using Lemma 3.5,

$$\lambda_n \int_{\Omega} V(x) e^{\underline{u}_n} \underline{u}_n^2 dx \leq (p-1) \int_{\Omega} |\nabla \underline{u}_n|^p dx = (p-1) \lambda_n \int_{\Omega} V(x) e^{\underline{u}_n} \underline{u}_n dx$$

Let us introduce the sets  $\varepsilon_n = \{x \in \Omega \mid \underline{u}_n > 2(p-1)\}$ . Then, in  $\Omega - \varepsilon_n$ ,  $0 < \underline{u}_n < 2(p-1)$  and

$$\begin{aligned} \lambda_n \int_{\Omega} V(x) e^{\underline{u}_n} \underline{u}_n^2 dx &\leq (p-1) \lambda_n \int_{\Omega \setminus \varepsilon_n} V(x) e^{\underline{u}_n} \underline{u}_n dx + \\ &+ (p-1) \lambda_n \int_{\varepsilon_n} V(x) e^{\underline{u}_n} \underline{u}_n dx \leq 2(p-1)^2 \lambda_n e^{2(p-1)} \int_{\Omega} V(x) dx + \\ &+ \frac{\lambda_n}{2} \int_{\varepsilon_n} V(x) e^{\underline{u}_n} \underline{u}_n^2 dx. \end{aligned}$$

Therefore

$$\int_{\Omega} V(x) e^{u_n} \underline{u}_n^2 dx \leq 4(p-1)^2 e^2 (p-1) \int_{\Omega} V(x) dx$$

and we get

$$\int_{\Omega} V(x) e^{u_n} \underline{u}_n dx \leq C, \quad \int_{\Omega} |\nabla \underline{u}_n|^p dx \leq C.$$

Then, if we take a subsequence  $\{\underline{u}_n\}$

- (1)  $\underline{u}_{n_k} \rightharpoonup u^*$  weakly in  $W_0^{1,p}(\Omega)$ .
- (2) By monotone convergence  $e^{\underline{u}_{n_k}} \rightarrow e^{u^*}$  in  $L^1(\Omega)$ .

Besides,  $\{\underline{u}_n\}$  is monotone (remember that  $\lambda_n$  is increasing). Hence the limit  $u^*$  is unique and the whole sequence converges. In order to prove that  $u^*$  is a singular solution of  $(P_{\lambda^*})$ , we consider the following inequalities:

$$\int_{\Omega} |\nabla \underline{u}_n|^{p-2} \langle \nabla \underline{u}_n, \nabla \varphi \rangle dx = \lambda \int_{\Omega} V(x) e^{u_n} \varphi dx \quad \forall \varphi \in W_0^{1,p}(\Omega),$$

$$\lambda \int_{\Omega} V(x) e^{u_n} \psi^2 dx \leq (p-1) \int_{\Omega} |\nabla \underline{u}_n|^{p-2} |\nabla \psi|^2 dx \quad \forall \psi \in W_0^{1,p}(\Omega).$$

If we take  $\varphi = (1/2\alpha)(e^{2\alpha u_n} - 1)$ ,  $\psi = e^{\alpha u_n} - 1$  in the above inequalities, we arrive to

$$\left( \frac{1}{(p-1)\alpha} - \frac{1}{2} \right) \int_{\Omega} V(x) e^{(2\alpha+1)u_n} dx \leq \frac{2}{(p-1)\alpha} \int_{\Omega} V(x) e^{(\alpha+1)u_n} dx.$$

Taking  $\alpha$  such that

$$\frac{1}{(p-1)\alpha} > \frac{1}{2} \quad \text{i.e.} \quad 2\alpha + 1 < \frac{3+p}{p-1}$$

we get, using the Young's inequality,

$$C(\alpha) \int_{\Omega} V(x) e^{(2\alpha+1)u_n} dx \leq \int_{\Omega} V(x) e^{u_n} dx \leq \int_{\Omega} V(x) e^{u^*} dx.$$

If we also assume that

$$2\alpha + 1 > \frac{p^*}{p^* - 1} = \frac{Np}{N(p-1) + p}$$

then we obtain by Hölder inequality

$$\begin{aligned} \int_{\Omega} (V(x) e^{\underline{u}_n})^{p^*/(p^*-1)} dx &= \\ &= \int_{\Omega} (V(x)^{1/2\alpha} e^{\underline{u}_n})^{p^*/(p^*-1)} (V(x)^{2\alpha/(2\alpha+1)})^{p^*/(p^*-1)} dx \leq \\ &\leq \left( \int_{\Omega} V(x) e^{(2\alpha+1)\underline{u}_n} dx \right)^{p^*/((p^*-1)(2\alpha+1))} \cdot \\ &\quad \cdot \left( \int_{\Omega} V(x)^{2\alpha p^*/((2\alpha+1)(p^*-1)-p^*)} dx \right)^{1/((p^*-1)(2\alpha+1)/p^*)}. \end{aligned}$$

As  $V \in L^q(\Omega)$ ,  $q > N/p > 1$ , this quantity is finite if

$$\frac{2\alpha p^*}{(2\alpha+1)(p^*-1)-p^*} \leq \frac{N}{p}$$

that is

$$a > \frac{1}{2} \frac{1}{N+p-1}.$$

Since the following is always true

$$\frac{Np}{N(p-1)+p} < \frac{Np}{N(p-1)} < \frac{3+p}{p-1}; \quad \frac{1}{2} \frac{1}{N+p-1} < \frac{2}{p-1}$$

all the requirements about the value of  $\alpha$  hold: there always exists some  $\alpha$  verifying them.

Then, we have proven that  $V(x) e^{\underline{u}_n} \in L^{p^*/(p^*-1)}(\Omega) \subset W_0^{1,p}(\Omega)$ , and  $V(x) e^{\underline{u}_n}$  converges in  $W^{-1,p'}(\Omega)$  by the monotone convergence theorem. The continuity of  $(-\Delta_p): W^{-1,p'}(\Omega) \rightarrow W_0^{1,p}(\Omega)$  implies that the sequence  $\{V(x) e^{\underline{u}_n}\}$  converges strongly in  $W_0^{1,p}(\Omega)$ . Therefore, if we take  $\varphi \in W_0^{1,p}(\Omega)$

$$\begin{aligned} \int_{\Omega} \langle |\nabla u^*|^{p-2} \nabla u^*, \nabla \varphi \rangle dx &= \lim_{n \rightarrow \infty} \int_{\Omega} \langle |\nabla \underline{u}_n|^{p-2} \nabla \underline{u}_n, \nabla \varphi \rangle dx = \\ &= \lim_{n \rightarrow \infty} \int_{\Omega} V(x) e^{\underline{u}_n} \varphi dx = \int_{\Omega} V(x) e^{u^*} \varphi dx. \quad \blacksquare \end{aligned}$$

The next result gives the conditions in which the limit minimal solution is regular or singular.

**THEOREM 3.7.** *If  $V$ ,  $\lambda^*$  and  $u^*$  are as in Theorem 3.6 and the dimension satisfies*

$$N < \frac{pq(3+p)}{4+q(p-1)} = \frac{p(3+p)}{4/q+p-1}$$

then  $u^* \in L^\infty(\Omega)$  and it is regular solution for  $(P_{\lambda^*})$ .

**PROOF.** We have to show that  $V(x)e^{u_n} \in L^r(\Omega)$  with  $r > N/p$ , since the Stampacchia's lemma [S] implies that  $\|u_n\|_\infty \leq C$  uniformly in  $\lambda$ , and so the limit  $u^*$  is regular. If we apply Hölder inequality

$$\begin{aligned} \int_{\Omega} (V(x)e^{u_n})^r dx &= \int_{\Omega} (V(x)^{1/(2\alpha+1)} e^{u_n})^r (V(x)^{2\alpha/(2\alpha+1)})^r dx \leq \\ &\leq \left( \int_{\Omega} V(x) e^{(2\alpha+1)u_n} dx \right)^{r/(2\alpha+1)} \left( \int_{\Omega} V(x)^{2\alpha r/(2\alpha+1-r)} dx \right)^{(2\alpha+1-r)/(2\alpha+1)}. \end{aligned}$$

We are assuming that  $V \in L^q(\Omega)$ ,  $q > N/p$ . then, the above quantity is finite if

$$\frac{2\alpha r}{2\alpha+1-r} < q$$

i.e.  $(r > N/p)$

$$\frac{2\alpha N/p}{2\alpha+1-N/p} = \frac{2\alpha N}{(2\alpha+1)p-N} < q$$

or

$$\alpha > \frac{(N-p)}{2(pq-N)}.$$

But we are also assuming that

$$\alpha < \frac{2}{p-1}.$$

Then

$$\frac{(N-p)q}{2(pq-N)} < \frac{2}{p-1} \quad \text{i.e.} \quad N < \frac{pq(3+p)}{4+q(p-1)}. \quad \blacksquare$$

REMARK. If we take  $V \in L^\infty$ , the last relationship transforms in

$$N < p + \frac{4p}{p-1}$$

This is the relationship appearing in [GPP], where  $V \equiv 1$ .

The previous results show that, under the regularity hypothesis above cited about  $V \geq 0$ , there exists at least one positive regular solution of (P), for  $1 < p < N$ . However, for the subcritical case  $p \geq N$ , we can do a variational argument.

**4. -  $V$  changing sign and  $p \geq N$ .**

We will assume in this section the following hypotheses:

(1)  $p \geq N$ .

(2)  $V(x) \in L^q(\Omega)$ ,  $q > 1$  for  $p = N$ ,  $q \geq 1$  for  $p > N$ .

(3) There exists an open ball  $B \subset \Omega$  such that  $V(x) > 0$  for  $x \in B$ .

In these hypotheses the comparison argument don't work in general. But the condition  $p \geq N$  allow us to state a result by critical points methods. More precisely we have the theorem:

**THEOREM 4.1.** *There exists a constant  $\lambda_0 > 0$  such that if  $\lambda < \lambda_0$ , problem (P) has two regular solutions at least.*

Hypothesis (3) imply that  $V^+ \not\equiv 0$ : it plays a fundamental role in the existence of two solutions: notice that for  $V < 0$  (i.e.  $V^+ \equiv 0$ ) there is only one negative solution. The proof of 4.1 follows the argument used in [GP] for the case of constant potential.

The energy functional corresponding to our problem is

$$J(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \lambda \int_{\Omega} V(x) e^u dx.$$

It satisfies the following inequality

$$J(u) \geq \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \lambda \|V\|_q C \exp \left\{ D \left( \int_{\Omega} |\nabla u|^p dx \right)^{N/p} \right\}$$

where  $C = (k_1 |\Omega|)^{1/q'}$  and  $D = k_2 (q')^{N-1} |\Omega|^{(p-1)/N}$  ( $k_1, k_2$  are the constants that appear in Trudinger's inequality [GT, p. 162]).

LEMMA 4.2. *The functional  $J$  verifies the Palais-Smale condition.*

PROOF. Let  $\{u_j\} \subset W_0^{1,p}(\Omega)$  be a Palais-Smale sequence for  $J$ ; i.e.

$$J(u_j) \rightarrow C,$$

$$J'(u_j) \rightarrow 0 \quad \text{in} \quad W^{-1,p'}(\Omega).$$

It is necessary to show that any Palais-Smale sequence contains a subsequence which converges strongly in  $W_0^{1,p}(\Omega)$ . If  $\varepsilon_j = J'(u_j)$  then  $\varepsilon_j \rightarrow 0$  in  $W^{-1,p'}(\Omega)$ ; therefore, we can assume that  $\|\varepsilon_j\|_{W^{-1,p'}(\Omega)} \leq 1$ , and

$$\begin{aligned} C &= \lim_{j \rightarrow \infty} \left\{ J(u_j) - \frac{1}{2p} \langle \varepsilon_j, u_j \rangle + \frac{1}{2p} \langle \varepsilon_j, u_j \rangle \right\} \geq \\ &\geq \lim_{j \rightarrow \infty} \left\{ \frac{1}{2p} \int_{\Omega} |\nabla u_j|^p dx + \lambda \int_{\Omega} V(x) e^{u_j} \left( \frac{u_j}{2p} - 1 \right) dx - \frac{1}{2p} \left( \int_{\Omega} |\nabla u_j|^p dx \right)^{1/p} \right\} \geq \\ &\geq \lim_{j \rightarrow \infty} \left\{ \frac{1}{2p} \int_{\Omega} |\nabla u_j|^p dx + \lambda \|V\|_q C_0 |\Omega|^{1/q'} - \frac{1}{2p} \left( \int_{\Omega} |\nabla u_j|^p dx \right)^{1/p} \right\} \end{aligned}$$

where

$$-C_0 = \min_{x \in \mathbf{R}} \left\{ e^x \left( \frac{x}{2p} - 1 \right) \right\} < 0$$

since

$$g(x) = e^x \left( \frac{x}{2p} - 1 \right)$$

is a continuous function defined in the whole  $\mathbf{R}$  that verifies  $g(x) \rightarrow 0^-$  as  $x \rightarrow -\infty$  and  $g(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . Hence  $g$  attains a minimum value  $-C_0 < 0$ .

Thus, the sequence  $\{u_j\}$  is bounded in  $W_0^{1,p}(\Omega)$ ; the rest of the proof of Lemma 4.2 is identical to the one appearing in [GP] for  $V \equiv 1$ . ■

As in [GP], it is easy to show the following properties of the functional  $J$  for  $0 < \lambda < \lambda_0$ :

(1) If  $J(0) = -\lambda \int_{\Omega} V(x) dx \leq 0$ , i.e. if  $\int_{\Omega} V(x) dx \geq 0$ , the functional  $J$  verifies that, for  $\lambda$  small enough there exist  $R_1 > 0$ ,  $\varrho \in \mathbf{R}$  such that if  $\|\nabla u\|_p = R_1$ , then  $J(u) > \varrho > J(0)$ : by taking  $R_1 = 1$ ,  $\varrho = 1/2p$ ,  $\lambda < (2p\|V\|_q Ce^D)^{-1}$ , we get

$$J(u) \geq \frac{1}{p} - \lambda \|V\|_q Ce^D > \frac{1}{2p} = \varrho > J(0).$$

(2) If  $\int_{\Omega} V(x) dx = -\alpha > 0$ , then  $J(0) = \lambda\alpha$ . If we take  $\|\nabla u\|_p = R_1$ , we have

$$J(u) \geq \frac{R_1^p}{p} - \lambda \|V\|_q Ce^{DR_1^N} > \frac{\alpha}{p} > \lambda\alpha = J(0)$$

whenever

$$\lambda < \min \left( \frac{R_1^p - \alpha}{p \|V\|_q Ce^{DR_1^N}}, \frac{1}{p} \right).$$

(3) There exists  $w_0 \in W_0^{1,p}(\Omega)$ , with  $\|\nabla w_0\|_p = R_2 > R_1$  and  $J(w_0) < J(0)$ ; for any  $w \in C_0^\infty(B) \subset W_0^{1,p}(\Omega)$ ,  $w \geq 0$ , where  $B \subset \Omega$  is the ball where  $V$  is positive, we have

$$\begin{aligned} J(tw) &= \frac{t^p}{p} \int_{\Omega} |\nabla w|^p dx - \lambda \int_{\Omega} V(x) e^{tw} dx = \\ &= \frac{t^p}{p} \int_{\Omega} |\nabla w|^p dx - \lambda \int_{\Omega} V^+(x) e^{tw} dx + \lambda \int_{\Omega} V^-(x) dx \leq \\ &\leq \frac{t^p}{p} \int_{\Omega} |\nabla w|^p dx - \lambda t^{2p} \int_{\Omega} V^+(x) w^{2p} dx + \lambda \int_{\Omega} V^-(x) dx \rightarrow -\infty \text{ as } t \rightarrow \infty \end{aligned}$$

since  $e^{tw} > (tw)^{2p}$  and  $\int_{\Omega} V^-(x) dx$  and  $\int_{\Omega} V^+(x) w^{2p} dx$  are bounded.

The graph of  $J$  is then contained in the region above the graph of

$$f(\|\nabla u\|) = \frac{1}{p} \|\nabla u\|_p^p - \lambda \|V\|_q Ce^{D\|\nabla u\|_p^N}$$

$f$  has two critical points: a local minimum near 0 and a local maximum i.e., the geometrical conditions required for the existence of critical

points of  $J$  are fulfilled. Therefore, we can use the Mountain Pass Lemma (see [AR]) to get

LEMMA 4.3. *There exists a constant  $\lambda_0 > 0$  such that if  $0 < \lambda < \lambda_0$ , problem (P) admits a solution corresponding to a critical point of the functional  $J$  with critical value*

$$c = \inf_{\varphi \in \mathcal{C}} \max_{t \in [0, 1]} J(\varphi(t))$$

where  $\mathcal{C} = \{\varphi \in (C([0, 1]), W_0^{1,p}(\Omega)): \varphi(0) = 0, \varphi(1) = w_0\}$  for some  $w_0 \in W_0^{1,p}(\Omega)$  such that  $J(w_0) \leq J(0)$ . Moreover,  $c > J(0)$ .

To obtain the other critical point, we make the truncation in the functional appearing in [GP]. Let us consider a cut-off function  $\tau \in C^\infty(\Omega)$  such that for the previously defined  $R_1$  and  $R_2$

$$\tau(x) \equiv \begin{cases} 1 & \text{if } x \leq R_1, \\ 0 & \text{if } x \geq R_2, \end{cases}$$

and  $\tau$  nonincreasing. Thus, we obtain the truncated functional

$$F(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \lambda \int_{\Omega} V(x) \tau(\|\nabla u\|_p) e^u dx$$

It has to be noted that

- (1)  $J$  and  $F$  are the same if  $\|\nabla u\|_p \leq R_1$ .
- (2) If  $\|\nabla u\|_p \geq R_2$ , then  $F(u) = (1/p) \int_{\Omega} |\nabla u|^p dx$ .
- (3)  $F(u) < J(0) = -\lambda \int_{\Omega} V(x) dx$ , then  $\|\nabla u\|_p \geq R_1$ , since  $F$  is increasing for  $u$  with  $\|\nabla u\|_p = R_1$ . Hence,  $J$  and  $F$  are the same in a neighborhood of  $u$  and the Palais-Smale condition for  $J$  implies the subsequent Palais-Smale condition for  $F$ .

LEMMA 4.4. *Let  $\{u_j\} \in W_0^{1,p}(\Omega)$  be such that*

$$\begin{cases} F(u_j) \rightarrow C < J(0), \\ F'(u_j) \rightarrow 0 & \text{in } W^{-1,p'}(\Omega). \end{cases}$$

*Then there exists a subsequence which converges strongly to  $u \in W_0^{1,p}(\Omega)$ ,*

LEMMA 4.5.

$$\inf_{u \in W_0^{1,p}(\Omega)} F(u) < J(0) = -\lambda \int_{\Omega} V(x) dx.$$

PROOF. Let  $w$  be a function in  $C_0^\infty(B) \in W_0^{1,p}(\Omega)$ ,  $w \geq 0$ , where  $B$  is again the open ball where  $V > 0$ . If  $\|\nabla w\|_p = 1$ , and  $\varrho < R_1$ , then

$$\begin{aligned} F(\varrho w) &= \frac{\varrho^p}{p} - \lambda \int_{\Omega} V^+(x) e^{\varrho w} dx + \lambda \int_{\Omega} V^-(x) dx \leq \\ &\leq \frac{\varrho^p}{p} - \lambda \int_{\Omega} V^+(x)(1 + \varrho w) dx + \lambda \int_{\Omega} V^-(x) dx = \\ &= \varrho \left\{ \frac{\varrho^{p-1}}{p} - \lambda \int_{\Omega} V^+(x) w dx \right\} - \lambda \int_{\Omega} V^+(x) dx + \lambda \int_{\Omega} V^-(x) dx < J(0) \end{aligned}$$

whenever  $\varrho$  is small enough  $\left( \int_{\Omega} V^+(x) w dx \text{ is bounded} \right)$ . ■

By the same argument as in [GP] we have the following result.

LEMMA 4.6. *There exists a positive constant  $\lambda_0 > 0$  such that if  $0 < \lambda < \lambda_0$ , then problem (P) has a solution corresponding to a critical point of the functional  $J$ , with critical value  $c' < J(0)$ .*

With the lemmas we conclude the proof of Theorem 4.1 in an immediate way.

### 5. - A remark on eigenvalues and nonexistence results.

Let  $\Omega$  be a bounded domain,  $\partial\Omega \in C^{2,\beta}$  and now we assume  $V(x) \geq 0$ ,  $V(x) \in L^q(\Omega)$  ( $q \geq 1$  if  $p > N$ ,  $q > 1$  if  $p = N$ ,  $q > N/p > 1$  otherwise), with  $|\{x \in \Omega: V(x) > 0\}| \neq 0$ .  $|\cdot|$  means the Lebesgue measure.

Let us consider the problem ( $1 < p < \infty$ ):

$$(P_\lambda) \quad \begin{cases} u \in W_0^{1,p}(\Omega), u \neq 0, \\ -\Delta_p u \equiv -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = \lambda V(x) |u|^{p-2} u, u|_{\partial\Omega} = 0. \end{cases}$$

DEFINITION 5.1. *We say that  $\lambda$  is an eigenvalue if  $(P_\lambda)$  admits*

a solution. such solution is said an eigenfunction corresponding to the eigenvalue  $\lambda$ .

We now define the first eigenvalue,  $\lambda_1$ , as

$$\lambda_1 = \inf \left\{ \int_{\Omega} |\nabla w|^p dx : w \in W_0^{1,p}(\Omega), \int_{\Omega} V(x) |u|^p dx = 1 \right\}.$$

This problem when  $V \in L^\infty$ , was studied by [An], [Bl] and [L]. We need the case  $V \in L^q$ , with  $q$  in the hypothesis above.

The proof follows closely the one in [An], and then we concentrate our attention only in the points that need some change.

The two next results are well known.

**LEMMA 5.2.** *If  $u$  is a solution of  $(P_\lambda)$ ,  $u \in C^{1,\alpha}(\Omega)$ . Moreover, if  $u \geq 0$  then  $u > 0$  in  $\Omega$  and  $\partial u / \partial \nu < 0$  on  $\partial\Omega$ , where  $\nu$  denotes the unit exterior normal vector to  $\partial\Omega$ . (Hopf's lemma).*

**LEMMA 5.3.**  *$\lambda_1$  is an eigenvalue and every eigenfunction  $u_1$  corresponding to  $\lambda_1$  does not change sign in  $\Omega$ : either  $u_1 > 0$  or  $u_1 < 0$ .*

We consider  $I(u, v)$  defined as

$$I(u, v) = \left\langle -\Delta_p u, \frac{u^p - v^p}{u^{p-1}} \right\rangle + \left\langle -\Delta_p v, \frac{v^p - u^p}{v^{p-1}} \right\rangle$$

with

$$(u, v) \in D_I = \left\{ (u, v) \in (W_0^{1,p}(\Omega))^2 : u, v \geq 0 \text{ and } \frac{u}{v} \in L^\infty(\Omega) \right\}$$

we get the following results (see [An, Prop. 1 and Th. 1]):

**PROPOSITION 5.4.**  *$\forall (u, v) \in D_I$ ,  $I(u, v) \geq 0$ . Moreover,  $I(u, v) = 0$  if and only if there exists  $\alpha \in (0, \infty)$  such that  $u = \alpha v$ .*

The proof of this proposition consists of the calculation of

$$\left\langle -\Delta_p u, \frac{u^p - v^p}{u^{p-1}} \right\rangle \quad \text{and} \quad \left\langle -\Delta_p v, \frac{v^p - u^p}{v^{p-1}} \right\rangle$$

to show that  $I(u, v)$  is the integral of a sum of two non-negative functions, hence  $I(u, v) \geq 0$ . Moreover,  $I(u, v)$  vanishes if there exists some  $\alpha \in (0, \infty)$  such that  $u = \alpha v$ . As a consequence we get

PROPOSITION 5.5.  $\lambda_1$  is simple, i.e. if  $u, v$  are two eigenfunctions corresponding to the eigenvalue  $\lambda_1$ , then  $u = \alpha v$  for some  $\alpha$ .

With respect to the isolation, we extend the results by Anane [An]:

PROPOSITION 5.6. If  $w$  is an eigenfunction corresponding to the eigenvalue  $\lambda, \lambda > 0, \lambda \neq \lambda_1$ , then  $w$  changes sign in  $\Omega$ :  $w^+ \neq 0, w^- \neq 0$  and

$$|\Omega^-| \geq (\lambda \|V\|_q C^p)^\sigma$$

where  $\Omega^- = \{x \in \Omega: w(x) < 0\}, \sigma = -2q'$  if  $p \geq N, \sigma = -qN/(qp - N)$  if  $1 < p < N$  and  $\lambda_1$  is the first eigenvalue for the  $p$ -laplacian with weight  $V$  in  $\Omega$ .

PROOF. Let  $u, w$  be two eigenfunctions corresponding to  $\lambda_1$  and  $\lambda$  respectively, with  $\|\nabla u\|_p = \|\nabla w\|_p = 1$ . If  $w$  does not change sign, by applying Lemmas 5.2, 5.3 and Proposition 5.4 we get

$$(u, w) \in D_I, \quad I(u, w) \geq 0.$$

But

$$0 \leq I(u, w) = \int_{\Omega} (\lambda_1 - \lambda) V(x)(u^p - w^p) dx = (\lambda_1 - \lambda) \left( \frac{1}{\lambda_1} - \frac{1}{\lambda} \right) < 0$$

and we arrive to a contradiction.

If  $w^-$  replaces to  $w$  in  $(P_\lambda)$ , we get

$$\|\nabla w^-\|_p^p = \lambda \int_{\Omega} V(w^-)^p dx \leq \lambda \|V\|_q \|(w^-)^q\|_\alpha |\Omega^-|^{1/\beta}$$

with  $1/q + 1/\alpha + 1/\beta = 1$ . Now we are considering two cases

(1)  $p \geq N$ . By Sobolev's inequality

$$\|(w^-)^p\|_\alpha = \|w^-\|_{\alpha p}^p \leq C \|\nabla w^-\|_p^p \quad (\alpha > 1).$$

Thus, if we take  $\alpha = \beta = 2q'$  then

$$|\Omega^-| \geq (\lambda \|V\|_q C^p)^{-2q'}$$

(2)  $1 < p < N$ . We take

$$\alpha = N/(N - p), \quad \beta = qN/(qp - N) (\|(w^-)^p\|_\alpha = \|w^-\|_{p^*}^p,$$

where  $p^* = Np/(N - p)$ ). by Sobolev's inequality

$$\begin{aligned} \|\nabla w^-\|_p^p &\leq \lambda \|V\|_q \|w^-\|_{p^*}^p |\Omega^-|^{(qp - N)/qN} \leq \\ &\leq \lambda \|V\|_q C^p \|\nabla w^-\|_p^p |\Omega^-|^{(qp - N)/qN}. \end{aligned}$$

Hence

$$|\Omega^-| \geq (\lambda \|V\|_q C^p)^{-qN/(qp - N)}. \quad \blacksquare$$

REMARK 1. *If  $q' \rightarrow 1$  ( $q \rightarrow \infty$ ), we obtain the estimation by Anane (see [An, Prop. 2]).*

REMARK 2. *As Lindqvist pointed out, we can also extend these results to any bounded domain (see [L]).*

In the hypotheses of Proposition 5.6 we obtain the next theorem which proof follow [An]. However, we include it for the sake of completeness.

THEOREM 5.7.  *$\lambda_1$  is isolated; that is,  $\lambda_1$  is the unique eigenvalue in  $[0, a]$  for some  $a > \lambda_1$ .*

PROOF. Let  $\lambda \geq 0$  be an eigenvalue and  $v$  be the corresponding eigenfunction. By the definition of  $\lambda_1$  (it is the infimum) we have  $\lambda \geq \lambda_1$ . Then,  $\lambda_1$  is left-isolated.

We are now arguing by contradiction. We assume there exists a sequence of eigenvalues  $(\lambda_k)$ ,  $\lambda_k \neq \lambda_1$  which converges to  $\lambda_1$ . Let  $(u_k)$  be the corresponding eigenfunctions with  $\|\nabla u_k\|_p = 1$ . we can therefore take a subsequence, denoted again by  $(u_k)$ , converging weakly in  $W^{1,p}$ , strongly in  $L^p(\Omega)$  and almost everywhere in  $\Omega$  to a function  $u \in W_0^{1,p}$ . Since  $u_k = -\Delta_p^{-1}(\lambda_k V|u_k|^{p-2}u_k)$ , the subsequence  $(u_k)$  converges strongly in  $W_0^{1,p}$ , and subsequently  $u$  is the eigenfunction corresponding to the first eigenvalue  $\lambda_1$  with norm equals to 1. Hence, by applying the Egorov's Theorem ([B, Th. IV.28]),  $(u_k)$  converges uniformly to  $u$  in the exterior of a set of arbitrarily small measure. Then, there exists a piece of  $\Omega$  of arbitrarily small measure in which exterior  $u_k$  is positive for  $k$  large enough, obtaining a contradiction with the conclusion of Proposition 4.6.  $\blacksquare$

Now we are ready to show a nonexistence result for the problem (P): if  $V(x) \in L^q(\Omega)$ , ( $q \geq 1$  for  $p > N$ ,  $q > 1$  for  $p = N$ ,  $q > N/p > 1$  otherwise)  $V \geq 0$ , and  $\lambda$  is large enough then problem (P) does not have a solution.

In the end of the proof we use the isolation of the first eigenvalue for  $-\Delta_p$  with weight  $V(x)$ .

**THEOREM 5.8.** *Problem (P) does not have a solution if*

$$\lambda > \max \left\{ \lambda_1, \lambda_1 \left( \frac{p-1}{e} \right)^{p-1} \right\}$$

where  $V \in L^q(\Omega)$ , ( $q \geq 1$  for  $p > N$ ,  $q > 1$  for  $p = N$ ,  $q > N/p$  otherwise),  $V \geq 0$  and  $\lambda_1$  is the first eigenvalue for the  $p$ -Laplacian with weight  $V(x)$ .

**PROOF.** If  $\lambda_1$  is the first eigenvalue for the  $p$ -Laplacian with weight  $V(x)$ , take  $\lambda_\varepsilon = \lambda_1 + \varepsilon$ ,  $\varepsilon > 0$ ,  $v_1$  a positive eigenfunction corresponding to  $\lambda_1$  with  $\|v_1\|_\infty \leq 1$ , and suppose that problem (P) has a solution  $u \in W_0^{1,p}(\Omega)$  and

$$\lambda > \max \left( \lambda_1, \lambda_1 \left( \frac{p-1}{e} \right)^{p-1} \right)$$

for small  $\varepsilon$ , we have ( $\lambda_\varepsilon < \lambda$ ):

$$\lambda_\varepsilon x^{p-1} < \lambda e^x, \quad \forall x \geq 0,$$

$$-\Delta_p v_1 = \lambda_1 V v_1^{p-1} < \lambda_\varepsilon V < \lambda V \leq \lambda V e^u = -\Delta_p u.$$

By using the weak comparison principle for the  $p$ -Laplacian we obtain

$$v_1 \leq u.$$

Let  $v_2$  be the solution of

$$\begin{cases} -\Delta_p v_2 = \lambda_\varepsilon V(x) v_1^{p-1} & \text{in } \Omega, \\ v_2|_{\partial\Omega} = 0. \end{cases}$$

We know by regularity results that  $v_2 \in C^{1,\alpha}(\Omega)$ . In addition

$$-\Delta_p v_2 = \lambda_\varepsilon V v_1^{p-1} < \lambda V e^{v_1} \leq \lambda V e^u = -\Delta_p u,$$

$$-\Delta_p v_1 \leq \lambda_\varepsilon V v_1^{p-1} = -\Delta_p v_2.$$

By applying again the weak comparison principle, we get  $v_1 \leq v_2 \leq u$ .

Now, let us consider the problems

$$\begin{cases} -\Delta_p v_k = \lambda_\varepsilon V(x) v_{k-1}^{p-1} & \text{in } \Omega, \\ v_k |_{\partial\Omega} = 0. \end{cases}$$

The solutions of these problems form an increasing sequence  $\{v_k\}$  such that

$$v_1 \leq v_k \leq v_{k+1} \leq u$$

Passing to the limit, we obtain a solution  $w \in W_0^{1,p}(\Omega)$  to the problem

$$\begin{cases} -\Delta_p v = \lambda_\varepsilon V(x) w^{p-1} & \text{in } \Omega, \\ v_k |_{\partial\Omega} = 0. \end{cases}$$

But this is impossible for  $\varepsilon$  small enough, because the first eigenvalue for the  $p$ -Laplacian with weight in  $L^q(\Omega)$  is isolated by Theorem 5.7. ■

This argument also shows the nonexistence of positive solutions to (P) for  $\lambda$  large enough when  $V$  changes sign in  $\Omega$ :

**COROLLARY 5.9.** *Suppose that  $V \in L^q(\Omega)$  ( $q \geq 1$  for  $p > N$ ,  $q > 1$  for  $p = N$ ,  $q > N/p$  otherwise),  $V$  changes sign in  $\Omega$  and there exists a ball  $B \subset \Omega$  such that  $V(x) > 0$  for  $x \in B$ . If*

$$\lambda > \max \left\{ \lambda_1, \lambda_1 \left( \frac{p-1}{e} \right)^{p-1} \right\}$$

*then (P) has no positive solutions,  $\lambda_1$  being the first eigenvalue for the  $p$ -Laplacian with weight  $V(x)$  in  $B$ .*

**PROOF.** Let  $w_1$  a positive eigenfunction corresponding to  $\lambda_1$  in  $B$  with  $\|w_1\|_\infty \leq 1$ , that is,  $w_1$  verifies

$$\begin{cases} -\Delta_p w_1 = \lambda_1 V(x) w_1^{p-1} & \text{in } B, \\ w_1 |_{\partial B} = 0. \end{cases}$$

Take  $\lambda_\varepsilon = \lambda_1 + \varepsilon$ ,  $\varepsilon > 0$ , and suppose that problem (P) has a positive solution  $u$  for

$$\lambda > \max \left( \lambda_1, \lambda_1 \left( \frac{p-1}{e} \right)^{p-1} \right).$$

Then, for small  $\varepsilon$ , we have ( $\lambda_\varepsilon < \lambda$ ):

$$\begin{aligned} \lambda_\varepsilon x^{p-1} &< \lambda e^x, \quad \forall x \geq 0, \\ -\Delta_p w_1 &= \lambda_1 V w_1^{p-1} < \lambda_\varepsilon V < \lambda V \leq \lambda V e^u = -\Delta_p u \quad \text{in } B, \\ w_1 &= 0 \leq u \quad \text{in } \partial B. \end{aligned}$$

By using the weak comparison principle for the  $p$ -Laplacian we obtain

$$w_1 \leq u.$$

Using the argument in the proof of Theorem 5.8, we obtain a solution to the following problem

$$\begin{cases} -\Delta_p w = \lambda_\varepsilon V(x) w^{p-1} & \text{in } B, \\ w|_{\partial B} = 0. \end{cases}$$

But this is impossible for  $\varepsilon$  small enough, because the first eigenvalue for the  $p$ -Laplacian with weight in  $L^q(B)$  is isolated by Theorem 5.7. ■

### 6. – Analysis of the radial solutions in a ball.

In this section, we consider the problem

$$(P_r) \quad \begin{cases} -\Delta_p u = \lambda \frac{e^u}{r^\alpha} & \text{in } B_1(0) \subset \mathbf{R}^N, \quad \alpha \in \mathbf{R}, \quad r = |x|, \\ u|_{\partial B_1(0)} = 0. \end{cases}$$

where  $B_1(0)$  denotes the unit ball in  $\mathbf{R}^N$  and  $\lambda > 0$ .

In order to study the existence of singular solutions, we just consider the case  $1 < p < N$  since for  $p \geq N$  every solution is a regular solution. In this hypothesis, it is easy to prove the nonexistence of singular solutions for some  $\alpha$ :

If we consider the problem

$$\begin{cases} -\Delta_p v = \frac{\lambda}{r^\alpha} & \text{in } B_1(0) \subset \mathbf{R}^N, \\ v|_{\partial B_1(0)} = 0 \end{cases}$$

with  $p < \alpha < N$ ,  $\lambda > 0$  and we try the solutions  $v(r) = \beta r^\gamma$ , we get that

$$\gamma = \frac{p - \alpha}{p - 1}, \quad \lambda = (-\beta\gamma)^{p-1}[(p - 1)(\gamma - 1) + N - 1],$$

and therefore the solution  $v$  is not bounded for  $p < \alpha < N$ . For  $\alpha = p$  we can take

$$v(r) = \beta \log r,$$

obtaining  $\beta = -(\lambda/(N - p))^{1/(p-1)}$  and so  $v$  is also not bounded.

If we assume now that  $u$  is a positive singular solution of  $(P_r)$ , i.e.  $u \in W_0^{1,p}(B_1(0))$ ,  $e^u r^{-\alpha} \in L^1(B_1(0))$ , being  $p < \alpha < N$ , then

$$-\Delta_p u = \lambda \frac{e^u}{r^\alpha} \geq \frac{\lambda}{r^\alpha} = -\Delta_p v$$

that is,  $u > v$ . But this leads to a contradiction, since

$$\frac{e^u}{r^\alpha} \geq \frac{e^v}{r^\alpha} \notin L^1(B_1(0))$$

On the other hand, if  $\alpha \geq N$  then the potential  $V(r) = r^{-\alpha}$  does not belong to  $L^1(B_1(0))$ .

Then we directly assume that  $\alpha < p$ , independently on the dimension. Following the procedure carried out in [GPP], we introduce the new variables

$$s = \log r,$$

$$r(s) = |u_s|^{p-2} u_s,$$

$$u(s) = -\lambda e^{u + (p-\alpha)s}.$$

In the plane  $(v, w)$  the radial solutions of  $(P_r)$  satisfy the following autonomous system

$$\begin{cases} \frac{dv}{ds} = w - (N - p)v, \\ \frac{dw}{ds} = (p - \alpha + |v|^{1/(p-1)} \text{sign}(v))w. \end{cases}$$

By the definition of the new variables, the region of interest is  $v < 0, w < 0$  (a radial solution of  $(P_r)$  is positive and its radial derivatives is negative). In this region we find two stationary points:  $P_1(0, 0)$  and  $P_2(- (p - \alpha)^{p-1}, - (p - \alpha)^{p-1}(N - p))$   $u \in W_0^{1,p}(\Omega)$ .

The point  $P_1$  is an unstable hyperbolic point. The  $v$ -axis is the stable manifold for this point, and the unstable manifold is tangent to the straightline  $w = (N - \alpha)v$ .

With respect to the point  $P_2$ , it is

- (1) A stable nodus if  $N \geq p + (4(p - \alpha))/(p - 1)$ .
- (2) A stable spiral point if  $p < N < p + (4(p - \alpha))/(p - 1)$ .

We can see also that a singular selfsimilar solution of  $(P_r)$  is

$$S(x) = \log \left( \frac{1}{|x|^{p-\alpha}} \right)$$

with  $\lambda$  being equal to  $\tilde{\lambda} = (p - \alpha)^{p-1}(N - p)$ . This singular solution corresponds to the critical point  $P_2$  in the phase portrait, and it verifies the following interesting property:

$$\frac{1}{|x|^\alpha} e^{S(x)} = \frac{1}{|x|^p}$$

for every  $\alpha$ .

We need some previous lemmas (their proofs are similar to those in [GPP]).

**LEMMA 6.1.** *Let  $u$  be a radial solution of  $(P_r)$  and  $(v, w)$  the corresponding trajectory of the autonomous system. Then,  $u$  is a regular solution of  $(P_r)$  ( $\lim_{s \rightarrow -\infty} u(s) = A < \infty$ ) if and only if  $\lim_{s \rightarrow -\infty} (v(s), w(s)) = (0, 0)$ .*

**LEMMA 6.2.** *The unique trajectory of the autonomous system corresponding to a solution of  $(P_r)$  such that  $\lim_{s \rightarrow -\infty} u(s) = \infty$  is the critical point  $P_2$ .*

Let  $\underline{u}(\lambda)$  be the minimal solution of  $(P_r)$ ,  $\lambda^* = \sup\{\lambda: (P_r) \text{ has solution}\}$ , and  $\tilde{\lambda} = (p - \alpha)^{p-1}(N - p)$ . In this way, we arrive to the

**THEOREM 6.3.** *i) If  $N \geq p + (4(p - \alpha))/(p - 1)$  then  $\lambda^* = \tilde{\lambda}$ , for each  $\lambda < \lambda^*$  we have a unique radial regular solution, and  $\lim_{\lambda \rightarrow \lambda^*} \underline{u}(\lambda) = u_*$  is a singular solution;*

*ii) If  $p < N < p + (4(p - \alpha))/(p - 1)$  then  $\tilde{\lambda} < \lambda^*$ , and for  $\lambda = \tilde{\lambda}$ , there are infinitely many regular radial solutions, the values at the origin going to infinity.*

Moreover, in case ii),  $\lim_{\lambda \rightarrow \lambda^*} \underline{u}(\lambda) = u_* \in L^\infty$ , and there exists a positive constant,  $\varepsilon_0 > 0$  such that, if  $0 < |\lambda - \bar{\lambda}| < \varepsilon_0$  then the corresponding problem  $(P_r)$  has a finite family of radial solutions.

PROOF. In case i) we show that the trajectory joining  $P_1$  and  $P_2$ , denoted by  $\phi$ , is a monotone curve contained in the region  $-(p - \alpha)^{p-1} < v < 0$ ,  $-(p - \alpha)^{p-1}(N - p) < w < 0$ . Thus, there exists a unique point of intersection for each line  $w = -\lambda$ , i.e., there exists a unique regular radial solution for each  $\lambda \in (0, (p - \alpha)^{p-1}(N - p))$ .

First, it is easy to see that  $\phi$  is below the line  $w = (N - p)v$ . We need a lower bound for  $\phi$ ; for that, we consider two different cases.

If  $N \geq \max\{p + (4(p - \alpha))/(p - 1), 3p - 2\alpha\}$ , and  $R$  is the line

$$w = \frac{N - p}{2} v - (p - \alpha)^{p-1} \frac{N - p}{2}$$

we will show that  $dw/dv < (N - p)/2$  along  $R$ , whenever

$$-(p - \alpha)^{p-1} < v < 0.$$

In this way the trajectories  $(v, w)$  in the phase plane must cross  $R$  from below; this implies that  $\phi$  cannot cut  $R$ , since it starts from above.

Then, it suffices to show that

$$\frac{dw}{ds} - \frac{N - p}{2} \frac{dv}{ds} > 0$$

when  $(v, w) \in R$ ,  $-(p - \alpha)^{p-1} < v < 0$  (it has to be noted that  $dv/ds < 0$  in the region  $-(p - \alpha)^{p-1} < v < 0$ ,  $-(p - \alpha)^{p-1}(N - p) < w < (N - p)v$ ). So

$$\frac{dw}{ds} - \frac{N - p}{2} \frac{dv}{ds} = ((p - \alpha)^{p-1} - |v|) \left\{ \left( \frac{N - p}{2} \right)^2 + \frac{N - p}{2} (p - \alpha - |v|^{1/(p-1)}) - (N - p)(p - \alpha)^{p-1} \frac{p - \alpha - |v|^{1/(p-1)}}{(p - \alpha)^{p-1} - |v|} \right\}.$$

The factor  $((p - \alpha)^{p-1} - |v|)$  is positive; if we write  $s = |v|^{1/(p-1)}/(p - \alpha)$ , and we suppose  $1 < p < 2$ , the function

$(1 - s)/(1 - s^{p-1})$  is increasing in  $(0, 1)$ . We obtain (remember that  $N \geq p + 4(p - \alpha)/(p - 1)$ )

$$\begin{aligned} \frac{N - p}{2} + (p - \alpha)(1 - s) - 2(p - \alpha) \frac{1 - s}{1 - s^{p-1}} &> \\ &> \frac{N - p}{2} - \frac{2(p - \alpha)}{p - 1} > 0. \end{aligned}$$

If  $p > 2$ , then

$$\frac{N - p}{2} + (p - \alpha)(1 - s) - 2(p - \alpha) \frac{1 - s}{1 - s^{p-1}} = \frac{1}{1 - s^{p-1}} f(s)$$

where  $f(s)$  is

$$f(s) = (p - \alpha)s^p - \left( \frac{N + p - 2\alpha}{2} \right) s^{p-1} + (p - \alpha)s + \frac{N - 3p + 2\alpha}{2}$$

for  $s \in (0, 1)$ . This function verifies the following properties:

- (1)  $f(0) = (N - 3p - 2\alpha)/2 > 0$ ,  $f'(0) = p - \alpha > 0$ .
- (2)  $f(1) = 0$  and  $f'(1) \leq 0$  since  $N \geq p + 4(p - \alpha)/(p - 1)$ .
- (3)  $f$  has two critical points, the first between 0 and 1, the second one greater or equal to 1.

This implies that

$$\frac{dw}{ds} - \frac{N - p}{2} \frac{dv}{ds} > 0$$

when  $(u, v) \in R$ ,  $-(p - \alpha)^{p-1} < v < 0$ , and therefore the trajectory  $\phi$  cannot cross  $R$ .

When  $p + 4(p - \alpha)/(p - 1) \leq N < 3p - 2\alpha$ , we can do a different argument. We consider now the curve

$$f(v) = -(p - \alpha)^{(p-1)/2} (N - p) |v|^{1/2}$$

contained in the region  $-(p - \alpha)^{p-1} < v < 0$ . Then  $f$  verifies

- (1)  $f(0) = 0$ ,  $f(-(p - \alpha)^{p-1}) = -(p - \alpha)^{p-1} (N - p)$ , that is,  $f$  connects the two singular points in the phase plane.
- (2)  $f$  is increasing and convex in  $(-(p - \alpha)^{p-1}, 0)$ .
- (3)  $dw/dv < f'(v)$  on  $(v, f(v))$ .

Then, it follows that  $f$  is a lower bound for the trajectory  $\phi$  and we conclude the analysis for i).

In case ii) the line  $w = -\lambda$  cross the manifold  $\phi$  infinitely many times. Each point of intersection  $s_j$  corresponds to a radial solution of  $(P_r)$  by scaling  $s$  in such a way that for  $s = 0$  we have as an initial value  $s_j$ . The rest is a consequence of the analysis carried out in this section. ■

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