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On a Pair of Automorphisms of C^* -Algebras.

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1. - Introduction.

During the last decade or so, a lot of work has been done on the operator equation $\alpha + \alpha^{-1} = \beta + \beta^{-1}$, where α and β are $*$ -automorphisms of a W^* -algebra or a C^* -algebra M (say). Among several decomposition results in this context, it is known (see e.g. [1,7,9]) that this operator equation in the commuting case (that is, when α and β commute) leads to a decomposition of the W^* -algebra M , namely, $\alpha = \beta$ on Mp and $\alpha = \beta^{-1}$ on $M(1-p)$ for a central projection p in M . In case M is a factor then $\alpha = \beta$ or $\alpha = \beta^{-1}$. Batty [2] has studied this operator equation for C^* -algebras. Recently, Bresar [3] has considered this operator equation in a more general context of rings and has obtained decomposition results analogous to the results of Batty [2] and Thaheem [1,7,8]. There are situations where this operator equation ensures the commutativity of α and β and consequently the commutativity condition can be relaxed from the hypothesis to obtain the decomposition. For instance, it has been proved in [7, Theorem 3.1] that if M is a commutative Banach algebra then the operator equation $\alpha + \alpha^{-1} = \beta + \beta^{-1}$ implies the commutativity of α and β . Also, it has been shown in [10] that if M is a C^* -algebra and α (or β) is inner, then also the operator equation $\alpha + \alpha^{-1} = \beta + \beta^{-1}$ implies the commutativity of α and β (see also [3, Corollary 3] for an analogous result for rings). Recently, in [4] we have studied this operator equation in a more general context. We consider there the linear combination $a\alpha + b\alpha^{-1}$ (a, b are complex

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numbers, $a^2 \neq b^2$ rather than the sum and obtain several decomposition results for W^* -algebras and C^* -algebras. If we put $b/a = c$, then the equation $aa + ba^{-1} = a\beta + b\beta^{-1}$ reduces to the equation $a + ca^{-1} = \beta + c\beta^{-1}$. The purpose of this note is to look for situations analogous to [7, 10] where the operator equation $a + ca^{-1} = \beta + c\beta^{-1}$ implies the commutativity of a and β for an appropriate choice of c . For instance, we show (Proposition 2.1) that if a and β are inner automorphisms of a C^* -algebra M such that $a + ca^{-1} = \beta + c\beta^{-1}$ where c is a complex number with $|c| > \max\{1, \|\alpha\|\}$, then a and β commute. In case a and β are automorphisms of a W^* -algebra M satisfying $a + ca^{-1} = \beta + c\beta^{-1}$ where $\|a - 1\| < 1$, $\|\beta - 1\| < 1$ and c is a complex number such that $|c| > 4$, then we are able to prove a stronger form of the result, namely, $a = \beta$. We conclude this note with a general result on the commutativity of the inner automorphisms a and β on a complex algebra even when the automorphisms satisfy the operator equation $a + ca^{-1} = \beta + c\beta^{-1}$ for certain specific elements. For more information about the operator equation we refer to [1, 8, 9, 10] which also contain further references. We shall follow Pedersen [5] and Sakai [6] for the general theory of W^* -algebras and C^* -algebras. All C^* -algebras considered here are assumed to have the identity element.

2. - Commutativity of automorphisms.

Recall that an automorphism α of a C^* -algebra M is said to be inner if there is an invertible element u in M such that $\alpha(x) = uxu^{-1}$ for all $x \in M$. We say that α is implemented by u . If u is unitary then α is a $*$ -automorphism.

We now prove a commutativity result analogous to a result of [10, Theorem 2.2] where c is equal to 1.

PROPOSITION 2.1. *Let M be a C^* -algebra and α, β be inner automorphisms of M which are implemented by u and v respectively. Assume that $a + ca^{-1} = \beta + c\beta^{-1}$, where c is any complex number such that $|c| > \max\{1, v\|\alpha\|\}$. Then α and β commute.*

PROOF. For any $x \in M$, $\alpha(x) + ca^{-1}(x) = \beta(x) + c\beta^{-1}(x)$ implies that $uxu^{-1} + cu^{-1}xu = vxv^{-1} + cv^{-1}xv$. For $x = v$, we get $uvu^{-1} + cu^{-1}vu = (1 + c)v$, or in other words, $\alpha(v) + ca^{-1}(v) = (1 + c)v$. This implies that $\alpha^2(v) + cv = (1 + c)\alpha(v)$ where α^2 means $\alpha \circ \alpha$. Then $(\alpha - c)(\alpha - 1)(v) = 0$. Put $(\alpha - 1)(v) = y$. Then $\alpha(v) = v + y$ and $(\alpha - c)(y) = 0$ or $\alpha(y) = cy$. This implies $\alpha^2(v) = \alpha(v) + \alpha(y) = v + y + cy = v + (1 + c)y$. Thus we obtain that $\alpha^n(v) = v + (1 + c +$

$+ c^2 + \dots + c^{n-1})y$ for any natural number $n \geq 1$. Then

$$\|y\| \frac{|c^n - 1|}{|c - 1|} = \|\alpha^n(v) - v\| \leq (\|\alpha\|^n + 1)\|v\|$$

and

$$\|y\| |(c^n - 1)(\|\alpha\|^n + 1)^{-1}(c - 1)^{-1}| \leq \|v\|.$$

From $|c| > \max\{1, \|\alpha\|\}$ we conclude that $\|y\| = 0$ and $y = 0$. So $\alpha(v) = v$ and hence $uvu^{-1} = v$. This implies that $uv = vu$ and consequently α and β commute. This completes the proof of the proposition.

PROPOSITION 2.2. *Let α, β be automorphism of a W^* -algebra M such that $\alpha + c\alpha^{-1} = \beta + c\beta^{-1}$, $\|\alpha - 1\| < 1$, $\|\beta - 1\| < 1$, where c is a complex number such that $|c| > 4$. Then $\alpha = \beta$.*

PROOF. By Sakai [6, Theorem 4.1.19], α and β are inner automorphisms. Therefore, $\alpha(x) = uxu^{-1}$ and $\beta(x) = vxv^{-1}$ for all $x \in M$, where u, v are invertible elements of M . But $\|\alpha\| < 2$, and $|c| > 4$. Therefore by Proposition 2.1, α and β commute. From the relation $\alpha + c\alpha^{-1} = \beta + c\beta^{-1}$ together with the commutativity of α, β , we get

$$(A) \quad (\alpha\beta - c)(\beta^{-1} - \alpha^{-1}) = 0.$$

We now prove that $N(\alpha\beta - c) = \{0\}$ where $N(\alpha\beta - c)$ denotes the null space of $(\alpha\beta - c)$. Let $x \in N(\alpha\beta - c)$. Then $\alpha\beta x = cx$. It follows that $cx - \beta x = \alpha\beta x - \beta x = (\alpha - 1)\beta x$. This implies that

$$\|cx - \beta x\| = \|(\alpha - 1)\beta x\| \leq \|\alpha - 1\| \|\beta x\| < \|\beta x\|.$$

That is

$$\|cx\| - \|\beta x\| < \|\beta x\|.$$

This implies that

$$|c| \|x\| < 2 \|\beta x\| \leq 2 \|\beta\| \|x\| < 4 \|x\|.$$

If $x \neq 0$, then we get $|c| < 4$, a contradiction. This implies that $x = 0$ and consequently $N(\alpha\beta - c) = \{0\}$. It follows from (A) and the commutativity of α, β that for any $x \in M$, $(\beta^{-1} - \alpha^{-1})(x) \in N(\alpha\beta - c) = \{0\}$. Thus we get that $\beta^{-1}x = \alpha^{-1}x$ for all $x \in M$ and hence by the commutativity of α and β , it follows that $\alpha(x) = \beta(x)$ for all $x \in M$. This completes the proof.

PROPOSITION 2.3. *Assume that α, β are $*$ -automorphisms of a C^* -algebra M and $\alpha + c\alpha^{-1} = \beta + c\beta^{-1}$ where c is a complex number with $|c| > 1$ and α is inner. Then $\alpha = \beta$.*

PROOF. Because α is inner, $\alpha(x) = u x u^{-1}$ for all $x \in M$ where u is an invertible element of M . Also, $\beta(x) + c\beta^{-1}(x) = \alpha(x) + c\alpha^{-1}(x)$ for all $x \in M$. In particular, when $x = u$, we get

$$\beta(u) + c\beta^{-1}(u) = (1 + c)u.$$

A procedure similar to the proof of Proposition 2.1 implies that $\beta(u) = u$. Therefore,

$$\begin{aligned} (\alpha\beta)(x) &= \alpha(\beta(x)) = u\beta(x)u^{-1} = \beta(u)\beta(x)\beta(u^{-1}) = \\ &= \beta(uxu^{-1}) = (\beta\alpha)(x), \quad \forall x \in M. \end{aligned}$$

Thus α, β commute. But α and β are *-automorphisms. Therefore $\|\alpha\| = \|\beta\| = 1$ and as in the proof of Proposition 2.2, we get that $N(\alpha\beta - c) = \{0\}$. The commutativity of α, β together with the equation (A) in the proof of Proposition 2.2 imply that $\alpha = \beta$ on M .

COROLLARY 2.4. *Assume that α, β are *-automorphisms of a type I W^* -algebra M such that $\alpha + c\alpha^{-1} = \beta + c\beta^{-1}$ where c is a complex number such that $|c| > 1$. Assume further that α leaves the center pointwise fixed. Then $\alpha = \beta$.*

PROOF. By Sakai [6, Corollary 2.9.32], α is inner and the result follows from Proposition 2.3.

We conclude the note with the following proposition which gives a more general result on the commutativity of automorphisms α and β on a complex algebra M even when the automorphisms satisfy the operator equation $\alpha + c\alpha^{-1} = \beta + c\beta^{-1}$ for certain specific elements of M .

PROPOSITION 2.5. *Let M be a complex algebra with identity 1; let α, β be automorphisms of M which are implemented by u and v respectively; and let c be a complex number different from -1 and 1 . Assume that*

- (i) $\alpha(x) + c\alpha^{-1}(x) = \beta(x) + c\beta^{-1}(x)$ for $x = u$ and v ,
- (ii) $\alpha(\beta(x)) = \beta(\alpha(x))$ for $x = u$ and v .

Then α and β commute.

PROOF. Let $k = v^{-1}u^{-1}vu$. Then

$$(1) \quad vu = uvk.$$

The proof is complete if we show that $k = 1$.

We first prove that k commutes with u and v . It follows from (ii) that

$$uvxv^{-1}u^{-1} = vuxu^{-1}v^{-1}.$$

Then

$$uvxv^{-1}u^{-1}vu = vux.$$

This implies $uvxk = vux = uvkx$ (from (1)). For $x = u$, we get

$$uvuk = uvku.$$

This implies that $uv(uk - ku) = 0$ and since u, v are invertible, we get

$$uk - ku = 0 \quad \text{or} \quad uk = ku.$$

Thus k commutes with u . Similarly k commutes with v . Thus (1) may be rewritten as

$$(1') \quad vu = kuv.$$

Put $x = v$ in (i) to obtain

$$(2) \quad uvu^{-1} + cu^{-1}vu = (1 + c)v.$$

Multiply (2) on the right by u , and apply (1') to get

$$(3) \quad uv + cu^{-1}vu^2 = (1 + c)kuv,$$

or what is the same

$$(4) \quad cu^{-1}vu^2 = (ck + k - 1)uv.$$

Multiply (4) on the left by u to get

$$(5) \quad cvu^2 = (ck + k - 1)u^2v.$$

Multiply (1') on the left by ku to get

$$(6) \quad kuvu = k^2u^2v.$$

Multiply (1') on the right by u to get

$$(7) \quad vu^2 = kuvu.$$

From (6) and (7) we get

$$(8) \quad cvu^2 = ck^2u^2v.$$

From (5) and (8) we obtain

$$(9) \quad ck^2u^2v = (ck + k - 1)u^2v,$$

or equivalently

$$(10) \quad (ck^2 - ck - k + 1)u^2v = 0.$$

But u^2v is invertible, so we have

$$(11) \quad (ck - 1)(k - 1) = ck^2 - ck - k + 1 = 0.$$

Repeating the above arguments with v in place of u , u in place of v , and k^{-1} in place of k , we get

$$(12) \quad (ck^{-1} - 1)(k^{-1} - 1) = 0.$$

Multiply (12) on the left with ck and on the right with k and combine it with (11) to obtain

$$(13) \quad (c^2 - 1)(k - 1) = 0.$$

But $c^2 \neq 1$, so $k = 1$ is clear and the proof is complete by (1).

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