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On Π -Normally Embedded Subgroups of Finite Soluble Groups.

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1. - Introduction and statement of results.

All groups considered in this paper are finite and soluble.

Let π be a set of primes. A subgroup H of a group G is said to be π -normally embedded in G if a Hall π -subgroup of H is also a Hall π -subgroup of some normal subgroup of G . A Hall π -subgroup of a normal subgroup of G is an obvious example of a π -normally embedded subgroup of G . This embedding property was studied in [1] and [2]. The present paper represents an attempt to carry this study further. In fact we analyze what properties related to p -normal embedding property (p a prime number) can be extended to a set of primes π .

Our first Theorem concerns \mathcal{F} -normalizers associated to a saturated formation \mathcal{F} . Chambers [3] proved that in a group G with abelian Sylow p -subgroups, the \mathcal{F} -normalizers of G are p -normally embedded in G , where \mathcal{F} is saturated formation. We prove:

THEOREM 1. *Let \mathcal{F} be a saturated formation. If G is a group with abelian Hall π -subgroups, then the \mathcal{F} -normalizers of G are π -normally embedded in G .*

Let π be a set of primes. A subgroup U of a group G is said to be π -pronormal in G if U and U^g are conjugate in $O^\pi(\langle U, U^g \rangle)$ for all $g \in G$. Note that the π -pronormality is just the \mathcal{F} -pronormality introduced in [6] when \mathcal{F} is the saturated formation of all soluble

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π -groups. This embedding property is closely related to π -normal embedding one as it is shown in the following Theorem.

THEOREM 2. *For a π -subgroup U of a group G , the following statements are equivalent:*

- (i) U is π -normally embedded in G .
- (ii) U is π -pronormal in G and a CAP subgroup of G .
- (iii) U permutes with every Hall π' -subgroup of G and is normalized by each Hall π -subgroup of G containing it.

Wood [5] proves the following result:

THEOREM (Wood). Let p be a prime and let G be a group. The following statements are pairwise equivalent:

- (a) All the maximal subgroups of G are p -normally embedded in G .
- (b) Every Sylow p -subgroup of every maximal subgroup of G is pronormal in G .
- (c) G has p -length at most one.

The obvious extension of Wood's Theorem to a set of primes π does not hold. Let H be the symmetric group of degree 3. It is known that H has an irreducible and faithful module over $\text{GF}(5)$, the finite field of 5 elements, V say. Let $\pi = \{2, 5\}$ and $G = [V]H$. Then the Hall π -subgroups of the maximal subgroups of G are pronormal in G . However the π -length of G is bigger than 1.

THEOREM 3. *Let π be a set of primes. Let G be a group. The following statements are equivalent:*

- (i) All maximal subgroups of G are π -normally embedded in G .
- (ii) Every Hall π -subgroup of a maximal subgroup of G is π -pronormal in G .

If either (i) or (ii) hold, then G has π -length at most one.

The symmetric group of degree 3 has π -length one for $\pi = \{2, 3\}$. However the maximal subgroup $\langle(12)\rangle$ of G is not π -normally embedded in G . So the assertion in Theorem 3: G has π -length at most one, does not imply (i) and (ii) in general.

2. – Preliminaries.

In this section we collect some results which are needed in proving our Theorems. All the known results concerning finite soluble groups which we will need appear in [4]. This book is the main reference for the notation and terminology.

From now on π will be a set of primes.

LEMMA 1 [2]. *Let U be a π -normally embedded subgroup of a group G , K a normal subgroup of G , and L a subgroup of G . Then:*

- (i) *If U is a subgroup of L , then U is π -normally embedded in L .*
- (ii) *UK is π -normally embedded in G and UK/K is π -normally embedded in G/K .*
- (iii) *If K is a subgroup of L and L/K is π -normally embedded in G/K , then L is π -normally embedded in G .*

LEMMA 2. *Assume that τ is a set of primes with $\pi \cap \tau = \emptyset$. Let P be a π -subgroup of a group G and let Q be a τ -subgroup of G . Suppose that P is π -normally embedded in G and Q is τ -normally embedded in G . If $\langle P, Q \rangle$ is a $(\pi \cup \tau)$ -group, then $PQ = QP$.*

PROOF. By virtue of Lemma 1, we have that P is a π -normally embedded subgroup in $T = \langle P, Q \rangle = P[P, Q]Q$. So P is a Hall π -subgroup of its normal closure $\langle P^T \rangle = P[P, Q]$. Since Q is a τ -group and $\tau \cap \pi = \emptyset$, it follows that P is a Hall π -subgroup of T . Analogously we have that Q is a Hall τ -subgroup of T . This implies that $T = PQ = QP$ because T is a $(\pi \cup \tau)$ -group.

LEMMA 3. *If a group G has an abelian Hall π -subgroup, then G has π -length at most one.*

PROOF. Consider the upper π' - π -series of G :

$$1 \trianglelefteq P_0 \trianglelefteq N_0 \trianglelefteq P_1 \trianglelefteq \dots \trianglelefteq G$$

where $N_0 = O_{\pi'}(G)$, $P_1 = O_{\pi', \pi}(G)$ and $N_1/P_1 = O_{\pi'}(G/P_1)$. Let H be a Hall π -subgroup of G . By ([4] [I; (3.2)]), we know that HN_0/N_0 is a Hall π -subgroup of G/N_0 . Since $P_1/N_0 = O_{\pi}(G/N_0)$ is a normal subgroup of G/N_0 , it follows that $P_1/N_0 \leq HN_0/N_0$. Now HN_0/N_0 is abelian. This implies that $HN_0/N_0 \leq C_{G/N_0}(P_1/N_0) \leq P_1/N_0$ by virtue of [7]. In particular $H \leq P_1$ and G/P_1 is then a π' -group. This means that $N_1 = G$ and G has π -length at most one.

3. - Proofs of the Theorems.

PROOF OF THEOREM 1. We argue by induction on $|G|$. Let D be an \mathcal{F} -normalizer of G associated to the Hall system Σ of G . It is clear that we can assume that $D < G$. Let N be a minimal normal subgroup of G . Then DN/N is an \mathcal{F} -normalizer of G associated to the Hall system $\Sigma N/N$ of G/N ([4] [V; (3.2)]). Since G/N has abelian Hall π -subgroups, it follows that DN/N is π -normally embedded in G/N and so DN is π -normally embedded in G . Assume that N_1 and N_2 are two distinct minimal normal subgroups of G . Then DN_i is π -normally embedded in G for $i \in \{1, 2\}$. Since Σ reduces in DN_1 and DN_2 , we have that $DN_1 \cap DN_2$ is π -normally embedded in G by ([2] [Th. 1]). Now, by ([4] [V; (3.2)]), D either covers or avoids N_i . If $N_i \leq D$ for some i , it follows that D is π -normally embedded in G and we are done. So D avoids N_1 and N_2 . This implies that $D = DN_1 \cap DN_2$ is π -normally embedded in G and we are done. Consequently G has a unique minimal normal subgroup, N say. Then $F(G)$ is a p -group for some prime p . Since DN is π -normally embedded in G , we have that $p \in \pi$. In particular, $O_{\pi'}(G) = 1$ and G has a normal Hall π -subgroup H because G has π -length at most one by Lemma 3. Since H is abelian and $F(G) \leq H$, it follows that $H \leq C_G(F(G)) \leq F(G)$. This means that $F(G)$ is an abelian Hall π -subgroup of G . By ([4] [V; (3.6) and (3.7)]), there exists a maximal subgroup M of G such that $G = F(G)M$ and D is an \mathcal{F} -normalizer of M . By induction, D is π -normally embedded in M . Let A be a Hall π -subgroup of D . Then A is a Hall π -subgroup of $\langle A^M \rangle$. Since $D \leq F(G)$ and $F(G)$ is abelian, we have that $\langle A^M \rangle = \langle A^G \rangle$ and so $\langle A^M \rangle$ is a normal subgroup of G . Therefore D is π -normally embedded in G and the Theorem is proved.

PROOF OF THEOREM 2. (i) implies (ii). Since U is a Hall subgroup of a normal subgroup of G , it follows that U is pronormal in G . Then, there exists $x \in J = \langle U, U^g \rangle$ with $U^g = U^x$. By Lemma 1, U is π -normally embedded in J and $UO^\pi(J)/O^\pi(J)$ is π -normally embedded in $J/O^\pi(J)$, which is a π -group. This implies that $J = UO^\pi(J)$. In particular $x = uz$, with $u \in U$ and $z \in O^\pi(J)$. So $U^g = U^x = U^z$ and U is π -pronormal in G .

Let H/K be a chief factor of G . If H/K is a π' -group, then U avoids H/K . Assume that H/K is a π -group. Let Σ be a Hall system of G reducing into U . Then Σ reduces into UK . By Lemma 1, UK is π -normally embedded in G . Since Σ also reduces into H , we have that $UK \cap H$ is π -normally embedded in G by ([2] [Th. 1]). In particular, $UK \cap H$ is also subnormal in G , we have that $UK \cap H \trianglelefteq G$, and U either covers or avoids H/K . Therefore U is a CAP subgroup of G .

(ii) implies (i). Assume that U is π -pronormal in G and a CAP subgroup of G . We prove that U is π -normally embedded in G by induction on $|G|$. Let N be a minimal normal subgroup of G . Since UN/N is π -pronormal in G/N and UN/N is a CAP subgroup of G , we have that UN is a π -normally embedded subgroup of G by induction. If N is a π' -group, then U is π -normally embedded in G and we are done. Hence we may assume $O_{\pi'}(G) = 1$. Now U is π -pronormal in UN and UN is π -group. This means that U is a normal subgroup of UN . Moreover, U either covers or avoids N . If $N \leq U$, then U is π -normally embedded in G and we are done. Thus $U \cap N = 1$ and $U \leq C_G(N)$.

Therefore we may assume that U centralizes every minimal normal subgroup of G . With the same arguments to those used in ([4] [I; (7.12)]), we conclude that U is normal in G and so U is π -normally embedded in G .

(i) implies (iii). Assume that U is π -normally embedded in G and let $G_{\pi'}$ be a Hall π' -subgroup of G . By Lemma 2, U permutes with $G_{\pi'}$, because $G_{\pi'}$ is π' -normally embedded in G .

Let G_{π} be a Hall π -subgroup of G such that $U \leq G_{\pi}$. Since U is π -normally embedded in G , we have that U is π -normally embedded in G_{π} . This means that U is normal in G_{π} . Hence G_{π} normalizes U .

(iii) implies (i). Suppose that $G_{\pi'}$ is a Hall π' -subgroup of G such that $G_{\pi'}$ permutes with U and let G_{π} be a Hall π -subgroup of G such that $U \leq G_{\pi}$. Then U is normalized by G_{π} . Moreover $G = G_{\pi}G_{\pi'}$ and hence $\langle U^G \rangle = \langle U^{G_{\pi'}} \rangle \leq UG_{\pi'}$. Since U is a Hall π -subgroup of $UG_{\pi'}$ and $U \leq \langle U^G \rangle$, it follows that U is a Hall π' -subgroup of $\langle U^G \rangle$ and U is π -normally embedded in G .

PROOF OF THEOREM 3. Since the maximal subgroups of G are CAP subgroups of G , it follows that every Hall π -subgroup of every maximal subgroup is a CAP subgroup of G . So the equivalence between (i) and (ii) in Theorem 3 follows from the equivalence between (i) and (ii) in Theorem 2. Assume that there exists a group G such that G has the property (i) but G does not have π -length at most one. Let us consider G of minimal order. Since the property (i) is invariant under epimorphic images, we have that G/N has π -length at most one for every minimal normal subgroup N of G . Since the class of all groups with π -length at most one is a saturated formation, it follows that G has a unique minimal normal subgroup, N say, such that $G = MN$ and $M \cap N = 1$ for some maximal subgroup M of G . By hypothesis M is π -normally embedded in G . If N is a π -group, then M should be a π' -group and then G has π -length at most one, a contradiction. So N must be a π' -group. But then G also has π -length at most one, a contradiction.

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