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## A Note on Natural Tensor Products Containing Complemented Copies of $c_0$ .

J. A. LÓPEZ MOLINA(\*) - M. J. RIVERA(\*)(\*\*)

**ABSTRACT** - Let  $H$  be a Fréchet lattice containing a positive sequence equivalent to the unit basis of  $c_0$ . We prove that the natural tensor product  $H \widehat{\otimes}_{\mu} X$  contains a complemented subspace isomorphic to  $c_0$  for every infinite dimensional Banach space  $X$ , which generalizes a previous result of Cembranos and Freniche.

### 0. - Introduction.

Cembranos [4] shows in 1984 that  $\mathcal{C}(K, X)$  contains a complemented subspace isomorphic to  $c_0$  if  $K$  is an infinite compact Hausdorff space and  $X$  an infinite dimensional Banach space. The theorem was extended in 1986 by E. Saab and P. Saab [14] who proved that the injective tensor product  $X \widehat{\otimes}_{\varepsilon} Y$  of two infinite dimensional Banach spaces  $X$  and  $Y$  contains a complemented copy of  $c_0$  if  $X$  or  $Y$  contains  $c_0$ , using a proof inspired by the Cembranos's one. However both results have been obtained also in 1984, indeed in a little more general version, in a paper of Freniche [8]. On the other hand, Emmanuele [7] showed in 1988 that if  $(\Omega, \mathcal{A}, \nu)$  is a not purely atomic measure space and  $X$  is a Banach space containing  $c_0$ , the space  $L^p(\mu, X)$ ,  $1 \leq p < \infty$ , of Bochner integrable  $X$ -valued functions, contains a complemented copy of  $c_0$ .

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The spaces used by Freniche, Cembranos and Emmanuele are particular cases of the Levin natural tensor product  $H \widehat{\otimes}_{\mu} X$  of a Banach lattice  $H$  and a Banach space  $X$  (see the original paper [11] and also [5]). *The main purpose of this paper is to extend the above results to this more general setting.* Our method will be useful also to deal with the same problem in the case of the Saphar tensor product  $X \widehat{\otimes}_{d_{\infty}} Y$  (see [15]).

The notation and terminology is standard (see [10] for general theory of locally convex spaces and [1] for Banach lattices). If  $H$  is a Fréchet space (resp. a Fréchet lattice),  $(\|\cdot\|_s)_{s=1}^{\infty}$  will be an increasing fundamental system of continuous seminorms (resp. continuous lattice seminorms) in  $H$  and  $U_s = \{x \in H \mid \|x\|_s \leq 1\}$ . The usual Schauder basis of  $\ell^1$  and  $c_0$  will be denoted by  $(e_n)_{n=1}^{\infty}$ . A sequence  $(h_i)$  in the Fréchet space  $H$  is said to be weakly absolutely summable if, for every  $n \in \mathbb{N}$ , the inequality

$$\varepsilon_n((h_i)) = \sup_{h' \in U_n^0} \sum_{i=1}^{\infty} |\langle h_i, h' \rangle| < \infty$$

holds. We denote by  $\ell^1(H)$  the set of all weakly absolutely summable sequences in  $H$  and by  $\ell_0^1(H)$  the set of all  $(h_i) \in \ell^1(H)$  such that, for every  $n \in \mathbb{N}$ , we have

$$\lim_{k \rightarrow \infty} \varepsilon_n((0, 0, \dots, 0, h_k, h_{k+1}, \dots)) = 0.$$

**1. - Levin natural tensor products.**

Let  $H$  be a Fréchet lattice and  $F$  a Fréchet space. The Levin (or natural) tensor product  $H \widehat{\otimes}_{\mu} F$  is the completion of  $H \otimes_{\mu} F$  where  $\mu$  is the topology defined by the family of seminorms  $\{\mu_s \mid s \in \mathbb{N}\}$  and

$$\mu_s(z) = \inf \left\{ \left\| \sum_{i=1}^n \|x_i\|_s h_i \right\|_s \mid z = \sum_{i=1}^n h_i \otimes x_i \in H \otimes X, h_i \geq 0 \right\}.$$

This tensor product was introduced by Levin [11] in the case of a Banach lattice  $H$  and a Banach space  $G$ , although with a different equivalent definition, (see [9] for a proof of the equivalence) and by Chaney [5] and Lotz [13] with the present definition. The  $\mu$ -topology verifies that  $\varepsilon \leq \mu \leq \pi$  ( $\varepsilon$  and  $\pi$  are the injective and projective topologies of Grothendieck respectively) but  $\mu$  is not a tensor norm since it does not verify in general the metric mapping property.

If  $H = \mathcal{C}(K)$  or  $H = L^\infty(\nu)$  then  $\mu$  coincides with  $\varepsilon$ . If  $H$  is an  $L^1(\mu)$  space,  $\mu = \pi$  (see [5]).

The importance of the Levin tensor product lies in the fact that many usual function spaces can be represented by means of such a product. For instance, let  $(\Omega, \mathcal{A}, \mu)$  be a complete  $\sigma$ -finite measure space and  $\mathfrak{N}(\Omega, \mathcal{A})$  be the set of classes, modulo equality almost everywhere, of measurable real functions on  $(\Omega, \mathcal{A})$ . A Köthe function space (or a Banach ideal function space) on  $(\Omega, \mathcal{A}, \mu)$  will be a Banach space  $H$  which is an ideal in  $\mathfrak{N}(\Omega, \mathcal{A})$ . Given a Banach space  $X$ , we define  $H(X)$  as the set of strongly measurable functions  $f: \Omega \rightarrow X$  such that  $\|f(\cdot)\| \in H$  endowed with the norm  $\|f\| = \|\|f(\cdot)\|_X\|_H$ . It can be proved (see [11], [2], [3]) that, when  $H$  has an order continuous norm, we have  $H(X) = H \widehat{\otimes}_\mu X$ . In particular, the familiar Lebesgue-Bochner spaces  $L^p(\mu, X)$ ,  $1 \leq p < \infty$ , are isometric to  $L^p(\Omega) \widehat{\otimes}_\mu X$ .

We shall need the following result about the representation of the elements of  $H \widehat{\otimes}_\mu X$ .

LEMMA 1. *Let  $H$  be a Fréchet lattice and  $F$  a Fréchet space. Every  $z \in H \widehat{\otimes}_\mu F$  has a representation  $z = \sum_{i=1}^\infty h_i \otimes x_i$  with  $h_i \geq 0$ , where  $(x_i)_{i=1}^\infty$  is a bounded sequence in  $F$  and  $(h_i)_{i=1}^\infty \in \ell_0^1(H)$ . Moreover, if  $n \in \mathbb{N}$ ,  $\mu_n(z) \leq \left(\sup_{i \in \mathbb{N}} \|x_i\|_n\right) \varepsilon_n((h_i))$  for every such representation of  $z$  and  $\mu_n(z) = \inf \left\{ \left\| \sum_{i=1}^\infty \|x_i\|_n h_i \right\|_n \right\}$ , where the infimum is taken over all representations of  $z$  as above.*

PROOF. If  $z \in H \widehat{\otimes}_\mu F$ , there exists a sequence  $(z_n)_{n=0}^\infty \subset H \otimes F$  such that  $\lim_{n \rightarrow \infty} \mu_k(z - z_n) = 0$  for each  $k \in \mathbb{N}$ . Choosing a subsequence if it is needed, we can suppose that  $z = z_0 + \sum_{n=1}^\infty (z_n - z_{n-1})$  with

$$\mu_n(z_n - z_{n-1}) < 2^{-2n}, \quad n \in \mathbb{N}.$$

For every  $n \in \mathbb{N}$  there exists a representation  $z_n - z_{n-1} = \sum_{i=1}^{j(n)} h_{ni} \otimes x_{ni}$ , such that  $h_{ni} \geq 0$  and  $\left\| \sum_{i=1}^{j(n)} \|x_{ni}\|_n h_{ni} \right\|_n < 2^{-2n}$ . By the monotony of the lattice seminorms,  $z$  can be represented by a convergent series in  $H \widehat{\otimes}_\mu F$ ,  $z = \sum_{k=1}^\infty h_k \otimes x_k$  with every  $h_k \geq 0$ ,  $x_k \neq 0$  and  $\sum_{k=1}^\infty \|x_k\|_n h_k$  converges in  $H$  for every  $n \in \mathbb{N}$ . We define a sequence  $(\alpha_k)_{k=1}^\infty$  in  $\mathbb{K}$  as follows: given  $k \in \mathbb{N}$ , there are  $n, i \in \mathbb{N}$ ,  $1 \leq i \leq j(n)$ , such that  $x_k = x_{ni}$ .

We put  $\alpha_k = \|x_{ni}\|_n$  if  $\|x_{ni}\|_n \neq 0$  and  $\alpha_k = 2^{-2n} \left( \left\| \sum_{i \in J_n} h_{ni} \right\|_n \right)^{-1}$ , where  $J_n = \{i \in \mathbb{N} \mid 1 \leq i \leq j(n) \text{ and } \|x_{ni}\|_n = 0\}$ , if  $\left\| \sum_{i \in J_n} h_{ni} \right\|_n \neq 0$ ; if  $\|x_{ni}\|_n = 0$  and  $\left\| \sum_{i \in J_n} h_{ni} \right\|_n = 0$ , we put  $\alpha_k = 1$ . Then  $z = \sum_{k=1}^{\infty} (\alpha_k h_k) \otimes (x_k/\alpha_k)$ , is a representation of the announced type. In fact, it is clear that  $(x_k/\alpha_k)_{k=1}^{\infty}$  is a bounded sequence in  $E$ . On the other hand,  $H'$  is a lattice with its canonical order. Denoting  $V_n = \{h \in H \mid \|h\|_n \leq 1\}$ , we have that  $V_n^0$  is a solid set in  $H'$ . Then, if  $k_n = 1 + \sum_{i=1}^{n-1} j(i)$ , we have

$$\begin{aligned} \sup_{h' \in V_n^0} \sum_{k=k_n}^{\infty} \alpha_k |\langle h_k, h' \rangle| &\leq 2 \left( \sup_{h' \in V_n^0, h' \geq 0} \sum_{k=k_n}^{\infty} \alpha_k \langle h_k, h' \rangle \right) \leq \\ &\leq 2 \left\| \sum_{k=k_n}^{\infty} \alpha_k h_k \right\|_n \leq 2 \sum_{k=n}^{\infty} \left\| \sum_{i=1}^{j(k)} \|x_{ki}\|_k h_{ki} \right\|_k < 4 \sum_{k=n}^{\infty} 2^{-2k} \leq 2^{-2n-3}, \end{aligned}$$

and thus,  $(\alpha_k h_k)_{k=1}^{\infty}$  is in  $\ell_0^1(H)$ .

Finally, we fix  $n \in \mathbb{N}$ . It is clear that  $\mu_n(z) \leq \left\| \sum_{k=1}^{\infty} \|x_k\|_n h_k \right\|_n$  for every representation of  $z$  of the above type. Given one of them, for every  $\varepsilon > 0$ , there is a  $k_0$  such for every  $k \geq k_0$ ,  $\left\| \sum_{j=1}^{\infty} \|x_j\|_n h_j \right\|_n < \varepsilon/3$ . Putting  $w_k = \sum_{j=1}^{k-1} h_j \otimes x_j$ , we have  $\mu_n(z - w_k) < \varepsilon/3$ . Now it is possible to take a new representation of  $w_k$ , say  $w_k = \sum_{j=1}^t \bar{h}_j \otimes \bar{x}_j$  with  $\bar{h}_j \geq 0$ , such that

$$\begin{aligned} \left\| \sum_{j=1}^t \|\bar{x}_j\|_n \bar{h}_j \right\|_n &\leq \\ &\leq \mu_n(w_k) + \varepsilon/3 \leq \mu_n(z - w_k) + \mu_n(z) + \varepsilon/3 \leq \mu_n(z) + 2\varepsilon/3. \end{aligned}$$

Then  $z = \sum_{j=1}^t \bar{h}_j \otimes \bar{x}_j + \sum_{j=k}^{\infty} h_j \otimes x_j$ , so we get another representation  $z = \sum_{i=1}^{\infty} h_i \otimes x_i$  of the above mentioned type, satisfying  $\left\| \sum_{i=1}^{\infty} \|x_i\|_n h_i \right\|_n \leq \mu_n(z) + \varepsilon$ . The remaining inequality follows easily. ■

**2. Complemented copies of  $c_0$  in Levin tensor products.**

Next result generalizes the theorem of Cembranos and Freniche quoted in the introduction from the lattice point of view.

**THEOREM 2.** *Let  $H$  be a Fréchet lattice which contains a positive sequence equivalent to the unit basis of  $c_0$ . Then for every infinite dimensional Banach space  $E$ ,  $H \widehat{\otimes}_\mu E$  contains a complemented subspace isomorphic to  $c_0$ .*

**PROOF.** Let  $(b_n)_{n=1}^\infty$  be a sequence in  $H^+$  equivalent to the standard basis  $(e_n)_{n=1}^\infty$  of  $c_0$  by an isomorphism  $\Phi: c_0 \rightarrow H$  such that  $\Phi(e_n) = b_n$ . As  $c_0$  is isomorphic to a subspace of  $H$ ,  $\ell^1$  is isomorphic to a quotient of  $H'_\beta$ . The sequence  $(e_n)_{n=1}^\infty$  is also bounded in  $\ell^1$  and verifies that  $\langle e_m, e_n \rangle = \delta_{nm}$ . As the quotients of the  $DF$  spaces lift bounded sets [10, 12.4.8], there exist a bounded sequence  $(b'_n)_{n=1}^\infty$  in  $H'_\beta$  such that  $\langle b'_m, b_n \rangle = \delta_{nm}$ . Since  $(b'_n)_{n=1}^\infty$  is equicontinuous, there is  $k \in \mathbb{N}$  such that,  $(b'_n)_{n=1}^\infty \subset U_k^0$  holds. By the theorem of Josefson-Nissenzweig, there is a  $\sigma(E', E)$  null sequence  $(a'_n)_{n=1}^\infty$  in  $E'$  such that  $\|a'_n\| = 1, \forall n \in \mathbb{N}$ . An application of the principle of local reflexivity gives us a sequence  $(a_n)_{n=1}^\infty$  in  $E$  such that  $\langle a'_n, a_n \rangle = 1$  and  $\|a_n\| \leq 2$ , for every  $n \in \mathbb{N}$ .

We define  $Q: c_0 \rightarrow H \widehat{\otimes}_\mu E$  such that  $Q((\xi_i)) = \sum_{i=1}^\infty \xi_i b_i \otimes a_i$ , for every  $(\xi_i) \in c_0$ . The map  $Q$  is well defined:  $(b_i)_{i=1}^\infty$  being equivalent to  $(e_i)_{i=1}^\infty$ , we have that  $(\xi_i b_i) \in \ell^1_0(H)$ . Moreover,  $Q$  is linear and continuous: in fact, given  $s \in \mathbb{N}$

$$\mu_s(Q((\xi_i))) \leq \sup_{x' \in U_s^0} \sum_{i=1}^\infty |\langle \xi_i \Phi(e_i), x' \rangle| \|a_i\| \leq$$

$$\leq 2 \sup_{i \in \mathbb{N}} |\xi_i| \sup_{x' \in U_s^0} \sum_{i=1}^\infty |\langle e_i, \Phi'(x') \rangle| \leq 2 \sup_{i \in \mathbb{N}} |\xi_i| \sup_{x' \in U_s^0} \|\Phi'(x')\|_{\ell^1}.$$

Then we define  $T: H \widehat{\otimes}_\mu E \rightarrow c_0$  in the following way. By Lemma 1, every  $H \widehat{\otimes}_\mu E$  can be represented as  $z = \sum_{i=1}^\infty u_i \otimes v_i$  where, for each  $n \in \mathbb{N}$

$$(1) \quad \lim_{m \rightarrow \infty} \sup_{h' \in U_n^0} \sum_{i=m}^\infty |\langle h', u_i \rangle| = 0.$$

and  $\sup_{i \in \mathbb{N}} \|v_i\| \leq M$  for some  $M > 0$ . Then we put

$$T(z) = (\langle z, b'_n \otimes a'_n \rangle)_{n=1}^\infty = \left( \sum_{i=1}^\infty \langle u_i, b'_n \rangle \langle v_i, a'_n \rangle \right)_{n=1}^\infty.$$

The map  $T$  is well defined: it is easy to see that  $b'_n \otimes a'_n \in \left( H \widehat{\otimes}_\mu E \right)'$ ; on the other hand, since  $(u_i)_{i=1}^\infty$  is bounded in  $H$ , we have  $M_k := \sup_{i \in \mathbb{N}} \|u_i\|_k < \infty$ ; as  $b'_n \in U_k^0$  for all  $n \in \mathbb{N}$ , from (1), given  $\varepsilon > 0$ , there is  $h \in \mathbb{N}$  such that, for all  $n \in \mathbb{N}$

$$\begin{aligned} \left| \sum_{i=1}^\infty \langle u_i, b'_n \rangle \langle v_i, a'_n \rangle \right| &\leq \sum_{i=1}^h |\langle u_i, b'_n \rangle \langle v_i, a'_n \rangle| + \\ &+ M \sum_{i=h+1}^\infty |\langle u_i, b'_n \rangle| \leq M_k \sum_{i=1}^h |\langle v_i, a'_n \rangle| + \frac{\varepsilon}{2} \end{aligned}$$

and since  $(a'_n)_{n=1}^\infty$  is  $\sigma(E', E)$  null, there is  $n_0$  such that for every  $n \geq n_0$

$$\left| \sum_{i=1}^\infty \langle u_i, b'_n \rangle \langle v_i, a'_n \rangle \right| \leq M_k \sum_{i=1}^h |\langle v_i, a'_n \rangle| + \frac{\varepsilon}{2} \leq \varepsilon,$$

and hence  $T(z) \in c_0$ .

Moreover  $T$  is continuous. In fact, by Lemma 1, for every  $z \in H \widehat{\otimes}_\mu E$ , given  $\varepsilon > 0$ , there exist a representation  $z = \sum_{i=1}^\infty h_i \otimes x_i$  with  $h_i \geq 0$ , such that

$$\mu_k(z) + \frac{\varepsilon}{2} > \sup_{h' \in U_k^0} \left| \sum_{i=1}^\infty \|x_i\| \langle h', h_i \rangle \right|.$$

As  $U_k^0$  is a solid set we have  $b'_n = (b'_n)^+ - (b'_n)^-$  with  $(b'_n)^+, (b'_n)^- \in U_k^0$  for each  $n \in \mathbb{N}$ . Then we obtain

$$\begin{aligned} \|T(z)\| &= \sup_{n \in \mathbb{N}} \left| \sum_{i=1}^\infty \langle h_i, b'_n \rangle \langle x_i, a'_n \rangle \right| \leq \\ &\leq 2 \sup_{n \in \mathbb{N}} \sup_{h' \in U_k^0} \left| \sum_{i=1}^\infty \langle h_i, h' \rangle \|x_i\| \right| \leq 2\mu_k(z) + \varepsilon, \end{aligned}$$

and hence  $\|T(z)\| \leq 2\mu_k(z)$ .

It is easy to see that  $TQ((\xi_i)) = (\xi_i)$  for every  $(\xi_i) \in c_0$ , and then  $Q(c_0) = QT\left(H \widehat{\otimes}_\mu E\right)$  is a complemented of  $H \widehat{\otimes}_\mu E$  isomorphic to  $c_0$ . ■

**REMARK 3.** If  $K$  is an infinite compact Hausdorff space,  $\mathcal{C}(K)$  contains a subspace isomorphic to  $c_0$  by a positive isometry, and this fact is implicit in the Cembranos's proof and more explicit in the Freniche's one. To see that, take a sequence  $(G_n)_{n=1}^\infty$  of open non empty pairwise disjoint sets in  $K$ , and a sequence  $(t_n)_{n=1}^\infty$ ,  $t_n \in G_n$ ,  $\forall n \in \mathbb{N}$ . The Urysohn lemma gives us a sequence  $(b_n)_{n=1}^\infty \subset \mathcal{C}(K)^+$  such that  $b_n: K \rightarrow [0, 1]$  with  $b_n(t_n) = 1$  and  $b_n(t) = 0$  if  $t \in K \setminus G_n$ . Let  $F$  be the linear span of  $\{b_n | n \in \mathbb{N}\}$ . The map  $Q: F \rightarrow c_0$  such that  $Q(b_n) = e_n$  is an isometry since if  $f = \sum_{i=1}^s \lambda_i b_{n_i} \in \mathcal{C}(K)$ , then  $\|f\| = \sup_{i \in \mathbb{N}} |\lambda_i| = \|Q(f)\|$ , and it is clearly positive. Moreover, if we take  $b'_n = \delta_{t_n}$  (the Dirac measure at  $t_n$ ), choosing  $(a_n)$  as in Theorem 2, we get an easy representation of the projection  $P$  from  $\mathcal{C}(K, X) = \mathcal{C}(K) \widehat{\otimes}_\epsilon X$  onto the closure of the linear span of  $\{b_n(\cdot) a_n | n \in \mathbb{N}\}$ .

**EXAMPLE 4.** If  $M(t)$  is an Orlicz function which does not satisfies the  $\Delta_2$  condition at 0, the Orlicz sequence space  $h_M$  has a sublattice order isomorphic to  $c_0$  (see [3] and [12]) and  $\ell^\infty$  is not a subspace of  $h_M$ . By Theorem 2, for every infinite dimensional Banach space  $X$ ,  $c_0$  is a complemented subspace of  $h_M \widehat{\otimes}_\mu X$ .

**COROLLARY 5.** Let  $(\Omega, \mathcal{A}, \mu)$  be a complete  $\sigma$ -finite measure space such that  $L^\infty(\mu)$  is infinite dimensional. Let  $X$  be a Banach space which does not contain  $c_0$ . Then  $L^\infty(\mu) \widehat{\otimes}_\mu X$  is not complemented in  $L^\infty(\mu, X)$ .

**PROOF.** By Theorem 2,  $c_0$  is complemented in  $L^\infty(\mu) \widehat{\otimes}_\mu X$ . By [6], if  $c_0$  were complemented in  $L^\infty(\mu, X)$ ,  $c_0$  would be a subspace of  $X$ . Then the result follows. ■

We note also, for the sake of completeness, the alternative result:

**THEOREM 6.** If  $H$  is an order continuous norm Köthe function space,  $H(X) = H \widehat{\otimes}_\mu X$  contains a complemented copy of  $c_0$  if  $X$  has a copy of  $c_0$ .

**PROOF.** It is essentially the same one of Emmanuele for  $L^p(\mu, X)$  in [7]. ■

Our method can be easily applied also in the context of the  $d_\infty$ -tensor product of Saphar (see [15] for definitions and details):



**THEOREM 7.** *Let  $F$  be a Fréchet space containing a subspace isomorphic to  $c_0$ . Then for every infinite dimensional Banach space  $X$ ,  $F \widehat{\otimes}_{d_\infty} X$  contains a complemented subspace isomorphic to  $c_0$ .*

**PROOF.** It is similar to the proof of Theorem 2, since the topology of  $F \widehat{\otimes}_{d_\infty} X$  is determined by the seminorms given, for each  $z \in F \widehat{\otimes}_{d_\infty} X$ , by

$$d_{\infty s}(z) = \inf \left\{ \left( \sup_{i \in \mathbb{N}} \|x_i\| \right) \varepsilon_s((y_i)) \right\} \quad s \in \mathbb{N},$$

where the infimum is taken over all representations  $z = \sum_{i=1}^{\infty} y_i \otimes x_i$ , such that  $\sup_{i \in \mathbb{N}} \|x_i\| < \infty$  and  $(y_i) \in \ell_0^1(F)$ . ■

A Grothendieck space is a Banach space  $X$  such that  $\sigma(X', X)$  and  $\sigma(X', X'')$  null sequences in  $X'$  are the same. In consequence a Grothendieck space can not contain a complemented copy of  $c_0$ . Then we have:

**COROLLARY 8.** *Let  $H, G$  be Banach lattices such that  $H$  contains a positive sequence equivalent to the standard basis of  $c_0$  and  $G$  is a Köthe function space with order continuous norm. Let  $X$  and  $Y$  be infinite dimensional Banach spaces such that  $Y$  contains a subspace isomorphic to  $c_0$ . Then neither  $G(Y)$ ,  $H \widehat{\otimes}_{\mu} X$  nor  $Y \widehat{\otimes}_{d_\infty} X$  are Grothendieck spaces.*

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