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The Lattice of Very-Well-Placed Subgroups.

MARTHA SABOYÁ-BAQUERO (*)(**)

1. – Introduction.

Every group will be finite and soluble. In this paper we study the well-placed subgroups of a soluble group. These subgroups are introduced by Hawkes in [6] and play an important role in the theory of finite soluble groups.

A natural question concerning the well-placed subgroups is the following: is the set of the well-placed subgroups of a group G a sublattice of the subgroup lattice of G ? The answer is negative in general. We introduce a special type of well-placed subgroup called very-well-placed subgroup and study its properties. We prove that the set, denoted by $GE_{\Sigma}(G)$ of the very-well-placed subgroups of a group G associated to a Hall system Σ of G is a sublattice of the subgroup lattice of G . Moreover, we describe completely all these sublattices. This allows us to obtain a new characterization of the $\underline{N}^i$ -normalizers of a group G , where i is a natural number smaller than or equal to the nilpotent length of G and $\underline{N}^i$ the class of groups with nilpotent length at most i .

For basic definitions as well as notation we refer the reader to ([2], [3], [7]). We denote that U is maximal with $U < G$.

2. – Preliminaries.

We collect in this section some definitions and results we need

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in the sequel. First of all recall the definition of well-placed subgroup.

DEFINITION ([6], Def. 5.1). A subgroup U of G is called *well-placed* in G , if there exists a chain of subgroups $U = U_r < U_{r-1} < \dots < U_0 = G$, such that for $i = 1, \dots, r$:

- a) U_i is maximal in U_{i-1} ;
- b) U_i is critical in U_{i-1} , which means that $U_{i-1} = U_i F_i(U_{i-1})$.

The $\underline{\underline{F}}$ -normalizers of a soluble group associated to a saturated formation $\underline{\underline{F}}$ are an example of well-placed subgroups (see [2]).

The following proposition contains some remarkable facts about the well placed subgroups.

PROPOSITION 2.1. Let U be a well-placed subgroup of a group G .

a) U either covers or avoids the chief factors of G . Moreover if U covers the chief factor H/K of G , then $H \cap U/K \cap U$ is a chief factor of U and

$$\text{Aut}_G(H/K) \cong \text{Aut}_U(H \cap U/K \cap U) \text{ (see [3], III, 6.6).}$$

b) U belongs to the formation generated by G (see [1]).

c) If $\underline{\underline{H}}$ is a Schunck class and R is an $\underline{\underline{H}}$ -projector of U , there exists an $\underline{\underline{H}}$ -projector H of G such that $R \leq H$. Moreover, if $\underline{\underline{H}}$ is closed under well-placed subgroups (which is always true if $\underline{\underline{H}}$ is a saturated formation) then H may be chosen such that $R = H \cap U$ (see [3], III, 6.7).

The set of the well placed subgroups of a soluble group G is not a sublattice of the subgroup lattice of G , as the following example shows.

EXAMPLE 2.2. Let $H := \langle a, b \rangle$ be an elementary abelian group of order 9. There exists $c \in \text{Aut } H$ such that $a^c = a^{-1}$ and $b^c = b^{-1}$. Let $M = [H]\langle c \rangle$ be the corresponding semidirect product.

Denote by $K = \langle ab \rangle$ a diagonal of H . Then M/K is isomorphic to the symmetric group of degree 3. Therefore M has an irreducible two-dimensional $GF(2)$ -module N such that $\text{Ker}(M \text{ on } N) = K$.

Set $G := [N]M$, $U := [N]\langle a, c \rangle$ and $V := [N]\langle b, c \rangle$. The subgroups U and V are critical, and therefore they are well-placed in G . However, $U \cap V = [N]\langle c \rangle$ is not well-placed in G .

However, by imposing extra conditions on the subgroups we consider, in particular by requiring that Hall systems reduce into them, we can produce sublattices.

DEFINITION. Let $U \leq G$ and α an embedding property of G . A Hall system Σ of G *reduces via α into U* , if there exists a chain of subgroups $U = U_r < U_{r-1} < \dots < U_0 = G$, such that

- a) U_i is maximal in U_{i-1} for $i = 1, \dots, r$.
- b) Σ reduces into U_i for $i = 0, \dots, r$.
- c) U_i is α -subgroup of U_{i-1} for $i = 1, \dots, r$.

Even, the set $W_\Sigma(G) = \{U \leq G \mid \Sigma \text{ reduces via critical into } U\}$ does not form a sublattice. We come back to Example 2.2. Let $\Sigma := \{\{1\}, L, H, G\}$ where $L = [N]\langle c \rangle$ and H as defined above. Clearly Σ reduces into U and into V , but $U \cap V$ is not well-placed in G .

LEMMA 2.3. Let L and M be maximal subgroups of G . Then

- a) L and M are conjugate if and only if $\text{Core}_G(L) = \text{Core}_G(M)$ ([3], A, 16.1).
- b) If L and M are not conjugate and $\text{Core}_G(L) \neq \text{Core}_G(M)$, then $L \cap M$ is a maximal subgroup of M ([3], A, 16.5).

DEFINITION. Let $\underline{F}$ be a formation. A maximal subgroup U of G is called **$\underline{F}$ -critical** in G if:

- a) U is $\underline{F}$ -abnormal in G (that is to say $G/\text{Core}_G(U) \notin \underline{F}$), and
- b) U is critical in G .

LEMMA 2.4 ([3], IV, 1.17). Let $\underline{F}$ be a formation and $G = UN$ where $U \leq G$ and N is a normal subgroup of G . Then

- a) $U^{\underline{F}}N = G^{\underline{F}}N$, and
- b) if N is a nilpotent group, then $U^{\underline{F}} \leq G^{\underline{F}}$.

The notion of $\underline{F}$ -normalizer of G plays an important role in this work. The following proposition gives a useful characterization of $\underline{F}$ -normalizers.

PROPOSITION 2.5 [2]. Let $\underline{F}$ be a saturated formation, where $\underline{N} \subseteq \underline{F}$. A subgroup D of G is an $\underline{F}$ -normalizer of a group G if and only if

- a) $D \in \underline{F}$ and

b) D can be joined to G by an $\underline{\underline{F}}$ -critical maximal chain, namely a chain of the form

$$(1) \quad D = G_r < G_{r-1} < \dots < G_1 < G_0 = G,$$

where G_i is an $\underline{\underline{F}}$ -critical subgroup of G_{i-1} ($i = 1, \dots, r$).

We recall from [2] that each Hall system Σ of G gives rise to a unique $\underline{\underline{F}}$ -normalizer $D_{\underline{\underline{F}}}(\Sigma)$ and from [8] that $D_{\underline{\underline{F}}}(\Sigma)$ can be characterized as the $\underline{\underline{F}}$ -normalizer of G defined by the chain (1) with the additional condition that Σ reduces into each G_i for $i = 1, \dots, r$.

LEMMA 2.6. Let $\underline{\underline{F}}$ be a saturated formation such that $\underline{\underline{N}} \subseteq \underline{\underline{F}}$ and Σ a Hall system of G .

a) If M is a $\underline{\underline{F}}$ -critical subgroup of G into which Σ reduces, then

$$D_{\underline{\underline{F}}}(\Sigma) = D_{\underline{\underline{F}}}(\Sigma \cap M) \quad ([3], \text{ V, } 3.7).$$

b) If W is a well-placed subgroup of G such that Σ reduces via critical into W , then

$$D_{\underline{\underline{F}}}(\Sigma \cap W) \leq D_{\underline{\underline{F}}}(\Sigma) \quad ([3], \text{ V, } 2.7).$$

3. – The lattice $GE_{\Sigma}(G)$.

In this section, we introduce the concept *very-well-placed* and prove that the set, denoted by $GE_{\Sigma}(G)$, of the very-well-placed subgroups of a group G associated to a Hall system Σ of G forms a sublattice of the subgroup lattice of G .

DEFINITIONS. Let G be a group with nilpotent length n and denote by $L_{-1}(G)$ the $\underline{\underline{N}}^{n-1}$ -residual of G (i.e. the smallest normal subgroup N of G such that $G/N \in \underline{\underline{N}}^{n-1}$). A subgroup U of G is said to be *strongly critical* if $UL_{-1}(G) = G$.

A subgroup U of G is said to be *very-well-placed* in G , if there exists a chain $U = U_r < U_{r-1} < \dots < U_0 = G$, such that for $i = 1, \dots, r$:

- a) U_i is maximal in U_{i-1} ;
- b) U_i is strongly critical in U_{i-1} .

The next counterexample shows that the set of all very-well-placed subgroups of a group G is not closed under intersections.

EXAMPLE 3.1. Let $V := S_3$ the symmetric group of degree 3 and $K := GF(3)$.

Let A_3 be the normal Sylow 3-subgroup of S_3 . Let P_1 be the principal indecomposable projective KV -module such that $P_1/P_1J(KV) \cong K \cong \text{Soc}(P_1)$.

Set $G := [P_1]V$ the semidirect product of V with P_1 . Since $F(G) = A_3 \times P_1$, it follows that the nilpotent length of G is 2.

Set $U := P_1H$, where H is a Sylow 2-subgroup of V . Clearly $UG \cong G$.

Hence U is a strongly critical maximal subgroup of G . Since $G/P_1 \cong S_3$, there exists $g \in G$ such that $U \cap U^g = P_1$. Clearly U^g is a strongly critical maximal subgroup of G , but P_1 is not very-well-placed in G .

Therefore, we restrict our discussion to the set

$$GE_\Sigma(G) = \{U \leq G \mid \Sigma \text{ reduces via strongly critical into } U\}.$$

REMARKS 3.2. a) The embedding property very-well-placed is transitive.

b) If G is a nilpotent group, then all subgroups of G are very-well-placed.

c) If U is a strongly critical maximal subgroup of G and $R := \text{Core}_G(U)$, then the nilpotent length of G and G/R are equal. Hence U is a $N^{n(G)-1}$ -critical subgroup of G .

d) If Σ is a Hall system of G and U, V are subgroups of G such that $U \leq V \leq G$ and Σ reduces into V , then Σ reduces into U if and only if the Hall system $\Sigma \cap V$ of V reduces into U .

e) Let $U \leq G$ and Σ a Hall system of G . Then

$$U \in GE_\Sigma(G) \text{ implies } GE_{\Sigma \cap U}(U) \subseteq GE_\Sigma(G).$$

LEMMA 3.3. Let $G = UN$ with N a nilpotent normal subgroup of G , and Σ a Hall system of G which reduces into U . If $V \leq G$ is such that $U \leq V \leq G$, then Σ reduces into V .

PROOF. Since a Hall system Σ reduces into a product of permutable subgroups, into which Σ reduces, (see [3], I, 4.22 b)), then Σ reduces into V because $U(V \cap N) = V$, U and $V \cap N$ permute, and Σ reduces into U and into the subnormal subgroup $V \cap N$ of G .

LEMMA 3.4. If U and V are strongly critical maximal subgroups of the group G , such that $U \neq V$, and Σ is a Hall system reducing into U and V , then $U \cap V$ is strongly critical maximal in U and V .

PROOF. If G is a nilpotent group, then the result is trivial.

Suppose $n(G) > 1$.

We prove first that $U \cap V \triangleleft V$ as well as $U \cap V \triangleleft U$.

Since Σ reduces into the maximal and therefore pronormal subgroups U and V , it follows from ([3], I, 6.6) that U and V are not conjugate subgroups of G . Therefore by Lemma 2.3 a), $R := \text{Core}_G(U) \neq \text{Core}_G(V) =: R^*$.

Assume $R \not\leq R^*$ without loss of generality. Hence from Lemma 2.3 b), we have $U \cap V \triangleleft V$.

We show now that $U \cap V \triangleleft U$.

Since $L_{-1}(G)$ is a nilpotent group and $V \triangleleft G$, it follows that $L_{-1}(G) \cap V \trianglelefteq G$. Hence $L_{-1}(G) \cap V \leq R^*$, and therefore $V/R^* \in \underline{N}^{n(G)-1}$ because $V/(V \cap L_{-1}(G)) \cong G/L_{-1}(G) \in \underline{N}^{n(G)-1}$.

Now assume that $R^* \leq R$. Hence $V/V \cap R \in \underline{N}^{n(G)-1}$ and since $G/R \cong VR/R \cong V/V \cap R$ we have $G/R \notin \underline{N}^{n(G)-1}$, a contradiction to Remark 3.2 c). Therefore $R^* \not\leq R$ and again from Lemma 2.3 b) it follows $U \cap V \triangleleft U$. Now we prove that $U \cap V$ is a strongly critical subgroup of U . The affirmation $U \cap V$ is strongly critical in V follows with the same arguments.

We prove that $n(U) = n(G)$.

Assume for a contradiction that $n(U) < n(G)$. By Proposition 2.5, U is a $\underline{N}^{n(G)-1}$ -normalizer of G , because $U \in \underline{N}^{n(G)-1}$ and U is a $\underline{N}^{n(G)-1}$ -critical subgroup of G (see Remark 3.2 c)). Since V is a $\underline{N}^{n(G)-1}$ -critical subgroup of G , V is a $\underline{N}^{n(G)-1}$ -normalizer of G too. This implies that U and V must be conjugate, a contradiction.

Now we have $UR^* = G$ and $n(U) = n(G)$. By Lemma 2.4, $U \underline{N}^{n(G)-1} R^* = G \underline{N}^{n(G)-1} R^*$. Finally, the desired conclusion follows from

$$U = G \cap U = VL_{-1}(G)R^* \cap U = VL_{-1}(U)R^* \cap U =$$

$$VL_{-1}(U) \cap U = (V \cap U)L_{-1}(U).$$

With the next theorem we show that $GE_\Sigma(G)$ forms a lattice.

THEOREM 3.5. Let Σ be a Hall system of the group G , and U, V subgroups belonging to $GE_\Sigma(G)$. Then $U \cap V$ and $\langle U, V \rangle$ belong to $GE_\Sigma(G)$.

PROOF. Since $U, V \in \mathbf{GE}_\Sigma(G)$, there exist chains

$$U = U_r \triangleleft U_{r-1} \triangleleft \dots \triangleleft U_0 = G$$

and

$$V = V_m \triangleleft V_{m-1} \triangleleft \dots \triangleleft V_0 = G,$$

where Σ reduces into U_i ($i = 0, \dots, r$) and V_j ($j = 0, \dots, m$).

We consider two cases:

If $U \leq V_i$, then it follows trivially that $U \in \mathbf{GE}_{\Sigma \cap V_1}(V_1)$. Moreover, clearly $V \in \mathbf{GE}_{\Sigma \cap V_1}(V_1)$. We have then by induction on $|G|$ that $U \cap V$ and $\langle U, V \rangle \in \mathbf{GE}_{\Sigma \cap V_1}(V_1)$, and therefore $U \cap V$ and $\langle U, V \rangle$ belong to $\mathbf{GE}_\Sigma(G)$.

If $U \not\leq V_1$, then it follows using Lemma 3.4 and induction on $|G|$ that $U \cap V_1 \in \mathbf{GE}_\Sigma(G)$ and therefore $U \cap V_1 \in \mathbf{GE}_{\Sigma \cap V_1}(V_1)$. Again, since $V \in \mathbf{GE}_{\Sigma \cap V_1}(V_1)$ it follows by induction on the order of G that $U \cap V \in \mathbf{GE}_{\Sigma \cap V_1}(V_1)$, and thus $U \cap V \in \mathbf{GE}_\Sigma(G)$.

We prove now that $\langle U, V \rangle \in \mathbf{GE}_\Sigma(G)$.

Assume $\langle U, V \rangle \neq G$ without loss of generality.

We show first that $n(U) = n(G)$. Assume for a contradiction that $n(U) < n(G)$. We choose $k \in \{0, \dots, r\}$ so that $n(U_k) < n(G)$ and $n(U_t) = n(G)$ for all $t = 0, \dots, k-1$. By Proposition 2.5, U_k is a $\underline{N}^{n(G)-1}$ -normalizer of G and therefore of U_{k-1} . Since $U_{k-1} \cap V_1$ is $\underline{N}^{n(G)-1}$ -critical in U_{k-1} it follows that $U_{k-1} \cap V_1$ must be a $\underline{N}^{n(G)-1}$ -normalizer of U_{k-1} . Hence U_k and $U_{k-1} \cap V_1$ are conjugate in U_{k-1} . This implies that $U_k = U_{k-1} \cap V_1$ because $\Sigma \cap U_{k-1}$ reduces into U_k and $U_{k-1} \cap V_1$.

Therefore $U \leq U_k \leq V_1$, a contradiction to our assumption. The fact $n(U) = n(G)$ implies trivially $n(U_i) = n(G)$ for $i = 0, \dots, r-1$. Hence by Lemma 2.4 b),

$$L_{-1}(U) \leq L_{-1}(U_{r-1}) \leq \dots \leq L_{-1}(G).$$

Therefore

$$G = U_1 L_{-1}(G) = U_2 L_{-1}(U_1) L_{-1}(G) = U_2 L_{-1}(G) = \dots = U L_{-1}(G),$$

and then $\langle U, V \rangle L_{-1}(G) = G$.

If $\langle U, V \rangle \triangleleft G$ then the result follows.

If $\langle U, V \rangle$ is not maximal in G , then choose $L \leq G$ such that $\langle U, V \rangle < L < G$. Clearly, L is a strongly critical maximal subgroup of G . Otherwise, Σ reduces into L by Lemma 3.3. Therefore, $U, V \in \mathbf{GE}_{\Sigma \cap L}(L)$. By

induction on the order of G , $\langle U, V \rangle \in \mathbf{GE}_{\Sigma \cap L}(L)$ and thus $\langle U, V \rangle \in \mathbf{GE}_{\Sigma}(G)$.

4. – Description of the lattice $\mathbf{GE}_{\Sigma}(G)$.

In this section we describe the sublattice $\mathbf{GE}_{\Sigma}(G)$ by determining the saturated formations for which the $\underline{\underline{F}}$ -normalizers belong to $\mathbf{GE}_{\Sigma}(G)$.

DEFINITION. Let $\underline{\underline{F}}$ be a saturated formation. The maximal subgroup U of G is called *strongly $\underline{\underline{F}}$ -critical* if:

- a) U is strongly critical in G , and
- b) U is $\underline{\underline{F}}$ -abnormal in G .

THEOREM 4.1. Let $\underline{\underline{F}}$ be a saturated formation such that $\underline{\underline{N}} \subseteq \underline{\underline{F}}$. Then the following conditions are equivalent.

- a) Every $G \notin \underline{\underline{F}}$ contains a strongly $\underline{\underline{F}}$ -critical subgroup.
- b) $\underline{\underline{F}} = \underline{\underline{S}}$ or there exists $n' \in \mathbb{N}$ such that $\underline{\underline{N}}^{n'-1} \subseteq \underline{\underline{F}} \subseteq \underline{\underline{N}}^{n'}$.

PROOF. $a) \Rightarrow b)$ We show first that for every n either $(\underline{\underline{N}}^n \cap \underline{\underline{F}}) \subseteq \underline{\underline{N}}^{n-1}$ or $\underline{\underline{N}}^{n-1} \subseteq \underline{\underline{F}}$.

Assume for a contradiction that there is a natural number m such that $(\underline{\underline{N}}^m \cap \underline{\underline{F}}) \not\subseteq \underline{\underline{N}}^{m-1}$ as well as $\underline{\underline{N}}^m \not\subseteq \underline{\underline{F}}$. Let $G \in (\underline{\underline{N}}^m \cap \underline{\underline{F}}) \setminus \underline{\underline{N}}^{m-1}$ and $H \in \underline{\underline{N}}^{m-1} \setminus \underline{\underline{F}}$ be minimal counter-examples. Clearly G and H are primitive groups.

Set $X = G \times H$. Since for every saturated formation $\underline{\underline{H}}$ we have $(G \times H)\underline{\underline{H}} = G\underline{\underline{H}} \times H\underline{\underline{H}}$, then $L_{-1}(X) = L_{-1}(G)$. Let U be a stabilizer of H . Since $GU \in \underline{\underline{F}}$ and GU is a $\underline{\underline{F}}$ -critical subgroup of X , then GU is a $\underline{\underline{F}}$ -normalizer of X by Proposition 2.5. Hence all $\underline{\underline{F}}$ -normalizers of X contain $L_1(X)$ because they are conjugate to GU .

By hypothesis, X contains a strongly $\underline{\underline{F}}$ -critical subgroup V , since $X \notin \underline{\underline{F}}$. Using the characterization of $\underline{\underline{F}}$ -normalizers, we deduce that V contains a $\underline{\underline{F}}$ -normalizer of X . Furthermore, V contains $L_{-1}(X)$ too, a contradiction to the choice of V .

Then let n' be maximal such that $\underline{\underline{N}}^{n'-1} \subseteq \underline{\underline{F}}$ (if $\underline{\underline{N}}^i \subseteq \underline{\underline{F}}$ for all i , then $\underline{\underline{F}} = \underline{\underline{S}}$). Hence $\underline{\underline{N}}^{n'} \not\subseteq \underline{\underline{F}}$ and it follows that $(\underline{\underline{N}}^{n'+1} \cap \underline{\underline{F}}) \subseteq \underline{\underline{N}}^{n'}$. This implies $\underline{\underline{F}} \subseteq \underline{\underline{N}}^{n'}$. Assume for a contradiction that $\underline{\underline{F}} \not\subseteq \underline{\underline{N}}^{n'}$. Then we can

choose $G \in \underline{F} \setminus \underline{N}^{n'}$ of minimal order and thus we have $G \in (\underline{N}^{n'+1} \cap \underline{F}) \subseteq \underline{N}^{n'}$.

b) $\Rightarrow$ c) If $\underline{F} = \underline{S}$, then the result is trivial.

Assume $\underline{F} \neq \underline{S}$. Let then m be the natural number such that $\underline{N}^{m-1} \subseteq \underline{F} \subseteq \underline{N}^m$. This implies that for any $n \in \mathbb{N}$ either $\underline{N}^{n-1} \subseteq \underline{F}$ ($n \leq m$) or $(\underline{F} \cap \underline{N}^n) \subseteq \underline{N}^{n-1}$ ($n > m$).

Let $G \notin \underline{F}$.

If $\Phi(G) \neq 1$, then $G/\Phi(G)$ contains by induction on $|G|$ a strongly $\underline{F}$ -critical subgroup $M/\Phi(G)$. Hence M is a strongly $\underline{F}$ -critical of G , because $L_{-1}(G/\Phi(G)) = L_{-1}(G)\Phi(G)/\Phi(G)$.

Assume then $\Phi(G) = 1$ and set $n' = n(G)$. Hence, by hypothesis, either $(\underline{F} \cap \underline{N}^{n'}) \subseteq \underline{N}^{n'-1}$ or $\underline{N}^{n'-1} \subseteq \underline{F}$.

If $(\underline{F} \cap \underline{N}^{n'}) \subseteq \underline{N}^{n'-1}$, then a maximal complement M to $L_{-1}(G)$ is $\underline{F}$ -abnormal in G and therefore strongly $\underline{F}$ -critical in G . M would be a $\underline{F}$ -normal subgroup of G , then $G/\text{Core}_G(M) \in \underline{F} \cap \underline{N}^{n'} \subseteq \underline{N}^{n'-1}$ and thus $L_{-1}(G) \leq M$, a contradiction to the choice of M .

Assume then that $\underline{N}^{n'-1} \subseteq \underline{F}$.

Since $\Phi(G) = 1$, the Fitting subgroup of G can be decomposed as follows: $F(G) = \text{Soc}(G) = N_1 \times \dots \times N_t$, where N_i is a minimal normal subgroup of G for all $i = 1, \dots, t$.

Set $N_i^* = N_1 \dots N_{i-1} N_{i+1} \dots N_t$ for all $i = 1, \dots, t$; and let M_i be a complement to $F(G)/N_i^*$.

Then $F(G) \cap (\cap \text{Core}_G(M_i)) \leq \cap N_i = 1$. Hence $\cap \text{Core}_G(M_i) = 1$.

Now suppose that M_i is $\underline{F}$ -normal in G for all $i = 1, \dots, t$. Therefore, $G/\text{Core}_G(M_i) \in \underline{F}$ and $G \in \underline{F}$ because $\underline{F}$ is a formation. This is a contradiction to the choice of G .

Let then M_j be a $\underline{F}$ -abnormal subgroup of G . Hence M_j is $\underline{F}$ -abnormal and therefore strongly $\underline{F}$ -critical in G .

Using the same argument as Carter and Hawkes in [2], a characterization of $\underline{F}$ -normalizers may be given.

LEMMA 4.2. Let $\underline{F}$ be a saturated formation such that $\underline{N}^{n-1} \subseteq \underline{F} \subseteq \underline{N}^n$ for some $n \in \mathbb{N}$, $n > 1$. The subgroup D is a $\underline{F}$ -normalizer of G if and only if

a) $D \in \underline{F}$ and

b) there exists a chain $D = G < G_{s-1} < \dots < G_0 = G$, where G_{i+1} is a strongly $\underline{F}$ -critical subgroup of G_i ($i = 1, \dots, s-1$).

Moreover, we have $D = D_{\underline{F}}(\Sigma)$ for a Hall system Σ of G if and only if $D \in \underline{F}$ and Σ reduces via strongly $\underline{F}$ -critical into D . This may be proved by using the same arguments as A. Mann in ([8], Theorem 6).

COROLLARY 4.3. The $\underline{N}^i$ -normalizers of a group G , where $i = 1, \dots, n(G)$, are very-well-placed in G .

THEOREM 4.4. Let Σ be a Hall system of G and $n := n(G)$. Set $D^i(\Sigma) = D_{\underline{N}^i}(\Sigma)$ for $i = 1, \dots, n$, and

$$M_i = \{U \leq G \mid D^i(\Sigma) \leq U \leq D^{i+1}(\Sigma) \quad \text{for } i \in \{1, \dots, n-1\}\}.$$

Then

$$GE_{\Sigma}(G) = \left(\bigcup_{i=1}^{n-1} M_i \right) \cup \{U \leq G \mid U \leq D^1(\Sigma)\}.$$

PROOF. « $\subseteq$ ». Let $U \in GE_{\Sigma}(G)$ and $r = n(U)$.

If $r = 1$, $U \leq D^1(\Sigma)$ from Lemma 2.6 b).

Thus, we assume $r > 1$ and prove that $U \in M_{r-1}$. Again by Lemma 2.6 b) we have that $U \leq D^r(\Sigma)$.

We show now that $D^{r-1}(\Sigma) \leq U$.

Let U_i be the penultimate link of a chain of strongly critical maximal subgroups from U to G .

By Remark 3.2 c) the subgroup U_i is $\underline{N}^{n(G)-1}$ -critical in G and therefore U_1 is $\underline{N}^{r-1}$ -critical in G . Hence $D^{r-1}(\Sigma \cap U_1) = D^{r-1}(\Sigma)$ by Lemma 2.6 a).

Finally, by induction on $|U_1|$ it follows that $D^{r-1}(\Sigma) = D^{r-1}(\Sigma \cap U_1) \leq U$.

« $\supseteq$ ». If $U \leq D^1(\Sigma)$, then U is very-well-placed in $D^1(\Sigma)$ (Remark 3.2 b)). By Lemma 4.2, $D^1(\Sigma) \in GE_{\Sigma}(G)$. Hence clearly $U \in GE_{\Sigma}(G)$.

Now we assume that $D^i(\Sigma) \leq U \leq D^{i+1}(\Sigma)$ for $i \in \{1, \dots, n-1\}$. Since $D^{i+1}(\Sigma) \in GE_{\Sigma}(G)$ by Lemma 4.2, it is enough to show that $U \in GE_{\Sigma \cap D^{i+1}}(D^{i+1}(\Sigma))$.

Let $U = U_t < U_{t-1} < \dots < U_0 = D^{i+1}(\Sigma)$ be a chain of subgroups, such that U_j is maximal in U_{j-1} for $j = 1, \dots, t$.

By Proposition V, 3.13 from [3], $D^i(\Sigma)$ is an $\underline{N}^i$ -normalizer of $D^{i+1}(\Sigma)$ and therefore $D^i(\Sigma)$ is an $\underline{N}^i$ -projector of $D^{i+1}(\Sigma)$ (see [2], Theorem 5.6). Hence $D^i(\Sigma)$ is an $\underline{N}^i$ -projector of U_j ($j = 1, \dots, t$) by the persistence of projector in intermediate subgroups. Therefore,

$D^i(\Sigma)L_{-1}(U_j) = U_j$ and thus $U_{j+1}L_{-1}(U_j) = U_j$ for all $j = 0, \dots, t-1$; which means that U_{j+1} is strongly critical in U_j .

Finally, since $\Sigma \cap D^{i+1}(\Sigma)$ reduces into $D^i(\Sigma)$, we conclude by Lemma 3.3 that $\Sigma \cap D^{i+1}(\Sigma)$ reduces into U_j and therefore Σ reduces into U_j for all $j = 0, \dots, t-1$.

COROLLARY 4.6. Let n be the nilpotent length of G . The subgroup U is an $\underline{N}^i$ -normalizer of G , $i \leq n$, if and only if U is a very-well-placed $\underline{N}^i$ -maximal subgroup of G .

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