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Differential Modules Defined by Systems of Equations (*).

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1. - Introduction.

Fix $Q_1, \ldots, Q_n$, convex integral polytopes (i.e., with vertices in $\boldsymbol{Z}^n$) of dimension n in $\boldsymbol{R}^n$, and put $J_i = \boldsymbol{Z}^n \cap Q_i$. Let F be a field of characteristic 0 and consider indeterminates λ_{ij_i} indexed by $i = 1, \ldots, n$ and $j_i \in J_i$. Let $K = F(\{\lambda_{ij_i}\}_{i, j_i})$ and for $i = 1, \ldots, n$ put

$$(1.1) \qquad f_i(x_1, \ldots, x_n) = \sum_{j_i \in J_i} \lambda_{ij_i} x^{j_i} \in K[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}],$$

where $j_i = (j_i(1), \ldots, j_i(n))$ and $x^{j_i} = x_1^{j_i(1)} \ldots x_n^{j_i(n)}$. We regard f_i as the generic Laurent polynomial with Newton polytope equal to Q_i. Let

$$L = K[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]/(f_1, \ldots, f_n),$$

where $(f_1, \ldots, f_n)$ denotes the ideal of $K[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ generated by the f_i. For $g \in K[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$, we let $\bar{g}$ denote its image in L. It is not hard to show (see Section 7) that L is a field. Furthermore, the degree of L over K equals the number of points in the intersection of the toric hypersurfaces $f_i = 0$, $i = 1, \ldots, n$. By Bernstein's

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theorem [3] we thus have

$$[L:K] = n!\,M(Q_1, \ldots, Q_n), \tag{1.2}$$

where $M(Q_1, \ldots, Q_n)$ denotes the Minkowski mixed volume of $Q_1, \ldots, Q_n$.

Partial differentiation with respect to λ_{ij_i} defines a derivation of the field K that we denote by ∂_{ij_i}. By the usual procedure of implicit differentiation, these derivations extend in a unique manner to L. Thus L becomes a (left) module over the algebra $\mathcal{D} = K[\{\partial_{ij_i}\}_{i, j_i}]$ of partial differential operators with rational function coefficients. The purpose of this article is to explicitly identify L as a $\mathcal{D}$-module of hypergeometric type.

We accomplish this by completing certain aspects of the work of Katz [5], specifically, we show that L is isomorphic to a certain Dwork cohomology space (Theorem 4.1). We postpone stating this result until we have reviewed the definition of Dwork cohomology. Instead, we state here a consequence of our work, which may be more immediately accessible.

Let $e_1, \ldots, e_n$ be the standard basis of $\boldsymbol{R}^n$ and define ($i = 1, \ldots, n$, $j_i \in J_i$)

$$E_{ij_i} = (j_i, e_i) \in \boldsymbol{R}^{2n}.$$

Let B be the lattice of relations of the E_{ij_i}:

$$B = \left\{ b = (b_{ij_i}) \in \boldsymbol{Z} \times \ldots \times \boldsymbol{Z} \;\middle|\; \sum_{i=1}^{n} \sum_{j_i \in J_i} b_{ij_i} E_{ij_i} = 0 \right\}.$$

To each $b = (b_{ij_i}) \in B$ we associate the constant coefficient differential operator

$$\Box_b = \prod_{b_{ij_i} > 0} (\partial_{ij_i})^{b_{ij_i}} - \prod_{b_{ij_i} < 0} (\partial_{ij_i})^{-b_{ij_i}} \in \mathcal{D}.$$

Write $E_{ij_i} = (E_{ij_i}(1), \ldots, E_{ij_i}(2n))$. We also consider the differential operators ($k = 1, \ldots, 2n$)

$$Z_k = \sum_{i=1}^{n} \sum_{j_i \in J_i} E_{ij_i}(k)\, \lambda_{ij_i} \partial_{ij_i} \in \mathcal{D}.$$

Thus for $k = 1, \ldots, n$,

$$Z_k = \sum_{i=1}^{n} \sum_{j_i \in J_i} j_i(k)\, \lambda_{ij_i} \partial_{ij_i}$$

$$Z_{n+k} = \sum_{j_k \in J_k} \lambda_{kj_k} \partial_{kj_k}.$$

Put $\mathcal{M} = \mathcal{D} \Big/ \left(\sum_{b \in B} \mathcal{D}\Box_b + \sum_{k=1}^{2n} \mathcal{D}Z_k \right)$, a (left) $\mathcal{D}$-module of hypergeometric type. Let $\mathcal{N}$ be the $\mathcal{D}$-submodule of $\mathcal{M}$ generated by the image of all products of the form $\partial_{1j_1} \dots \partial_{nj_n}$, $j_i \in J_i$ for $i = 1, \dots, n$.

The rational function field K is obviously a $\mathcal{D}$-submodule of L. In fact, it is a direct summand. Let $\overline{K}$ be an algebraic closure of K and let G be the set of imbeddings of L into $\overline{K}$ over K. Then the «averaged» trace map $T_K : L \to K$ defined by

$$T_K(\xi) = \frac{1}{\operatorname{card}(G)} \sum_{\sigma \in G} \sigma(\xi)$$

splits the inclusion $K \hookrightarrow L$ since the σ's commute with the ∂_{ij_i}'s.

Let $C \subseteq \boldsymbol{R}^{2n}$ be the real cone generated by $\{E_{ij_i} \mid i = 1, \dots, n, \ j_i \in J_i\}$ and let $M \subset \boldsymbol{R}^{2n}$ be the monoid these lattice points generate:

$$M = \left\{ \sum_{i=1}^{n} \sum_{j_i \in J_i} b_{ij_i} E_{ij_i} \mid b_{ij_i} \in \boldsymbol{Z}_{\geq 0} \text{ for all } i, \ j_i \right\}.$$

THEOREM 1.3. *Suppose that* $M = \boldsymbol{Z}^{2n} \cap C$. *Then there is an isomorphism of* $\mathcal{D}$*-modules* $\mathcal{N} \simeq L/K$, *where* L/K *denotes the quotient of the* $\mathcal{D}$*-module* L *by its* $\mathcal{D}$*-submodule* K.

REMARK. The hypothesis of the theorem can be weakened somewhat, although we shall not address that question here. Some condition is needed, however. One can construct examples to show that without any hypothesis, $\mathcal{N}$ and L/K need not be isomorphic.

The isomorphism of the theorem can be made explicit as follows. Since $\mathcal{N}$ is generated as $\mathcal{D}$-module by the $\partial_{1j_1} \dots \partial_{nj_n}$, it suffices to give the element of L/K associated to $\partial_{1j_1} \dots \partial_{nj_n}$. Let $J \in K[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$ be the «toric» Jacobian of $f_1, \dots, f_n$, i.e.,

$$J = \det \left(x_s \frac{\partial f_r}{\partial x_s} \right)_{r, s = 1, \dots, n} . \tag{1.4}$$

Then $\overline{J}$ is an invertible element of L. The isomorphism is given by

$$\partial_{1j_1} \dots \partial_{nj_n} \mapsto \frac{\overline{x}^{j_1 + \dots + j_n}}{\overline{J}} \qquad (\text{mod } K).$$

Theorem 1.3 will be proved in Section 6.

The fact that an algebraic function satisfies a system of hypergeometric differential equations (the case $n = 1$ of Corollary 2.30 below) was first pointed out to us by B. Sturmfels. His interest and comments provided the original stimulus for our consideration of the problem treated here.

2. - Results of Katz.

The results of this section all appear in Katz [5], modulo the fact that we work in the toric case whereas Katz worked in the projective case. To our knowledge these results have not been published before (except for the hypersurface case, which is treated in [6]).

Consider the ring

$$R = K[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}, y_1, \dots, y_n].$$

We make R into a $\mathcal{D}$-module by letting ∂_{ij_i} act as $\partial/\partial\lambda_{ij_i} + x^{j_i} y_i$, where

$$\frac{\partial x_r}{\partial \lambda_{ij_i}} = \frac{\partial y_s}{\partial \lambda_{ij_i}} = 0$$

for all r, s, i, j_i. In particular,

$$\partial_{ij_i}(x^u y^v) = x^{u+j_i} y^{v+e_i}. \tag{2.1}$$

Set $S = \{1, \dots, n\}$ and define for $A \subseteq S$

$$R^A = \Big(\prod_{i \in A} y_i\Big) R.$$

The same definition makes the R^A into $\mathcal{D}$-modules.

Katz defined a $\mathcal{D}$-module homomorphism Θ: $R^S \to L$ as follows. Grade the R^A using the grading given by the y_i's, namely, if $x^u y^v \in R^A$, $v = (v_1, \dots, v_n)$, define

$$\deg(x^u y^v) = v_1 + \dots + v_n$$

and let $R^{A,(d)}$ denote the K-subspace of R^A spanned by the monomials of degree d. The map Θ will be defined inductively on the $R^{S,(d)}$. For each monomial $x^u y^v \in R^{S,(d)}$ we define $\Theta(x^u y^v)$ and extend to all of $R^{S,(d)}$ by K-linearity. Since every monomial in R^S has degree $\geqslant n$, we start with $d = n$. Define

$$\Theta(x^u y_1 \dots y_n) = \bar{x}^u / \bar{J}. \tag{2.2}$$

Now suppose Θ has been defined on $R^{S,(d-1)}$ and let $x^u y^v \in R^{S,(d)}$,

$d > n$. By (2.1) we have

$$\partial_{ij_i}(x^{u-j_i}y^{v-e_i}) = x^u y^v,$$

and since $d > n$ there is some choice of i such that $x^{u-j_i}y^{v-e_i} \in R^{S,(d-1)}$. Since we want Θ to be a homomorphism of $\mathcal{D}$-modules, set

$$\Theta(x^u y^v) = \partial_{ij_i}\Theta(x^{u-j_i}y^{v-e_i}).$$

It remains to show that Θ is well-defined, i.e., that if $x^{u-j_i}y^{v-e_i}, x^{u-j_l'}y^{v-e_l} \in R^{S,(d-1)}$, then

$$\partial_{ij_i}\Theta(x^{u-j_i}y^{v-e_i}) = \partial_{lj_l'}\Theta(x^{u-j_l'}y^{v-e_l}). \tag{2.3}$$

The first step is to reduce to the case $d = n + 1$, $i = l$. If $i \neq l$, then $d \geqslant n + 2$ and

$$x^{u-j_i-j_l'}y^{v-e_i-e_l} \in R^{S,(d-2)}. \tag{2.4}$$

Since ∂_{ij_i}, $\partial_{lj_l'}$ commute, we have by the induction hypothesis

$$\begin{aligned}\partial_{ij_i}\Theta(x^{u-j_i}y^{v-e_i}) &= \partial_{ij_i}\Theta(\partial_{lj_l'}(x^{u-j_i-j_l'}y^{v-e_i-e_l})) = \\ &= \partial_{ij_i}\partial_{lj_l'}\Theta(x^{u-j_i-j_l'}y^{v-e_i-e_l}) = \partial_{lj_l'}\partial_{ij_i}\Theta(x^{u-j_i-j_l'}y^{v-e_i-e_l}) = \\ &= \partial_{lj_l'}\Theta(\partial_{ij_i}(x^{u-j_i-j_l'}y^{v-e_i-e_l})) = \partial_{lj_l'}\Theta(x^{u-j_l'}y^{v-e_l}),\end{aligned}$$

which is the desired result.

Now suppose $i = l$. If $v_i \geqslant 3$, then (2.4) holds and the same argument can be repeated. So suppose $v_i = 2$. For each $r \in S$, $r \neq i$, choose $j_r \in J_r$. Then

$$\prod_{\substack{r=1\\ r\neq i}}^{n}(\partial_{rj_r})^{v_r-1}\left(x^{u-j_i-\sum_{\substack{r=1\\ r\neq i}}^{n}(v_r-1)j_r}y_1\dots y_n\right) = x^{u-j_i}y^{v-e_i}$$

and

$$\prod_{\substack{r=1\\ r\neq i}}^{n}(\partial_{rj_r})^{v_r-1}\left(x^{u-j_i'-\sum_{\substack{r=1\\ r\neq i}}^{n}(v_r-1)j_r}y_1\dots y_n\right) = x^{u-j_i'}y^{v-e_i}.$$

Using again the commutativity of the ∂'s and the induction hypothesis, we are reduced to showing that

$$\partial_{ij_i}\Theta\left(x^{u-j_i-\sum_{\substack{r=1\\ r\neq i}}^{n}(v_r-1)j_r}y_1\dots y_n\right) = \partial_{ij_i'}\Theta\left(x^{u-j_i'-\sum_{\substack{r=1\\ r\neq i}}^{n}(v_r-1)j_r}y_1\dots y_n\right),$$

i.e., we are reduced to proving (2.3) when $d = n + 1$ and $i = l$. From the definition of Θ when $d = n$, this means we must prove

$$\partial_{ij_i}(\bar{x}^{u-j_i}/\bar{J}) = \partial_{ij_i'}(\bar{x}^{u-j_i'}/\bar{J}). \tag{2.5}$$

Put $u' = u - j_i - j_i' - (1, \dots, 1)$ and let $J_0 = \det(\partial f_r/\partial x_s)_{r,s=1,\dots,n} \in K[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$. Equation (2.5) may be rewritten as

$$\partial_{ij_i}(\bar{x}^{j_i'}\bar{x}^{u'}/\bar{J}_0) = \partial_{ij_i'}(\bar{x}^{j_i}\bar{x}^{u'}/\bar{J}_0). \tag{2.6}$$

LEMMA 2.7. *Let $g \in K[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$ be a Laurent polynomial whose coefficients are independent of λ_{1j_1} and $\lambda_{1j_1'}$, i.e., whose coefficients are killed by ∂_{1j_1} and $\partial_{1j_1'}$. Then*

$$\bar{x}^{j_1'}\partial_{1j_1}(\bar{g}) = \bar{x}^{j_1}\partial_{1j_1'}(\bar{g}).$$

PROOF. Using the usual rules for differentiation, we are reduced to proving

$$\bar{x}^{j_1'}\partial_{1j_1}(\bar{x}_i) = \bar{x}^{j_1}\partial_{1j_1'}(\bar{x}_i) \tag{2.8}$$

for $i = 1, \dots, n$. Let A be the Jacobian matrix, i.e., the matrix whose entry in row r, column s is $\partial f_r/\partial x_s \in K[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$. The usual procedure of implicit differentiation gives

$$\bar{A}\begin{bmatrix} \partial_{1j_1}(\bar{x}_1) \\ \vdots \\ \partial_{1j_1}(\bar{x}_n) \end{bmatrix} = \begin{bmatrix} -\bar{x}^{j_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2.9}$$

Multiplying by $\bar{x}^{j_1'}$ gives

$$\bar{A}\begin{bmatrix} \bar{x}^{j_1'}\partial_{1j_1}(\bar{x}_1) \\ \vdots \\ \bar{x}^{j_1'}\partial_{1j_1}(\bar{x}_n) \end{bmatrix} = \begin{bmatrix} -\bar{x}^{j_1+j_1'} \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2.10}$$

Switching the roles of j_1, j_1', we see that both sides of (2.8) satisfy the same system of equations (2.10). Since $\det\bar{A} = \bar{J}_0 \neq 0$, they must be equal.

Applying the product formula for differentiation and Lemma 2.7 to

(2.6), we see that (2.6) is equivalent to

$$\partial_{ij_i}(\bar{x}^{j_i'}/\bar{J}_0) = \partial_{ij_i'}(\bar{x}^{j_i}/\bar{J}_0). \tag{2.11}$$

Solving (2.9) by Cramer's Rule gives

$$\begin{bmatrix} \partial_{1j_1}(\bar{x}_1) \\ \vdots \\ \partial_{1j_1}(\bar{x}_n) \end{bmatrix} = -\frac{\bar{x}^{j_1}}{\bar{J}_0}\begin{bmatrix} \bar{C}_{11} \\ \vdots \\ \bar{C}_{1n} \end{bmatrix}, \tag{2.12}$$

where C_{rs} denotes the (r, s)-cofactor of A, i.e., C_{rs} equals $(-1)^{r+s}$ times the determinant of the matrix obtained from A by deleting row r and column s. An analogous equation holds with j_1 replaced by j_1'. Applying $\partial_{1j_1'}$ to (2.12), ∂_{1j_1} to the analogous equation with j_1 replaced by j_1', and using $\partial_{1j_1}\partial_{1j_1'} = \partial_{1j_1'}\partial_{1j_1}$, we conclude that

$$\partial_{1j_1'}\left(\frac{\bar{x}^{j_1}}{\bar{J}_0}\begin{bmatrix} \bar{C}_{11} \\ \vdots \\ \bar{C}_{1n} \end{bmatrix}\right) = \partial_{1j_1}\left(\frac{\bar{x}^{j_1'}}{\bar{J}_0}\begin{bmatrix} \bar{C}_{11} \\ \vdots \\ \bar{C}_{1n} \end{bmatrix}\right). \tag{2.13}$$

But for $l = 1, \ldots, n$, C_{1l} satisfies the hypothesis of Lemma 2.7, hence

$$\frac{\bar{x}^{j_1}}{\bar{J}_0}\partial_{1j_1'}(\bar{C}_{1l}) = \frac{\bar{x}^{j_1'}}{\bar{J}_0}\partial_{1j_1}(\bar{C}_{1l}).$$

Applying the product formula for differentiation to (2.13) and using this relation gives

$$\partial_{1j_1'}\left(\frac{\bar{x}^{j_1}}{\bar{J}_0}\right)\begin{bmatrix} \bar{C}_{11} \\ \vdots \\ \bar{C}_{1n} \end{bmatrix} = \partial_{1j_1}\left(\frac{\bar{x}^{j_1'}}{\bar{J}_0}\right)\begin{bmatrix} \bar{C}_{11} \\ \vdots \\ \bar{C}_{1n} \end{bmatrix}. \tag{2.14}$$

But the column vector appearing on both sides of this equation is the first column of the adjoint matrix of $\bar{A}$, hence taking the «inner product» with the first row of $\bar{A}$ gives

$$\bar{J}_0\,\partial_{1j_1'}(\bar{x}^{j_1}/\bar{J}_0) = \bar{J}_0\,\partial_{1j_1}(\bar{x}^{j_1'}/\bar{J}_0). \tag{2.15}$$

Equation (2.11) follows since $\bar{J}_0$ is invertible in L.

REMARK. If $b \in B$, it is easily seen that $\Box_b(x^u y^v) = 0$ for all $x^u y^v \in R$. Since Θ is a $\mathcal{O}$-module homomorphism, it follows that for all $x^u y^v \in R^S$, $\Box_b(\Theta(x^u y^v)) = 0$. In particular, $\Box_b(\bar{x}^u / \bar{J}) = 0$ for all $u \in \boldsymbol{Z}^n$.

We define differential operators $D_{x_i}, D_{y_i}\colon R^A \to R^A$ for $i = 1, \ldots, n$ and all $A \subseteq S$ by

$$D_{x_i} = x_i \frac{\partial}{\partial x_i} + \sum_{k=1}^{n} y_k x_i \frac{\partial f_k}{\partial x_i}, \tag{2.16}$$

$$D_{y_i} = y_i \frac{\partial}{\partial y_i} + y_i f_i . \tag{2.17}$$

Note that $D_{y_i}(R^A) \subseteq R^{A \cup \{i\}}$. Put

$$g = y_1 f_1(x_1, \ldots, x_n) + \ldots + y_n f_n(x_1, \ldots, x_n) \in R . \tag{2.18}$$

Since formally $D_{x_i} = \exp(-g) \circ x_i \partial/\partial x_i \circ \exp g$, $D_{y_i} = \exp(-g) \circ y_i \partial/\partial y_i \circ \exp g$, and $\partial_{ij_i} = \exp(-g) \circ \partial/\partial \lambda_{ij_i} \circ \exp g$, all these operators on R commute with one another. In particular, the D_{x_i} and D_{y_i} are $\mathcal{O}$-module endomorphisms of R^A.

LEMMA 2.19. *The kernel of Θ contains $D_{x_i}(R^S)$ and $D_{y_i}(R^{S \setminus \{i\}})$ for $i = 1, \ldots, n$.*

PROOF. For this proof, we fix a choice of $j_i \in J_i$ for each $i = 1, \ldots, n$. Let $x^u y^v \in R^{S \setminus \{i\}}$. Then

$$x^u y^v = (\partial_{ij_i})^{v_i} \prod_{\substack{k=1 \\ k \neq i}}^{n} (\partial_{kj_k})^{v_k - 1} \left(x^{u - v_i j_i - \sum_{k=1, k \neq i}^{n} (v_k - 1) j_k} y_1 \cdots \hat{y}_i \cdots y_n \right).$$

Since D_{y_i} commutes with the ∂'s,

$$D_{y_i}(x^u y^v) = (\partial_{ij_i})^{v_i} \prod_{\substack{k=1 \\ k \neq i}}^{n} (\partial_{kj_k})^{v_k - 1} \left(x^{u - v_i j_i - \sum_{k=1, k \neq i}^{n} (v_k - 1) j_k} y_1 \cdots y_n f_i \right).$$

And since Θ is a $\mathcal{O}$-module homomorphism,

$$\Theta(D_{y_i}(x^u y^v)) = (\partial_{ij_i})^{v_i} \prod_{\substack{k=1 \\ k \neq i}}^{n} (\partial_{kj_k})^{v_k - 1} \left(\bar{x}^{u - v_i j_i - \sum_{k=1, k \neq i}^{n} (v_k - 1) j_k} \bar{f}_i / \bar{J} \right) = 0$$

since $\bar{f}_i = 0$ in L.

Now let $x^u y^v \in R^S$ and write

$$x^u y^v = \prod_{k=1}^{n} (\partial_{kj_k})^{v_k - 1} \left(x^{u - \sum_{k=1}^{n} (v_k - 1) j_k} y_1 \cdots y_n \right).$$

As before we then have

$$\Theta(D_{x_i}(x^u y^v)) = \prod_{k=1}^{n} (\partial_{kj_k})^{v_k - 1}\left(\Theta\left(D_{x_i}\left(x^{u - \sum_{k=1}^{n}(v_k - 1)j_k} y_1 \dots y_n\right)\right)\right).$$

Thus it suffices to prove

$$\Theta(D_{x_i}(x^u y_1 \dots y_n)) = 0 \tag{2.20}$$

for all $u \in \boldsymbol{Z}^n$.

We have

$$D_{x_i}(x^u y_1 \dots y_n) = u_i x^u y_1 \dots y_n + \sum_{k=1}^{n} y_k x_i \frac{\partial f_k}{\partial x_i} x^u y_1 \dots y_n .$$

But

$$y_k x_i \frac{\partial f_k}{\partial x_i} x^u y_1 \dots y_n = \partial_{kj_k}\left(x^{u - j_k} x_i \frac{\partial f_k}{\partial x_i} y_1 \dots y_n\right) - j_k(i)\, x^u y_1 \dots y_n ,$$

thus

$$D_{x_i}(x^u y_1 \dots y_n) = \left(u_i - \sum_{k=1}^{n} j_k(i)\right) x^u y_1 \dots y_n + \\ + \sum_{k=1}^{n} \partial_{kj_k}\left(x^{u + e_i - j_k} \frac{\partial f_k}{\partial x_i} y_1 \dots y_n\right)$$

and

$$\Theta(D_{x_i}(x^u y_1 \dots y_n)) = \left(u_i - \sum_{k=1}^{n} j_k(i)\right) \bar{x}^u / \bar{J} + \\ + \sum_{k=1}^{n} \partial_{kj_k}\left(\bar{x}^{u + e_i - j_k} \overline{\frac{\partial f_k}{\partial x_i}} / \bar{J}\right). \tag{2.21}$$

We must show this expression vanishes.

Let G be the $n \times n$ matrix whose entry in row r, column s is $\partial_{sj_s}(\bar{x}_r) \in L$. The usual procedure of implicit differentiation gives

$$\bar{A} G = -\operatorname{diag}[\bar{x}^{j_1}, \dots, \bar{x}^{j_n}], \tag{2.22}$$

where the notation on the right-hand side designates the $n \times n$ diagonal matrix whose diagonal entries are $\bar{x}^{j_1}, \ldots, \bar{x}^{j_n}$. Thus

$$\bar{J}_0 = \det \bar{A} = (-1)^n \bar{x}^{j_1 + \cdots + j_n} / \det G .$$

Since $\bar{J} = \bar{x}_1 \ldots \bar{x}_n \bar{J}_0$, we may substitute the resulting expression for $\bar{J}$ into the right-hand side of (2.21). Discarding the factor $(-1)^n$ and putting $w = u - \sum_{k=1}^{n} j_k - (1, \ldots, 1)$ to simplify notation, we get

$$(w_i + 1)\, \bar{x}^w \det G + \sum_{k=1}^{n} \partial_{kj_k} \left(\bar{x}^{w + e_i} \bar{x}^{-j_k} \overline{\frac{\partial f_k}{\partial x_i}} \det G \right). \tag{2.23}$$

Note that

$$\partial_{kj_k}(\bar{x}^{w + e_i}) = \sum_{l=1}^{n} (w + e_i)(l)\, \bar{x}^{w + e_i - e_l} \partial_{kj_k}(\bar{x}_l) .$$

Substituting into (2.23) gives the expression

$$\begin{aligned} &(w_i + 1)\, \bar{x}^w \det G + \\ &\quad + \sum_{l=1}^{n} (w + e_i)(l)\, \bar{x}^{w + e_i - e_l} \det G \sum_{k=1}^{n} \partial_{kj_k}(\bar{x}_l)\, \bar{x}^{-j_k} \overline{\frac{\partial f_k}{\partial x_i}} + \\ &\quad + \sum_{k=1}^{n} \bar{x}^{w + e_i} \partial_{kj_k} \left(\bar{x}^{-j_k} \overline{\frac{\partial f_k}{\partial x_i}} \det G \right). \end{aligned} \tag{2.24}$$

We rewrite (2.22) as

$$\bar{A}(G \operatorname{diag}[\bar{x}^{-j_1}, \ldots, \bar{x}^{-j_n}]) = -I_n ,$$

or, equivalently,

$$G(\operatorname{diag}[\bar{x}^{-j_1}, \ldots, \bar{x}^{-j_n}]\, \bar{A}) = -I_n . \tag{2.25}$$

But this equation says exactly that for $i, l = 1, \ldots, n$,

$$\sum_{k=1}^{n} \partial_{kj_k}(\bar{x}_l)\, \bar{x}^{-j_k} \overline{\frac{\partial f_k}{\partial x_i}} = -\delta_{i,l} .$$

Substituting into (2.24), we see that this expression simplifies to

$$\bar{x}^{w + e_i} \sum_{k=1}^{n} \partial_{kj_k} \left(\bar{x}^{-j_k} \overline{\frac{\partial f_k}{\partial x_i}} \det G \right). \tag{2.26}$$

Let G^* be the adjoint matrix of G, i.e., the entry in row r, column s of G^* is the (s, r)-cofactor of G. Since $GG^* = (\det G) I_n$, we have from

(2.25) that

$$\operatorname{diag}[\bar{x}^{-j_1}, \ldots, \bar{x}^{-j_n}]\, \bar{A} \det G = -G^*.$$

Thus $\bar{x}^{-j_k}\overline{(\partial f_k/\partial x_i)} \det G$ is the entry in row k, column i of $-G^*$, i.e., is the negative of the (i, k)-cofactor of G. Proving (2.26) vanishes is thus equivalent to proving

$$\sum_{k=1}^{n} \partial_{kj_k}((i, k)\text{-cofactor of } G) = 0. \tag{2.27}$$

Take $i = 1$ to fix ideas. For typographical convenience, we temporarily write ∂_k in place of ∂_{kj_k}. Set

$$\bar{X} = \begin{bmatrix} \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix}, \qquad \partial_k(\bar{X}) = \begin{bmatrix} \partial_k(\bar{x}_2) \\ \vdots \\ \partial_k(\bar{x}_n) \end{bmatrix}.$$

We must show

$$\sum_{k=1}^{n} (-1)^k \partial_k (\det[\partial_1(\bar{X}), \ldots, \partial_{k-1}(\bar{X}), \partial_{k+1}(\bar{X}), \ldots, \partial_n(\bar{X})]) = 0.$$

We rewrite the left-hand side as

$$\begin{aligned} &\sum_{k=1}^{n} (-1)^k \sum_{l<k} \det[\partial_1(\bar{X}), \ldots, \partial_{l-1}(\bar{X}), \\ &\qquad \partial_k\partial_l(\bar{X}), \partial_{l+1}(\bar{X}), \ldots, \partial_{k-1}(\bar{X}), \partial_{k+1}(\bar{X}), \ldots, \partial_n(\bar{X})] + \\ &\qquad + \sum_{k=1}^{n} (-1)^k \sum_{l>k} \det[\partial_1(\bar{X}), \ldots, \partial_{k-1}(\bar{X}), \partial_{k+1}(\bar{X}), \ldots, \partial_{l-1}(\bar{X}), \\ &\qquad \partial_k\partial_l(\bar{X}), \partial_{l+1}(\bar{X}), \ldots, \partial_n(\bar{X})]. \end{aligned} \tag{2.28}$$

Fix a pair (r, s), $1 \leqslant r < s \leqslant n$. Taking $k = s$, $l = r$ in the first double sum gives a contribution

$$\begin{aligned} &(-1)^s \det[\partial_1(\bar{X}), \ldots, \partial_{r-1}(\bar{X}), \partial_s\partial_r(\bar{X}), \partial_{r+1}(\bar{X}), \ldots, \partial_{s-1}(\bar{X}), \\ &\qquad\qquad \partial_{s+1}(\bar{X}), \ldots, \partial_n(\bar{X})], \end{aligned}$$

whereas taking $k = r$, $l = s$ in the second double sum gives a contribution

$$(-1)^r \det[\partial_1(\overline{X}), \ldots, \partial_{r-1}(\overline{X}), \partial_{r+1}(\overline{X}), \ldots, \partial_{s-1}(\overline{X}),$$
$$\partial_r \partial_s(\overline{X}), \partial_{s+1}(\overline{X}), \ldots, \partial_n(\overline{X})].$$

Keeping in mind that $\partial_r \partial_s = \partial_s \partial_r$, it is clear that these two contributions cancel. Hence the entire expression (2.28) vanishes. This completes the proof of Lemma 2.19.

For $A \subseteq S$, let

$$\wp^A = R^A \Big/ \left(\sum_{i=1}^{n} D_{x_i}(R^A) + \sum_{i=1}^{n} D_{y_i}(R^{A \setminus \{i\}}) \right).$$

We summarize the results of this section.

THEOREM 2.29. *The map Θ: $R^S \to L$ induces a surjective homomorphism of $\mathcal{D}$-modules $\overline{\Theta}$: $\wp^S \to L$.*

PROOF. The existence of $\overline{\Theta}$ follows immediately from Lemma 2.19. It is surjective because $\Theta(x^u J y_1 \ldots y_n) = \overline{x}^u$ and L is spanned as K-vector space by the $\overline{x}^u$, $u \in \boldsymbol{Z}^n$.

COROLLARY 2.30. *For $x^u y^v \in R^S$, $\Theta(x^u y^v)$ satisfies the following differential relations.*

1) *For all $b \in B$, $\Box_b(\Theta(x^u y^v)) = 0$.*
2) *For $i = 1, \ldots, n$, $Z_i(\Theta(x^u y^v)) = -u_i \Theta(x^u y^v)$.*
3) *For $i = 1, \ldots, n$, $Z_{n+i}(\Theta(x^u y^v)) = -v_i \Theta(x^u y^v)$.*

In particular, taking $v = (1, \ldots, 1)$, we have $\Box_b(\overline{x}^u / \overline{J}) = 0$ for all $b \in B$ and $Z_i(\overline{x}^u / \overline{J}) = -u_i \overline{x}^u / \overline{J}$, $Z_{n+i}(\overline{x}^u / \overline{J}) = -\overline{x}^u / \overline{J}$ for $i = 1, \ldots, n$ and all $u \in \boldsymbol{Z}^n$.

PROOF. The first assertion has already been observed in the remark following (2.15). Note that for $x^u y^v \in R^S$,

$$D_{x_i}(x^u y^v) = u_i x^u y^v + \sum_{k=1}^{n} \sum_{j_k \in J_k} j_k(i) \lambda_{kj_k} x^{u+j_k} y^{v+e_k} = u_i x^u y^v + Z_i(x^u y^v),$$

$$D_{y_i}(x^u y^v) = v_i x^u y^v + \sum_{j_i \in J_i} \lambda_{ij_i} x^{u+j_i} y^{v+e_i} = v_i x^u y^v + Z_{n+i}(x^u y^v).$$

Applying Θ to both sides and using Lemma 2.19 gives the second and third assertions.

3. - Reduction to $\widehat{R}$.

We shall eventually show that

$$\dim_K \mathcal{W}^S = n!\, M(Q_1, \ldots, Q_n) \qquad (= \dim_K L),$$

hence $\overline{\Theta}$ is an isomorphism. For this it is convenient to replace R by a closely related but more manageable ring. Let $C \subseteq \boldsymbol{R}^{2n}$ be the cone generated by $\{E_{ij_i} \mid i = 1, \ldots, n,\ j_i \in J_i\}$ and put

$$\widehat{R} = \left\{ \sum_{u,v} g_{u,v}(\lambda)\, x^u y^v \in R \,\middle|\, (u, v) \in \boldsymbol{Z}^{2n} \cap C \right\}.$$

Note that the hypothesis of Theorem 1.3 is equivalent to the requirement that $\widehat{R}$ be generated as subring of R by the $x^{j_i} y^{e_i}$, $i = 1, \ldots, n$, $j_i \in J_i$. For $A \subseteq S$ we put $\widehat{R}^A = \widehat{R} \cap R^A$. Note that the D_{x_i}, D_{y_i}, ∂_{ij_i} are stable on all $\widehat{R}^A$ and that $D_{y_i}(\widehat{R}^A) \subseteq \widehat{R}^{A \cup \{i\}}$. Put

$$\widehat{\mathcal{W}}^A = \widehat{R}^A \Big/ \left(\sum_{i=1}^n D_{x_i}(\widehat{R}^A) + \sum_{i=1}^n D_{y_i}(\widehat{R}^{A \setminus \{i\}}) \right).$$

The natural inclusion $\iota\colon \widehat{R} \hookrightarrow R$ induces $\mathcal{O}$-module homomorphisms $\bar{\iota}^A\colon \widehat{\mathcal{W}}^A \to \mathcal{W}^A$ for all $A \subseteq S$.

LEMMA 3.1. *For all $A \subseteq S$, $\bar{\iota}^A$ is surjective.*

PROOF. Given $x^u y^v \in R^A$ we need to show there exists $\xi \in \widehat{R}^A$ such that

$$x^u y^v \equiv \xi \quad \left(\bmod \sum_{i=1}^n D_{x_i}(R^A) + \sum_{i=1}^n D_{y_i}(R^{A \setminus \{i\}}) \right). \tag{3.2}$$

Let $l_1, \ldots, l_s$ be real linear forms in $2n$ variables defining the cone C, i.e., for $(\alpha, \beta) \in \boldsymbol{R}^{2n}$, $(\alpha, \beta) \in C$ if and only if $l_k(\alpha, \beta) \geqslant 0$ for $k = 1, \ldots, s$. If $l_k(u, v) \geqslant 0$ for $k = 1, \ldots, s$, then $(u, v) \in C$ so $x^u y^v \in \widehat{R}^A$ and there is nothing to prove. Suppose, say, $l_1(u, v) < 0$. Consider $l_1(D) = l_1(D_{x_1}, \ldots, D_{y_n})$, the differential operator obtained by replacing the $2n$ variables in the linear form l_1 by the differential operators $D_{x_1}, \ldots, D_{x_n}, D_{y_1}, \ldots, D_{y_n}$. A calculation gives

$$l_1(D)(x^u y^v) = l_1(u, v)\, x^u y^v + \sum_{i=1}^n \sum_{j_i \in J_i} l_1(j_i, e_i)\, \lambda_{ij_i}\, x^{u + j_i} y^{v + e_i}. \tag{3.3}$$

Consider the terms in the double sum. Since $(j_i, e_i) \in C$ for all i and j_i, either $l_1(j_i, e_i) = 0$ and the term vanishes or $l_1(u + j_i, v + e_i) > l_1(u, v)$. Also, $l_k(u + j_i, v + e_i) \geqslant l_k(u, v)$ for $k = 2, \ldots, s$. Solving (3.3) for $x^u y^v$

we get

$$(3.4) \qquad x^u y^v \equiv \sum_{u',v'} g_{u',v'}(\lambda) x^{u'} y^{v'} \left(\bmod \sum_{i=1}^{n} D_{x_i}(R^A) + \sum_{i=1}^{n} D_{y_i}(R^A) \right),$$

where for all u', v' we have $g_{u',v'}(\lambda) \in K$, $l_1(u', v') > l_1(u, v)$, and $l_k(u', v') \geqslant l_k(u, v)$ for $k = 2, \ldots, s$. Iterating this procedure, we eventually arrive at a relation of this type with $l_1(u', v') \geqslant 0$, $l_k(u', v') \geqslant l_k(u, v)$ for $k = 2, \ldots, s$ and all u', v'. Repeating the same argument successively for $l_2, \ldots, l_s$, we arrive at a relation of the type (3.4) with $l_k(u', v') \geqslant 0$ for $k = 1, \ldots, s$ and all u', v'. Since $x^u y^v \in R^A$, we have by construction that $x^{u'} y^{v'} \in R^A$ for all u', v'. Hence

$$\sum_{u',v'} g_{u',v'}(\lambda) x^{u'} y^{v'} \in \widehat{R}^A .$$

4. – The main theorem.

The results of the previous two sections give us surjections $\widehat{\wp}^S \xrightarrow{\bar{\imath}^S} \wp^S \xrightarrow{\bar{\Theta}} L$.

THEOREM 4.1. *We have* $\dim_K \widehat{\wp}^S = n!\, M(Q_1, \ldots, Q_n)$, *hence by* (1.2) *both* $\bar{\imath}^S$ *and* $\bar{\Theta}$ *are isomorphisms.*

We shall prove a slightly more general result. Fix r, $0 \leqslant r \leqslant n$ and let $T = \{1, \ldots, r\}$ ($T = \emptyset$ if $r = 0$). For $i \in T$, let $\theta_i \colon \widehat{R} \to \widehat{R}$ be the map «set $y_i = 0$», i.e., θ_i is defined by K-linearity and the condition

$$\theta_i(x^u y^v) = \begin{cases} x^u y^v & \text{if } v_i = 0, \\ 0 & \text{if } v_i > 0. \end{cases}$$

For $A \subseteq T$, put $\theta_A = \prod_{i \in T \setminus A} \theta_i$ and define $\widehat{R}_A = \theta_A(\widehat{R})$. For $B \subseteq A \subseteq T$, define $\widehat{R}_A^B = \widehat{R}_A \cap \widehat{R}^B$. Let $K.(\widehat{R}_A)$ be the Koszul complex on $\widehat{R}_A$ defined by the $n + |A|$ operators

$$D_{A, x_i} = x_i \frac{\partial}{\partial x_i} + \sum_{k \in A} y_k x_i \frac{\partial f_k}{\partial x_i} \qquad (i = 1, \ldots, n),$$

$$D_{y_i} = y_i \frac{\partial}{\partial y_i} + y_i f_i \qquad (i \in A).$$

For $i \in A$, the surjective map $\theta_i \colon \widehat{R}_A \to \widehat{R}_{A \setminus \{i\}}$ induces a surjective homomorphism of complexes $K.(\widehat{R}_A) \to K.\,(\widehat{R}_{A \setminus \{i\}})$. We denote the kernel of this homomorphism by $K.(\widehat{R}_A^{\{i\}})$. More generally, suppose $K.(\widehat{R}_A^B)$ has been defined for all $B \subsetneq A$ with $|B| \leqslant b$. Let $i \in A \backslash B$, $|B| = b$, and define $K.(\widehat{R}_A^{B \cup \{i\}})$ to be the kernel of the surjection $K.(\widehat{R}_A^B) \to \to K.(\widehat{R}_{A\backslash\{i\}}^B)$ induced by θ_i. Thus there is a short exact sequence of complexes

$$0 \to K.(\widehat{R}_A^{B \cup \{i\}}) \to K.(\widehat{R}_A^B) \to K.(\widehat{R}_{A\backslash\{i\}}^B) \to 0. \tag{4.2}$$

When $\dim_K H_l(K.(\widehat{R}_A^B))$ is finite for all l and vanishes for all but finitely many l, we define an Euler characteristic

$$\chi(A, B) = \sum_{l \geqslant 0} (-1)^l \dim_K H_l(K.(\widehat{R}_A^B)).$$

Since $\widehat{\mathfrak{W}}^S = H_0(K.(\widehat{R}_S^S))$, Theorem 4.1 is the special case $r = n$ of the following.

THEOREM 4.3. *We have*

1) $\dim_K H_l(K.(\widehat{R}_T^T)) < \infty$ *for all* l.
2) $H_l(K.(\widehat{R}_T^T)) = 0$ *for* $l > n - r$.
3) $\chi(T, T) = n! \sum_{\substack{i_1 + \ldots + i_r = n \\ i_k \geqslant 1 \text{ for all } k}} M(Q_1, i_1; \ldots; Q_r, i_r)$, *where* $M(Q_1, i_1; \ldots$ $\ldots; Q_r, i_r)$ *denotes the Minkowski mixed volume of the* n *polytopes obtained by listing the polytope* $Q_k\, i_k$ *times.*

We begin with a lemma. Let $K'.(\widehat{R}_A)$, $K'.(\widehat{R}_A^B)$ be defined analogously to $K.(\widehat{R}_A)$, $K.(\widehat{R}_A^B)$ but with the operators D_{A, x_i}, $i = 1, \ldots, n$, and D_{y_i}, $i \in A$, replaced by the operators of multiplication by $\sum_{k \in A} y_k x_i \partial f_k / \partial x_i$, $i = 1, \ldots, n$, and $y_i f_i$, $i \in A$. Let $\Delta_A \subset \boldsymbol{R}^{n + |A|}$ be the convex hull of the origin and the points $\{E_{ij_i} | i \in A, j_i \in J_i\}$ and let $\mathrm{vol}(\Delta_A)$ denote its volume with respect to Lebesgue measure on $\boldsymbol{R}^{n + |A|}$.

LEMMA 4.4. *For* $B \subsetneq A \subsetneq T$, $B \neq A$, *we have*

1) $H_l(K'.(\widehat{R}_A^B)) = 0$ *for* $l > 0$.
2) $\dim_K H_0(K'.(\widehat{R}_A^B)) = \sum_{I \subsetneq B} (-1)^{|I|} (n + |A| - |I|)! \, \mathrm{vol}(\Delta_{A \backslash I})$.

PROOF. It follows from Kouchnirenko [9, Théorème 6.1] that $y_1 f_1 + + \ldots + y_r f_r$ is nondegenerate, thus this lemma is a special case of [2, Theorem 2.17]. (Although the coefficient field of [2, Theorem 2.17] is finite, the argument given there is valid over any field.)

We regard the $\hat{R}_A^B$ as graded by the grading defined in Section 2. Since the operators defining the complex $K'_.(\hat{R}_A^B)$ are multiplication by homogeneous elements of degree 1, there is an induced grading on the complex $K'_.(\hat{R}_A^B)$. (The grading is shifted at each step of the complex so that the boundary maps have degree zero.) The grading on $\hat{R}_A^B$ gives rise in a natural way to an increasing filtration, the k-th term in the filtration being the sum of the graded pieces of degree $\leq k$. This determines in an obvious manner a filtration on the complex $K.(\hat{R}_A^B)$. It is clear from the definitions that the graded complex $K'_.(\hat{R}_A^B)$ is the associated graded of the filtered complex $K.(\hat{R}_A^B)$. Thus there is a convergent E_1 spectral sequence [11, Chapter 9] with

$$E^1_{k,l} = H_{k+l}(K'_.(\hat{R}_A^B))^{(k)},$$

$$E^\infty_{k,l} = \mathrm{gr}^{(k)} H_{k+l}(K.(\hat{R}_A^B)).$$

Assertion 1 of Lemma 4.4 implies that $E^1_{k,l} = 0$ for $k + l > 0$ or $k + l < 0$, hence all the differentials of the spectral sequence are zero. It follows that $E^1_{k,l} \simeq E^\infty_{k,l}$ for all k, l. Thus by Lemma 4.4 we have the following.

LEMMA 4.5. *For $B \subseteq A \subseteq T$, $B \neq A$, we have*

1) $H_l(K.(\hat{R}_A^B)) = 0$ *for* $l > 0$;
2) $\dim_K H_0(K.(\hat{R}_A^B)) = \sum_{I \subseteq B} (-1)^{|I|}(n + |A| - |I|)!\,\mathrm{vol}(\Delta_{A \setminus I})$.

PROOF OF THEOREM 4.3. The proof is by induction on $|T|$. When $T = \emptyset$, $\hat{R}_\emptyset^\emptyset = K$ and $D_{\emptyset, x_i} = x_i \partial/\partial x_i$ is the zero operator on K. Thus $K.(\hat{R}_\emptyset^\emptyset)$ is the complex

$$0 \to K^{\binom{n}{n}} \to K^{\binom{n}{n-1}} \to \ldots \to K^{\binom{n}{1}} \to K^{\binom{n}{0}} \to 0,$$

where all the maps are zero. Thus $H_l(K.(\hat{R}_\emptyset^\emptyset)) = K^{\binom{n}{l}}$ for all l and all assertions of the theorem are obvious. Now let $T = \{1, \ldots, r\}$ and consider the short exact sequence of complexes

$$0 \to K.(\hat{R}_T^T) \to K.(\hat{R}_T^{T\setminus\{r\}}) \to K.(\hat{R}_{T\setminus\{r\}}^{T\setminus\{r\}}) \to 0. \tag{4.6}$$

Using the long exact homology sequence and applying the induction hypothesis to $K.(\hat{R}_{T\setminus\{r\}}^{T\setminus\{r\}})$ and Lemma 4.5 to $K.(\hat{R}_T^{T\setminus\{r\}})$ shows that

$K.(\widehat{R}_T^T)$ satisfies assertions 1) and 2) of Theorem 4.3 and that

$$(4.7)\qquad \chi(T, T) = \sum_{I \subseteq T\setminus\{r\}} (-1)^{|I|} (n + r - |I|)! \operatorname{vol}(\Delta_{T\setminus I}) - \\ - n! \sum_{\substack{i_1 + \ldots + i_{r-1} = n \\ i_k \geq 1 \text{ for all } k}} M(Q_1, i_1; \ldots; Q_{r-1}, i_{r-1}).$$

LEMMA 4.8. *For $I \subseteq T$, $I \neq T$,*

$$(n + r - |I|)! \operatorname{vol}(\Delta_{T\setminus I}) = n! \sum_{\substack{\sum_{i \in T\setminus I} l_i = n \\ l_i \geq 0}} M(\{Q_i, l_i\}_{i \in T\setminus I}).$$

PROOF. We outline the proof when $I = \emptyset$, the general case being analogous. The projection of Δ_T on $\boldsymbol{R}^r$ is the simplex

$$\Sigma = \{(\sigma_1, \ldots, \sigma_r) \mid \sigma_1 + \ldots + \sigma_r \leq 1, \ \sigma_i \geq 0 \text{ for all } i\}.$$

The fiber of Δ_T over $(\sigma_1, \ldots, \sigma_r) \in \Sigma$ is the Minkowski sum $\sigma_1 Q_1 + \ldots + \sigma_r Q_r$. Thus

$$\operatorname{vol}(\Delta_T) = \int_\Sigma \operatorname{vol}(\sigma_1 Q_1 + \ldots + \sigma_r Q_r)\, d\sigma_1 \ldots d\sigma_r.$$

The lemma now follows by applying the definition of Minkowski mixed volume to express $\operatorname{vol}(\sigma_1 Q_1 + \ldots + \sigma_r Q_r)$ as a polynomial in $\sigma_1, \ldots, \sigma_r$ and then evaluating the above integral.

A straightforward calculation using this lemma shows that the first sum on the right-hand side of (4.7) equals

$$n! \sum_{\substack{i_1 + \ldots + i_r = n \\ i_k \geq 1 \text{ for } i = 1, \ldots, r-1}} M(Q_1, i_1; \ldots; Q_r, i_r).$$

Assertion 3) of Theorem 4.3 is now immediate.

5. – Relations between $\wp$-spaces.

Let $\widehat{\wp}_A^B = H_0(K.(\widehat{R}_A^B))$, i.e.,

$$\widehat{\wp}_A^B = \widehat{R}_A^B \Big/ \left(\sum_{i=1}^n D_{A, x_i}(\widehat{R}_A^B) + \sum_{i \in A} D_{y_i}(\widehat{R}_A^{B\setminus\{i\}}) \right).$$

When $A = S$, we drop the subscript S and write $\widehat{\wp}^B$ in place of $\widehat{\wp}_S^B$. When $B = \emptyset$, we drop the superscript $\emptyset$ and write $\widehat{\wp}_A$ in place of $\widehat{\wp}_A^\emptyset$. In particular, $\widehat{\mathcal{W}}_S^{\emptyset}$ will be denoted simply $\widehat{\mathcal{W}}$. This is consistent with our earlier notation.

For $B_1 \subseteq B_2 \subseteq A$, the natural inclusion ι: $\widehat{R}_A^{B_2} \hookrightarrow \widehat{R}_A^{B_1}$ induces a homomorphism of $\mathcal{O}$-modules $\widehat{\mathcal{W}}_A^{B_2} \to \widehat{\mathcal{W}}_A^{B_1}$.

LEMMA 5.1. *If $B_2 \neq A$, the above map $\widehat{\mathcal{W}}_A^{B_2} \to \widehat{\mathcal{W}}_A^{B_1}$ is injective.*

PROOF. It suffices by induction to prove the lemma when $B_2 = B_1 \cup \{i\}$. From (4.2) we have a short exact sequence of complexes

$$0 \to K.(\widehat{R}_A^{B_2}) \to K.(\widehat{R}_A^{B_1}) \to K.\ (\widehat{R}_{A\setminus\{i\}}^{B_1}) \to 0\,.$$

Since $B_2 \neq A$, we have $B_1 \neq A\setminus\{i\}$ so by Lemma 4.5 all these complexes are acyclic in positive dimension. The associated long exact homology sequence thus reduces to the short exact sequence

$$0 \to \widehat{\mathcal{W}}_A^{B_2} \to \widehat{\mathcal{W}}_A^{B_1} \to \widehat{\mathcal{W}}_{A\setminus\{i\}}^{B_1} \to 0\,, \tag{5.2}$$

which establishes the lemma.

From (4.2) we have also the short exact sequence of complexes ($i \in A$)

$$0 \to K.(\widehat{R}_A^{A}) \to K.(\widehat{R}_A^{A\setminus\{i\}}) \to K.(\widehat{R}_{A\setminus\{i\}}^{A\setminus\{i\}}) \to 0\,. \tag{5.3}$$

By Lemma 4.5 the middle complex is acyclic in positive dimension, hence the associated long exact homology sequence gives an exact sequence

$$0 \to H_1(K.(\widehat{R}_{A\setminus\{i\}}^{A\setminus\{i\}})) \to \widehat{\mathcal{W}}_A^{A} \to \widehat{\mathcal{W}}_A^{A\setminus\{i\}} \to \widehat{\mathcal{W}}_{A\setminus\{i\}}^{A\setminus\{i\}} \to 0 \tag{5.4}$$

and isomorphisms for $k \geqslant 1$

$$H_{k+1}(K.(\widehat{R}_{A\setminus\{i\}}^{A\setminus\{i\}})) \simeq H_k(K.(\widehat{R}_A^{A}))\,. \tag{5.5}$$

Suppose $A = \{\alpha_1, \ldots, \alpha_a\}$ with, say, $\alpha_1 < \ldots < \alpha_a$. Applying (5.5) inductively we have isomorphisms

$$H_a(K.(\widehat{R}_{\emptyset}^{\emptyset})) \simeq H_{a-1}(K.(\widehat{R}_{\{\alpha_1\}}^{\{\alpha_1\}})) \simeq \ldots \simeq H_1(K.(\widehat{R}_{A\setminus\{\alpha_a\}}^{A\setminus\{\alpha_a\}}))\,, \tag{5.6}$$

hence the exact sequence (5.4) becomes

$$0 \to H_a(K.(\widehat{R}_\emptyset^\emptyset)) \to \widehat{\mathcal{W}}_A^A \to \widehat{\mathcal{W}}_A^{A\setminus\{\alpha_a\}} \to \widehat{\mathcal{W}}_{A\setminus\{\alpha_a\}}^{A\setminus\{\alpha_a\}} \to 0\,. \tag{5.7}$$

We describe the image of $H_a(K.(\widehat{R}_\emptyset^\emptyset))$ in $\widehat{\mathcal{W}}_A^A$.

Since $\widehat{R}_\emptyset^\emptyset = K$, $H_a(K.(\widehat{R}_\emptyset^\emptyset))$ can be represented as

$$H_a(K.(\widehat{R}_\emptyset^\emptyset)) = \bigoplus_{\substack{I \subseteq S \\ |I| = a}} Ke_I\,, \tag{5.8}$$

where e_I is a formal symbol. Write $I = \{i_1, \ldots, i_a\}$, with $i_1 < \ldots < i_a$. For $J \subseteq I$, $|J| = k$, write $J = \{j_1, \ldots, j_k\}$, with $j_1 < \ldots < j_k$, and write $I\setminus J = \{l_1, \ldots, l_{a-k}\}$, with $l_1 < \ldots < l_{a-k}$. Define $\operatorname{sgn}(J, I) = \pm 1$ by the equation

$$\operatorname{sgn}(J, I)\, dx_{j_1} \wedge \ldots \wedge dx_{j_k} \wedge dx_{l_1} \wedge \ldots \wedge dx_{l_{a-k}} = dx_{i_1} \wedge \ldots \wedge dx_{i_a}\,.$$

To describe the image of $H_a(K.(\widehat{R}_\emptyset^\emptyset))$, it suffices by (5.8) to give the image of Ke_I. A straightforward calculation using the definition of Koszul complexes and the connecting homomorphism shows that the image of Ke_I in $H_{a-k}(K.(\widehat{R}_{\{\alpha_1, \ldots, \alpha_k\}}^{\{\alpha_1, \ldots, \alpha_k\}}))$ is the K-span of the homology class defined by the cycle

$$y_{\alpha_1} \cdots y_{\alpha_k} \sum_{\substack{J \subseteq I \\ |J| = k}} \operatorname{sgn}(J, I) \det\left(x_{j_s} \frac{\partial f_{\alpha_r}}{\partial x_{j_s}} \right)_{r, s = 1, \ldots, k} e_{I\setminus J}\,, \tag{5.9}$$

where $J = \{j_1, \ldots, j_k\}$ with $j_1 < \ldots < j_k$. Taking $k = a$, we see in particular that the image of Ke_I in $\widehat{\mathcal{W}}_A^A$ is the K-span of the homology class defined by

$$y_{\alpha_1} \cdots y_{\alpha_a} \det\left(x_{i_s} \frac{\partial f_{\alpha_r}}{\partial x_{i_s}} \right)_{r, s = 1, \ldots, a} e_\emptyset\,. \tag{5.10}$$

In the special case $A = S$, $k = a = n$, this says that the image of $H_n(K.(\widehat{R}_\emptyset^\emptyset)) = Ke_S$ in $\widehat{\mathcal{W}}^S$ is the K-span of the homology class defined by $Jy_1 \ldots y_n$, where J is as in (1.4).

By the short exact sequence (5.4) with $A = S$ and repeated use of Lemma 5.1, we get an exact sequence

$$0 \to K \cdot [Jy_1 \cdots y_n] \to \widehat{\mathcal{W}}^S \to \widehat{\mathcal{W}}\,, \tag{5.11}$$

where $[Jy_1 \cdots y_n] \in \widehat{\mathcal{W}}^S$ denotes the homology class defined by $Jy_1 \cdots y_n$.

6. - Proof of Theorem 1.3.

Assume that $M = \mathbf{Z}^{2n} \cap C$. Then by Dwork-Loeser [4] (see also [1, Theorem 4.4]), the map $\mathcal{D} \to \widehat{R}$ defined by K-linearity and the rule

$$\prod_{i=1}^{n} \prod_{j_i \in J_i} (\partial_{ij_i})^{b_{ij_i}} \mapsto x^{\sum_{i=1}^{n} \sum_{j_i \in J_i} b_{ij_i} j_i} y^{\sum_{i=1}^{n} \sum_{j_i \in J_i} b_{ij_i} e_i} \tag{6.1}$$

is a homomorphism of $\mathcal{D}$-modules that on passage to quotients induces an isomorphism $\mathcal{M} \simeq \widehat{\mathcal{W}}$, where $\mathcal{M}$ is as defined in Section 1. Now $\widehat{\mathcal{W}}^S$ is a quotient of $\widehat{R}^S$ and $\widehat{R}^S$ is generated as $\mathcal{D}$-module by the products $x^{j_1} y_1 \dots x^{j_n} y_n$, hence the image of $\widehat{\mathcal{W}}^S$ in $\widehat{\mathcal{W}}$ is generated by these same products. Under the isomorphism induced by (6.1), this image is thus isomorphic to $\mathcal{N}$, the $\mathcal{D}$-submodule of $\mathcal{M}$ generated by the image of all products of the form $\partial_{1j_1} \dots \partial_{nj_n}$.

On the other hand, the exact sequence (5.11) shows that this image is isomorphic to $\widehat{\mathcal{W}}^S / K \cdot [J y_1 \dots y_n]$. Under the isomorphism $\overline{\Theta} \circ \bar{\imath}^S \colon \widehat{\mathcal{W}}^S \simeq L$ (cf. Theorem 4.1) the homology class $[J y_1 \dots y_n]$ is mapped to $1 \in L$, hence $\widehat{\mathcal{W}}^S / K \cdot [J y_1 \dots y_n] \simeq L/K$ as $\mathcal{D}$-modules. We conclude that $\mathcal{N} \simeq L/K$ as $\mathcal{D}$-modules.

7. - Appendix.

THEOREM 7.1. *Let the notation be as in Section* 1. *For* $r = 1, \dots, n$, *the quotient ring* $K[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]/(f_1, \dots, f_r)$ *is a regular ring of dimension* $n - r$.

PROOF. For each $i = 1, \dots, n$, fix $A_i \in J_i$. Consider the ring

$$H = F[\{\lambda_{ij_i}\}_{i, j_i}, x_1, x_1^{-1}, \dots, x_n, x_n^{-1}].$$

Define an automorphism ϕ of the ring H by taking ϕ to be the identity on $F[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$ and on λ_{ij_i} for $j_i \neq A_i$ and by setting $\phi(\lambda_{iA_i}) = f_i$ for $i = 1, \dots, r$. Then ϕ induces an isomorphism

$$H/(\lambda_{1A_1}, \dots, \lambda_{rA_r}) \simeq H/(f_1, \dots, f_r),$$

thus $H/(f_1, \dots, f_r)$ is clearly a regular ring. The theorem now follows by inverting the elements of $F[\{\lambda_{ij_i}\}_{i, j_i}] \subset H$.

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