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Oscillation and asymptotic behaviour of neutral equations with distributed delay

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Oscillation and Asymptotic Behaviour of Neutral Equations with Distributed Delay.

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ABSTRACT - Consider the neutral differential equation

$$\left[x(t) + \int_0^{\sigma(t)} x(t-s)\, d_s r_1(t, s)\right]' + \int_0^{\sigma(t)} x(t-s)\, d_s r_2(t, s) = 0 .$$

The asymptotic properties of the nonoscillatory solutions of the equation are studied. Sufficient conditions are also given to guarantee that all solutions oscillate.

1. - Introduction.

In the recent few years a considerable number of papers were published, devoted to the oscillatory properties of first order linear neutral differential equations. Up to now equations of the following form have been investigated

$$[x(t) + px(t-\tau)]' + qx(t-\sigma) = 0 ,$$

$$[x(t) + p(t)\,x(t-\tau)]' + q(t)\,x(t-\sigma) = 0 ,$$

$$\left[x(t) + \sum_1^k p_i x(t-\tau_i)\right]' + \sum_1^m q_i x(t-\sigma_i) = 0 ,$$

$$\left[x(t) + \sum_1^k p_i(t)\,x(t-\tau_i)\right]' + \sum_1^m q_i(t)\,x(t-\sigma_i) = 0 .$$

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To these equations the papers [1]-[3], [6]-[10] were devoted. We shall note that nonautonomous neutral differential equations with distributed delay have not been studied up to now.

In the present paper the equation

$$(1)\qquad \left[x(t) + \int_0^{\sigma(t)} x(t-s)\,d_s r_1(t,s)\right]' + \int_0^{\sigma(t)} x(t-s)\,d_s r_2(t,s) = 0$$

is investigated with initial function $\varphi(t) \in C([a, t_0], \mathrm{R})$, $(a = \inf_{t \geqslant t_0}\{t - \sigma(t)\})$, where the integrals in (1) are in the sense of Riemann-Stieltjes. Some ideas of [2] are developed, the asymptotic behaviour of (1) is investigated and sufficient conditions for oscillation of all solutions of (1) are obtained.

2. - Preliminary notes.

We shall say that conditions (A) are met if the following conditions hold:

$A1)$ $\sigma(t) \in C([t_0, \infty), (0, \infty))$,

$A2)$ $\lim_{t\to\infty} (t - \sigma(t)) = \infty$.

Conditions (A) imply that $a = \inf_{t \geqslant t_0} (t - \sigma(t)) > -\infty$.

Introduce conditions (B):

$B1)$ $x \in C([a, \infty), \mathrm{R})$,

$B2)$ $x(t) + \int_0^{\sigma(t)} x(t-s)\,d_s r_1(t,s) \in C^1([t_0, \infty), \mathrm{R})$.

Consider equation (1) with the initial condition

$$(2)\qquad x(t) = \varphi(t), \qquad t \in [a, t_0].$$

DEFINITION 1. The function $x(t)$, satisfying conditions (B) is said to be a solution of the initial value problem (1)-(2) if $x(t)$ satisfies (1) for $t \geqslant t_0$ and if the relation (2) holds.

Introduce the following conditions (C):

$C1)$ $r_i(t, 0) = 0$, $t \in [t_0, \infty)$, $i = 1, 2$.

$C2)$ $r_i(t, \sigma(t)) \in C([t_0, \infty), \mathrm{R})$, $i = 1, 2$.

$C3)$ $r_1(t, s)$ is continuous at $s = 0$ for any fixed $t \in [t_0, \infty)$.

$C4$) The functions $v_i(t) = \sup_{s \in [0, \sigma(t)]} |r_i(t, s)|$, $t \geq t_0$, $i = 1, 2$ are bounded.

$C5$) For any fixed $t \geq t_0$, $r_i(t, s)$ are functions of bounded variation with respect to s in $[0, \sigma(t)]$.

$C6$) $\lim_{t_1 \to t} \int_0^{\min\{\sigma(t_1), \sigma(t)\}} |r_i(t_1, s) - r_i(t, s)| ds = 0$, $i = 1, 2$.

LEMMA 1. Let conditions (A) and (C) hold. Then for any initial function $\varphi(t) \in C([a, t_0], \mathrm{R})$ the initial value problem (1)-(2) has an unique solution.

Lemma 1 is obtained as a corollary of [4].

Introduce the following conditions (D):

$D1$) $r_1(t, s)$ is nonincreasing with respect to s for $s \in [0, \sigma(t)]$.

$D2$) $r_2(t, s)$ is nondecreasing with respect to s for $s \in [0, \sigma(t)]$.

$D3$) $\int_{t_0}^{\infty} r_2(t, \sigma(t))\, dt = \infty$.

DEFINITION 2. The solution $x(t)$ of (1) is said to oscillate if there exists an increasing sequence $\{t_n\}_1^\infty$, such that $\lim_{n \to \infty} t_n = \infty$ and $x(t_n) = 0$, $n \in \mathrm{N}$. Otherwise it is said to be nonoscillatory.

DEFINITION 3. The function $x(t)$ is said to eventually have the property K, if there exists t_0 such that for $t \geq t_0$ the function has the property K.

By Definition 3 the nonoscillatory solutions of (1) are characterized as being eventually positive or eventually negative.

Let

$$z(t) = x(t) + \int_0^{\sigma(t)} x(t - s)\, d_s r_1(t, s). \tag{3}$$

Then

$$z'(t) = - \int_0^{\sigma(t)} x(t - s)\, d_s r_2(t, s). \tag{4}$$

We shall prove several lemmas which are essentially used in the proof of the main results.

LEMMA 2. Let conditions (A), (C) and (D) hold and let

$$r_1(t, \sigma(t)) \geqslant p > -1. \tag{5}$$

If $x(t)$ is an eventually positive solution of (1), then $x(t)$ is a bounded function.

PROOF. Let $x(t)$ be an eventually positive solution of (1). From (4) it follows that $z'(t) \leqslant 0$ eventually and $z(t)$ is a nonincreasing function. $D3)$ implies that $z(t)$ is not an eventually constant function and thus either $z(t) < 0$ or $z(t) > 0$ eventually. Suppose that $z(t) < 0$. Then from (3), (5), $C1)$ and $D1)$ there follows the estimate

$$0 > x(t) + \int_0^{\sigma(t)} x(t-s)\, d_s r_1(t, s) \geqslant x(t) - \max_{[t-\sigma(t),\, t]} x(s) v_1(t) =$$

$$= x(t) + \max_{[t-\sigma(t),\, t]} x(s) r_1(t, \sigma(t)) > x(t) - \max_{[t-\sigma(t),\, t]} x(s).$$

From the above inequalities it follows that there exists $t_1 > t_0$, such that for $t > t_1$ the inequality

$$x(t) < \max_{[t-\sigma(t),\, t]} x(s) \tag{6}$$

holds. By virtue of condition $A2)$ we can choose $\bar{t}$ such that $t - \sigma(t) \geqslant t_0$ for $t \geqslant \bar{t}$. Then from (6) we have

$$x(t) < \max_{[t-\sigma(t),\, t]} x(s). \tag{7}$$

Suppose that $x(t)$ is unbounded. Then $\limsup_{t\to\infty} x(t) = \infty$ and there exists a sequence $\{t_n\}_{n=1}^{\infty}$, such that $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to\infty} x(t_n) = \infty$ and $\max_{[t_0,\, t_n]} x(s) = x(t_n)$. The last inequality however contradicts (7).

Let $z(t) > 0$ eventually. Since $z(t)$ is a nonincreasing function, there exists the finite limit $\lim_{t\to\infty} z(t) = c \geqslant 0$. We shall prove that $\liminf_{t\to\infty} x(t) = 0$. Suppose that this is not true, i.e. $d = \liminf_{t\to\infty} x(t) > 0$. There exists $\bar{\bar{t}} \geqslant \bar{t}$ such that $x(t) > d/2$ for $t \geqslant \bar{\bar{t}}$. From (4) and $D2)$ it follows that

$$z'(t) \leqslant -\min_{[t-\sigma(t),\, t]} x(s) r_2(t, \sigma(t)).$$

By virtue of $A2)$ we can choose $\tilde{t}$ such that $t - \sigma(t) > \bar{\bar{t}}$ for $t \geqslant \tilde{t}$. Then from the above estimate we have

$$z'(t) < -\frac{d}{2} r_2(t, \sigma(t)), \qquad t \geq \tilde{t}.$$

Integrate the last inequality from $\tilde{t}$ to t and obtain

$$z(t) \leq z(\tilde{t}) - \frac{d}{2} \int_{t}^{\tilde{t}} r_2(s, \sigma(s))\, ds.$$

$D3$) implies that $\lim_{t \to \infty} z(t) = -\infty$, which contradicts the inequality $z(t) > 0$ eventually. Thus $\liminf_{t \to \infty} x(t) = 0$. Suppose that $x(t)$ is an unbounded function. As above we choose a sequence $\{t_n\}_1^\infty$ with the respective properties. Since $\liminf_{t \to \infty} x(t) = 0$, there exists a sequence $\{\tau_k\}_1^\infty$, such that $\lim_{k \to \infty} \tau_k = \infty$ and $\lim_{k \to \infty} x(\tau_k) = 0$. Let $n, k \in \mathbb{N}$ be large enough and such that $t_n > \tau_k$. Then the following estimate is valid:

$$z(t_n) - z(\tau_k) > x(t_n) - x(\tau_k) + \int_0^{\sigma(t_n)} x(t_n - s)\, d_s r_1(t_n, s) \geq$$

$$\geq x(t_n) - x(\tau_k) + \max_{[t_n - \sigma(t_n),\, t_n]} x(s) r_1(t_n, \sigma(t_n)) \geq x(t_n) - x(\tau_k) + px(t_n).$$

Thus we have

$$z(t_n) - z(\tau_k) > x(t_n)(1 + p) - x(\tau_k).$$

The choice of the sequences $\{t_n\}$ and $\{\tau_k\}$ and (5) imply that $z(t_n) - z(\tau_k) > 0$ and since $t_n > \tau_k$, we get to a contradiction with the fact that $z(t)$ is an eventually nonincreasing function. ■

LEMMA 3 ([5]). Let $p, \tau \in C([t_0, \infty), (0, \infty))$ and $\lim_{t \to \infty} (t - \tau(t)) = \infty$. If

$$\liminf_{t \to \infty} \int_{t - \tau(t)}^{t} p(s)\, ds > \frac{1}{e},$$

then the inequality

$$y'(t) + p(t) y(t - \tau(t)) \leq 0$$

has no eventually positive solutions.

3. - Main results.

THEOREM 1. Let conditions (*A*), (*C*), (*D*) and (5) hold. Then each non-oscillatory solution $x(t)$ of (1) tends to 0 as $t \to \infty$.

PROOF. Let $x(t)$ be an eventually positive solution of (1). From Lemma 2 it follows that the function $x(t)$ is bounded. Then $b = \limsup_{t\to\infty} x(t) < \infty$. We shall prove that $b = 0$. Suppose that this is not true and choose $\varepsilon > 0$ so that $\varepsilon < (b(1+p))/(2-p)$. There exists a sequence $\{t_n\}_1^\infty$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} x(t_n) = b$. Then we can choose $\bar{t} \geq t_0$ and N so that $|x(t_n) - b| < \varepsilon$ for $n > N$ and $x(t) - b < \varepsilon$ for $t > \bar{t}$. Since $x(t)$ is a bounded function, $z(t)$ is bounded too. Then the fact that $z(t)$ is a nonincreasing function implies the existence of the finite limit $\lim_{t\to\infty} z(t) = c$. As in the proof of Lemma 2 it is shown that $\liminf_{t\to\infty} x(t) = 0$ and then we can choose a sequence $\{\tau_k\}_1^\infty$ such that $\lim_{k\to\infty} \tau_k = \infty$ and $\lim_{k\to\infty} x(\tau_k) = 0$. There exists K such that for $k > K$ we have $x(\tau_k) < \varepsilon$. From the sequences $\{t_n\}_1^\infty$ and $\{\tau_k\}_1^\infty$ choose the pair τ_j, t_i so that $j > K$, $i > N$, $\tau_j < t_i$ and $t_i - \sigma(t_i) > t_0$. Then the following estimate is valid:

$$
\begin{aligned}
z(t_i) - z(\tau_j) &= x(t_i) - x(\tau_j) + \\
&\quad + \int_0^{\sigma(t_i)} x(t_i - s)\, d_s r_1(t_i, s) - \int_0^{\sigma(\tau_j)} x(\tau_j - s)\, d_s r_1(\tau_j, s) \geq \\
&\geq x(t_i) - x(\tau_j) + \int_0^{\sigma(t_i)} x(t_i - s)\, d_s r_1(t_i, s) \geq \\
&\geq x(t_i) - x(\tau_j) + \max_{[t_i - \sigma(t_i),\, t_i]} x(s)\, r_1(t_i, \sigma(t_i)) \geq \\
&\geq b - \varepsilon - \varepsilon + p(b + \varepsilon) > 0\,.
\end{aligned}
$$

(The last inequality follows from the choice of ε.) Thus, for $\tau_j < t_i$ we obtained that $z(\tau_j) < z(t_i)$, which contradicts the fact that the function $z(t)$ is nonincreasing. Hence $\limsup_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} x(t) = 0$. If $x(t)$ is an eventually negative solution of (1), then since (1) is a linear equation, $-x(t)$ is an eventually positive solution of (1), which implies that in this case as well $\lim_{t\to\infty} x(t) = 0$. ■

In the same way the following theorem is proved:

THEOREM 2. Let conditions (*A*), (*C*), *D*2) and *D*3) hold and let $r_1(t, s)$ be nondecreasing with respect to s for $s \in [0, \sigma(t)]$.

If $r_1(t, \sigma(t)) \leqslant p < 1$, then each nonoscillatory solution of (1) tends to 0 as $t \to \infty$.

REMARK 1. Theorem 1 generalizes or generalizes and extends a number of known results, for instance Theorem 3 iv) ([2]), Theorem 1 ([10]), Theorem 5 ([7]), Corollary 3*b* ([1]).

REMARK 2. The condition $r_1(t, \sigma(t)) \leqslant p < 1$ in Theorem 2 is essential. We shall illustrate this fact with the following example:

EXAMPLE 1 [8]. Consider the equation

$$[x(t) + x(t-2)]' + q(t)x(t) = 0, \tag{8}$$

where

$$q(t) = \frac{(1/t^2) + 1/(t-2)^2}{\psi(t) + 1/t}$$

and $\psi(t)$ is the 4-periodic function

$$\psi(t) = \begin{cases} 0, & t \in [0, 1], \\ t-1, & t \in (1, 2], \\ 1, & t \in (2, 3], \\ 4-t, & t \in (3, 4]. \end{cases}$$

Clearly, equation (8) is a particular case of (1), moreover $r_1(t, \sigma(t)) \equiv 1$. It is immediately verified that $x(t) = \psi(t) + 1/t$ is a nonoscillatory solution of (8), yet the limit $\lim_{t \to \infty} x(t)$ does not exist. On the other hand all conditions of Theorem 2, except the condition $r_1(t, \sigma(t)) \leqslant p < 1$ are met.

The question whether the assertion of Theorem 2 is still valid without the condition $r_1(t, \sigma(t)) \leqslant p < 1$, if *D*3) is replaced with the more restrictive condition $r_2(t, \sigma(t)) \geqslant m > 0$, $t \geqslant t_0$, is open. For example, this is true (Theorem 2 ([2])) for the equation

$$[x(t) + p(t)x(t-\tau)]' + q(t)x(t-\sigma) = 0,$$

where p, q are continuous, nonnegative functions and τ, $\sigma \geqslant 0$. (In this case $D3)$ reduces to $\int_{t_0}^{\infty} q(t)\,dt = \infty$ and the condition $r_2(t, \sigma(t)) \geqslant$ $\geqslant m > 0$ reduces to $q(t) \geqslant q > 0$.)

Define the function $\tau(t)$

$$\tau(t) = \sup\{s \in [0, \sigma(t)]/r_2(t, s) = 0\}, \qquad t \geqslant t_0 .$$

THEOREM 3. Let conditions (A), (C), (D) and (5) hold and let $\tau(t) \in$ $\in C([t_0, \infty), (0, \infty))$. If

$$\liminf_{t \to \infty} \int_{t-\tau(t)}^{t} r_2(s, \sigma(s))\,ds > \frac{1}{e}, \tag{9}$$

then each solution of (1) is oscillatory.

PROOF. Suppose that (1) has at least one nonoscillatory solution $x(t)$. Without loss of generality, let $x(t)$ be eventually positive. From Theorem 1 it follows that $\lim_{t \to \infty} x(t) = 0$. (3), (5) and $D1)$ imply that $\lim_{t \to \infty} z(t) = 0$. On the other hand, $z(t)$ is an eventually nonincreasing nonconstant function. Hence $z(t) > 0$ eventually. (3) implies that $z(t) <$ $< x(t)$ and then from (1) we have

$$z'(t) + \int_0^{\sigma(t)} z(t-s)\,d_s r_2(t, s) \leqslant 0 .$$

By virtue of the definition of $\tau(t)$, the last inequality takes the form

$$0 \geqslant z'(t) + \int_{\tau(t)}^{\sigma(t)} z(t-s)\,d_s r_2(t, s) \geqslant z'(t) + z(t - \tau(t))\,r_2(t, \sigma(t)) .$$

Thus we obtained that the eventually positive function $z(t)$ is a solution of the inequality

$$y'(t) + r_2(t, \sigma(t))\,y(t - \tau(t)) \leqslant 0 .$$

Then, by condition (9) we get to a contradiction with the assertion of Lemma 3. ■

REMARK 3. Theorem 3 generalizes Theorem 5 ([2]).

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