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## On Self-Centralizing Sylow Subgroups of Order Four.

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### 1. – Introduction.

A well-known result of Gorenstein and Walter [3], which confirms a conjecture of R. Brauer [1], states that if  $G$  is a finite group of order  $4g$ ,  $g$  odd, with a self-centralizing Sylow 2-subgroup, then it contains a normal subgroup  $N$  of odd order such that  $G/N$  is isomorphic either to a Sylow 2-subgroup of  $G$  or to  $PSL(2, q)$  where  $q$  is a prime power,  $q \equiv 3, 5 \pmod{8}$ . Moreover (see [2, p. 348 and p. 356]),  $N$  must be metanilpotent.

The objective of this paper is to improve upon this result in the non soluble case by giving more precise information on the structure of  $N$ . Indeed, we have:

**THEOREM** *Let  $G$  be a non soluble finite group with a self-centralizing Sylow 2-subgroup of order 4 and let  $N = O(G)$  be the maximal normal subgroup of  $G$  of odd order. Then one of the following holds:*

- a)  $G/N \cong PSL(2, q)$  with  $q = p^f > 5$ ,  $q \equiv 3, 5 \pmod{8}$ , and  $N$  is a  $p$ -group;*
- b)  $G/N \cong PSL(2, 5)$  and  $N$  is nilpotent.*

This result is best possible in the sense that, in Case *b*) of the theorem,  $N$  need not be a 5-group (see Example 2.4). Moreover the derived length of  $N$  in *a*) and *b*) is not bounded. In fact in Example 3.3. groups  $G$  are constructed with a self-centralizing Sylow 2-subgroup of order 4

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such that  $N$  is a  $p$ -group of arbitrary derived length and  $G/N \cong \cong PSL(2, p)$  where  $p$  is a prime such that  $p \equiv 3, 5 \pmod{8}$ .

In the course of the proof of the theorem, we shall analyze actions of  $Q = PSL(2, p^f)$  on a group  $N$  of odd order such that a Sylow 2-subgroup of  $Q$  acts fixed point freely on  $N$ . Note that a series of papers deal with a somewhat similar situation in which the orders of  $Q$  and  $N$  are coprime, and  $C_N(Q) = 1$ , see for example [9], [13] and the references given there. By contrast with our result, in that situation  $N$  need not be nilpotent.

In [4] it was shown that if  $Q = PSL(2, 2^f)$ ,  $f \geq 2$ , acts on a 2-group  $N$  such that an element of order 3 acts fixed point freely, then  $N$  is elementary abelian. If  $f = 2$  and if an element of order 5 acts fixed point freely on  $N$ , then the nilpotent class of  $N$  is  $\leq 3$  (see [5] and [10]). In these cases,  $N$  is of bounded nilpotent class (note that this contrasts with Example 3.3).

In the following, we denote by  $G = [N]H$  that  $G$  is a split extension of its normal subgroup  $N$  by a complement  $H$ . Moreover,  $A_n$  is the alternating group on  $n$  letters, hence  $A_4 \cong PSL(2, 3)$  and  $A_5 \cong PSL(2, 5)$ . In addition,  $p$  will denote a prime and  $q$  is always a power of  $p$ . The order of the element  $g$  is denoted by  $o(g)$ .

All other notation is standard and can be found in [2], [6] and [8], for example. All groups in this paper are finite.

## 2. - Actions of $PSL(2, q)$ .

We start by noting some well-known facts that will be used several times in the sequel.

**LEMMA 2.1.** *Let  $G$  be a group and let  $N = O(G)$ . Assume that  $G/N \cong PSL(2, q)$  where  $q \equiv 3, 5 \pmod{8}$ . If  $S \in \text{Syl}_2(G)$  and  $C_G(S) = S$ , then  $|N| = |C_N(\sigma)|^3$  for every involution  $\sigma \in S$ .*

**PROOF.** Since all involutions of  $G$  are conjugate, the result follows from [2, p. 347]. ■

**PROPOSITION 2.2.** *Let  $G$  be a Frobenius group with kernel  $N$  and let  $F$  be a finite field of characteristic  $r$  not dividing  $|N|$ . Let  $M$  be an  $FG$ -module and assume that  $C_M(N) = 0$ . Let  $A$  be a Frobenius complement for  $G$ . Then  $M$  has a basis which is permuted by  $A$  with orbits of size  $|A|$ . In particular, if  $|M| = r^t$ , we have  $|C_M(A)| = r^{t/|A|}$ .*

**PROOF.** See [8, p. 270]. ■

We now deal with extensions of a group  $N$  of odd order by  $PSL(2, q)$ , which have a self-centralizing Sylow 2-subgroup. The following result can be read off from the (modular) character table of  $PSL(2, q)$  and may already be known. We present here an elementary proof.

**PROPOSITION 2.3.** *Let  $G = PSL(2, q)$  with  $q = p^f$ ,  $q \equiv 3, 5 \pmod{8}$  and assume  $q > 5$ . Let  $S \in \text{Syl}_2(G)$  and let  $M$  be a nontrivial and irreducible module for  $G$  over a finite field  $F$  of characteristic  $r$ , where  $r \neq 2$ ,  $r \neq p$ . Then  $C_M(S) \neq 0$ .*

**PROOF.** We proceed by way of contradiction. Suppose that  $C_M(S) = 0$ . First, Lemma 2.1 implies  $|M| = r^{3h}$  for some positive integer  $h$ , and for each involution  $\sigma \in S$ , we have  $|C_M(\sigma)| = r^h$ . Take  $u \in G$  with  $o(u) = (q-1)/2$ . Since  $q \not\equiv 3$ , we have that  $N_G(\langle u \rangle)$  is a dihedral group and we can write  $u = \gamma_1 \gamma_2$  where  $\gamma_1, \gamma_2$  are suitable involutions in  $N_G(\langle u \rangle)$ . We have  $[M, \gamma_1] \cap [M, \gamma_2] \leq C_M(u)$ . Set  $C^* = [M, \gamma_1] \cap [M, \gamma_2]$ . Then  $|C^*| \geq r^h$ , since  $|[M, \gamma_i]| = r^{2h}$  for  $i = 1, 2$ . Now let  $P$  be a Sylow  $p$ -subgroup of  $G$ , normalized by  $u$ . Then  $M_1 := [M, P]$  and  $M_2 := C_M(P)$  are invariant under the action of  $u$ . Since  $M = M_1 \oplus M_2$ , we have  $C_M(u) = C_{M_1}(u) \oplus C_{M_2}(u)$ . In particular, every element of  $C^*$  can be written in a unique way as sum of an element of  $C_{M_1}(u)$  and an element of  $C_{M_2}(u)$ . We have  $C^* \cap C_{M_2}(u) = 0$ . In fact, if  $x \in C^* \cap C_{M_2}(u)$  then  $\langle x \rangle$  is invariant by  $P$  and  $N_G(u)$ . So  $\langle x \rangle$  is invariant for  $G$ . Since  $M$  is irreducible for  $G$ , we get  $x = 0$ . Hence we have  $|C_{M_1}(u)| \geq r^h$ . On the other hand, since  $M_1$  is a faithful module for  $N_G(P)$  which satisfies the conditions of Proposition 2.2, it follows that  $|C_{M_1}(u)| = r^{6h/(q-1)}$ , a final contradiction. ■

It may be observed that there are modules for  $A_4$  and  $A_5$  such that the previous proposition does not hold:

**EXAMPLE 2.4.** *a) Let  $F$  be a field of characteristic different from 2 and let*

$$G = \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$$

*with entries in  $F$ . Then  $G \cong A_4$ . Let  $M$  be the natural vector space on which  $G$  acts and let  $S \in \text{Syl}_2(G)$ . Then  $M$  is an irreducible and faithful  $FG$ -module and we have  $C_M(S) = 0$ .*

*b) Now let  $G \cong A_5$ , choose  $P \in \text{Syl}_5(G)$  and set  $H = N_G(P)$ . Let  $F$*

be a field of characteristic different from 2. Suppose that  $M_1$  is the non-trivial  $FH$ -module of dimension 1. Let  $M = M_1^G$  be the induced module. It is easy to prove that if  $S \in \text{Syl}_2(G)$ , then  $C_M(S) = 0$ . So  $C_{\bar{M}}(S) = 0$  for every composition factor  $\bar{M}$  of  $M$ .

If we take two such modules  $M_1, M_2$  over fields of different odd characteristic, then the natural split extension of  $M_1 \oplus M_2$  by  $G$  shows that in part b) of the theorem, the group  $N$  need not be a  $p$ -group for no prime whatsoever.

The following result provides a criterion when every proper subgroup of  $PSL(2, q)$  has trivial intersection with at least one Sylow 2-subgroup. It will be seen in the proof of 2.6 that the strange-looking hypothesis of the following lemma is satisfied in our case.

LEMMA 2.5. *Let  $G = PSL(2, q)$ , where  $q \equiv 3, 5 \pmod{8}$  and assume that  $q > 5$ . Suppose that the proper subgroups of  $G$  are either soluble or isomorphic to  $A_5$ . If  $H < G$ , then there exists  $S \in \text{Syl}_2(G)$  such that  $H \cap S = 1$ .*

PROOF. Of course it suffices to consider the case when  $|H|$  is even. By Dickson's theorem (see [6, p. 213 f.]) and our hypothesis, the subgroups of  $G$  are the following:

- 1) Dihedral groups  $D_z$  of order  $2z$  with  $z \mid (q \pm 1)/2$ .
- 2) Groups isomorphic to  $A_4$ .
- 3) Groups isomorphic to  $A_5$ .
- 4) A subgroup  $Q$  of  $N_G(P)$  where  $P \in \text{Syl}_p(G)$  if  $q \equiv 5 \pmod{8}$ .

If  $\sigma$  is an involution of  $G$  and  $S, \bar{S}$  are Sylow 2-subgroups of  $G$  satisfying  $\sigma \in S \cap \bar{S}$ , then  $\langle S, \bar{S} \rangle \leq C_G(\sigma)$ . If we denote by  $n_\sigma$  the number of Sylow 2-subgroups containing  $\sigma$ , we have:

- a)  $n_\sigma = (q + 1)/4$  if  $q \equiv 3 \pmod{8}$ ,
- b)  $n_\sigma = (q - 1)/4$  if  $q \equiv 5 \pmod{8}$ .

Now let  $\mu_H$  be the number of involutions which are contained in the subgroup  $H$  of  $G$ . Then we have:

$$\begin{aligned} \mu_{D_z} &\leq \frac{q+1}{2} \quad \text{if } z \mid \frac{q \pm 1}{2}, \\ \mu_{A_4} &= 3, \\ \mu_{A_5} &= 15, \\ \mu_Q &\leq q \quad \text{when } q \equiv 5 \pmod{8}. \end{aligned}$$

So if  $\mu^*$  is the maximum number of involutions which are contained in a proper subgroup of  $G$ , we have:

$$\mu^* \leq \max \{q, 15\} \quad \text{if } q \equiv 5 \pmod{8} \text{ and}$$

$$\mu^* \leq \max \left\{ \frac{q+1}{2}, 15 \right\} \quad \text{if } q \equiv 3 \pmod{8}.$$

If  $m$  denotes the number of Sylow 2-subgroups which intersect non trivially with a proper subgroup  $H$  of  $G$ , we have:

$$m \leq \mu^* \cdot n_\sigma \leq \begin{cases} \mu^* \cdot \frac{q+1}{4} & \text{if } q \equiv 3 \pmod{8}, \\ \mu^* \cdot \frac{q-1}{4} & \text{if } q \equiv 5 \pmod{8}. \end{cases}$$

On the other hand, the number  $\gamma$  of Sylow 2-subgroups of  $G$  is equal to  $q(q+1)(q-1)/24$ . By an easy calculation we get  $m < \gamma$  for  $q \geq 11$ . Thus if  $q > 5$  and  $H$  is a proper subgroup of  $G$ , there is a Sylow 2-subgroup  $S$  such that  $H \cap S = 1$ . ■

The next two results deal with modules for the groups occurring in the theorem. They will be used to exclude nonnilpotent normal subgroups  $O(G)$ .

**PROPOSITION 2.6.** *Let  $\mathfrak{A}$  be the class of all groups  $G$  for which there exists a normal subgroup  $N$  of  $G$  such that:*

- a)  *$N$  is an abelian  $p$ -group (possibly the identity) where  $p \neq 2$ ;*
- b)  *$G/N \cong \text{PSL}(2, q)$  where  $q = p^f$ ,  $q \neq 3, 5$  and  $q \equiv 3, 5 \pmod{8}$ .*

*If  $G \in \mathfrak{A}$  and  $M$  is an  $FG$ -module where  $F$  is a finite field of characteristic  $\neq 2, \neq p$ , then for all  $S \in \text{Syl}_2(G)$  we have  $C_M(S) \neq 0$ .*

**PROOF.** By way of contradiction, assume that there exists a counterexample  $(G, M)$  where  $G \in \mathfrak{A}$  and  $M$  is an  $FG$ -module satisfying the hypothesis of the proposition. Choose this pair such that  $|G| + |M|$  is minimal. Then  $(G, M)$  has the following properties:

- 1) If  $p \neq 3, 5$ , then  $G/N \cong \text{PSL}(2, p)$ .

In fact, by Dickson's theorem there exists a subgroup  $H \leq G$  such that  $H/N \cong \text{PSL}(2, p)$ . If  $p \neq 3, 5$ , we have  $H \in \mathfrak{A}$  and  $H$  contains some Sylow 2-subgroup  $S$  of  $G$ . If  $H < G$  then  $C_M(S) \neq 0$  by minimality of  $(G, M)$ . But this is a contradiction and so we have  $G = H$ .

- 2) If  $p = 3$  or  $p = 5$ , then  $G/N \cong PSL(2, 3^f)$  or  $G/N \cong PSL(2, 5^m)$  where  $f$  and  $m$  are primes.

In fact, if  $G \in \mathcal{A}$  we have  $f > 1$ , and so there is a prime  $t$  with  $t|f$  (in a similar way there is a prime  $\bar{t}$  such that  $\bar{t}|m$ ). By Dickson's theorem there exists  $H \leq G$  such that  $H/N \cong PSL(2, 3^t)$  or  $H/N \cong PSL(2, 5^{\bar{t}})$ . As in 1), it follows that  $G = H$ .

We observe that by 1) and 2), the only subgroups of  $G/N$  are either soluble or isomorphic to  $A_5$ , so that the hypothesis of Lemma 2.5 holds.

- 3)  $M$  is an irreducible and faithful  $FG$ -module.

In fact, let  $M_1$  be an irreducible  $FG$ -module with  $M_1 < M$ . By minimality of  $(G, M)$ , we have  $C_{M_1}(S) \neq 0$  and so  $C_M(S) \neq 0$ , a contradiction. Thus  $M_1 = M$  and  $M$  is irreducible. Let  $K$  be the kernel of the action of  $G$  on  $M$ . Of course, we have  $K \leq N$  and  $(G/N, M)$  satisfies the hypotheses of the proposition. If  $K \neq 1$ , then  $0 \neq C_M(SK/K) = C_M(S)$  by minimality of  $(G, M)$ , a contradiction. So we have  $K = 1$ .

- 4)  $N \neq 1$ .

This follows from Proposition 2.3.

- 5)  $N$  is not contained in  $Z(G)$ .

In fact, suppose  $N \leq Z(G)$ . Then, by properties of the Schur multiplier of  $PSL(2, q)$  (see [6, p. 646] and [12, p. 257]), we have  $G = NL$  for a suitable subgroup  $L \neq G$ . Also  $(L, M)$  is a counterexample, but this is against the minimality of  $(G, M)$ .

- 6)  $N$  is not cyclic.

Otherwise  $N$  would be central, but this contradicts 5).

- 7)  $M$  is an induced module.

Let  $\bar{M}$  be a homogeneous component of  $M$ , considered as  $FN$ -module. Suppose that  $\bar{M} = M$ . Since  $N$  is not cyclic, the kernel of the action of  $G$  on  $\bar{M}$  is nontrivial against the faithfulness of the action of  $G$  on  $M$ . So  $\bar{M} \neq M$  and [6; p. 565] implies that  $M$  is induced.

- 8) Final contradiction.

Let  $I$  be the stabilizer of  $\bar{M}$  in  $G$ . So 7) implies  $M = (\bar{M})^G$ . We have  $I/N < G/N \cong PSL(2, q)$ . By Lemma 2.5, there exists  $SN/N \in \text{Syl}_2(G/N)$  with  $SN/N \cap I/N = N/N$ , i.e.  $SN \cap I = N$ . Then  $N(S \cap I) = N$ . Since

( $|S|, |N|$ ) = 1 it follows that  $S \cap I = 1$ . Let  $T$  be a set of double coset representatives with respect to  $S$  and  $I$  in  $G$ . We may assume  $1 \in T$ . By Mackey's theorem [6, p. 557], we have:

$$M|_S = \bigoplus_{t \in T} (\overline{M} \otimes t|_{I^t \cap S})^S = (\overline{M} \otimes 1|_{I \cap S})^S \oplus \left\{ \bigoplus_{t \neq 1} (\overline{M} \otimes t|_{I^t \cap S})^S \right\}.$$

Since  $I \cap S = 1$ , we have that  $(\overline{M} \otimes 1|_{I \cap S})^S$  is direct sum of regular  $FS$ -modules. Therefore the above implies that  $C_{\overline{M}}(S) \neq 0$  and so  $C_M(S) \neq 0$ . ■

The following deals with groups having  $A_5$  as nonsoluble chief factor:

**PROPOSITION 2.7.** *Let  $\mathcal{B}$  be the class of all groups  $G$  for which there exists a nontrivial normal subgroup  $N$  such that:*

- a)  $G/N \cong A_5$ ;
- b)  $N$  is abelian of odd order;
- c) If  $S$  is a Sylow 2-subgroup of  $G$ , then  $C_N(S) = 1$ .

*If  $G \in \mathcal{B}$  and  $M$  is a faithful and irreducible  $FG$ -module where  $F$  is of odd characteristic, then  $C_M(S) \neq 0$ .*

**PROOF.** Let  $G \in \mathcal{B}$  and let  $S \in \text{Syl}_2(G)$ . Assume that the  $FG$ -module  $M$  is a counterexample, that is  $C_M(S) = 0$ . We then have:

1)  $M$  is an induced module.

Since  $C_N(S) = 1$ , we see that  $N$  is not cyclic. As in part 7) of the proof of Proposition 2.6, it follows that if  $\overline{M}$  is a homogeneous component of  $M$ , restricted to  $N$ , then  $\overline{M} \neq M$ . So the stabilizer  $I$  of  $\overline{M}$  is properly contained in  $G$  and  $M = \overline{M}^G$ .

Let  $L = G/N \cong A_5$  and let  $H$  be the normalizer in  $L$  of a Sylow 5-subgroup  $P$  of  $L$ .

2) We have  $I/N \cong H$ .

An inspection of the proper subgroups of  $A_5$  shows that  $I/N \cong H$ , because otherwise, there would exist a Sylow 2-subgroup  $S$  of  $G$  such that  $I/N \cap SN/N = N/N$ . As in part 8) of the proof of Proposition 2.6, we get  $C_M(S) \neq 0$ . But this is a contradiction.

3) For all involutions  $\sigma \in I$ , we have  $C_{\overline{M}}(\sigma) = 0$ . In particular  $\sigma$  is not contained in the kernel  $K$  of the representation of  $I$  on  $\overline{M}$ .

In fact, suppose  $C_{\overline{M}}(\sigma) \neq 0$ . Let  $T$  be a right transversal of  $I$  in  $G$ .

Then 1) implies that  $M = \bigoplus_{t \in T} \overline{M} \otimes t$ . We may assume that  $1, \tau \in T$ , where  $\langle \sigma, \tau \rangle = S$ . We then have

$$0 \neq \langle x \otimes 1 + x \otimes \tau \mid x \in C_{\overline{M}}(\sigma) \rangle \leq C_M(S),$$

a contradiction.

4)  $[N, \sigma] \subseteq K$ .

Otherwise, the group  $[[N, \sigma]/([N, \sigma] \cap K)]\langle \sigma \rangle$  is a Frobenius group. But here, [7, p. 411] implies  $C_{\overline{M}}(\sigma) \neq 0$ , against 3).

5)  $|N|$  is not divisible by 5.

In fact, otherwise there would exist a minimal normal 5-subgroup  $R$  of  $G$  with  $R \leq N$ . Now  $R$  is a faithful and irreducible module for  $L$ , and  $S$  acts fixed point freely on  $R$ . So, from Lemma 2.1 and [7, p. 38 ff] it follows that  $|R| = 5^3$ . Moreover, considering the action of  $L$  on  $R$  it can be seen that  $|[R, \sigma]| = 5^2$ . By 4) we have  $[R, \sigma] \subseteq K$ . Since  $M$  is a faithful module, we get  $[R, \sigma] = K \cap R \leq I$ . But this is a contradiction because  $[R, \sigma]$  is not normalized by  $P$ .

6)  $C_N(P) \cap C_N(\sigma) \neq 1$ .

Since  $(|N|, |H|) = 1$ , it follows that  $N = (N \cap K) \oplus N_0$  where  $N_0$  is  $H$ -invariant. Moreover  $N_0$  must be cyclic. Also we have  $N_0 = [N_0, P] \oplus \oplus C_{N_0}(P)$ . Since  $[N_0, P]$  is invariant for the nonabelian group  $H$ , we get  $[N_0, P] = 1$ , so  $N_0 \subseteq C_N(P)$ . On the other hand, by 4), we have  $N_0 \subseteq C_N(\sigma)$ .

7) Final contradiction.

Let  $N = N_1 > N_2 > \dots > N_h = 1$  be part of a chief series of  $G$ . Since  $C_N(H) \neq 1$  and  $(|N|, |H|) \neq 1$  it follows that there exists a chief factor  $N_e/N_{e+1}$  of  $G$  such that  $C_{N_e/N_{e+1}}(H) \neq 1$ . Without loss of generality, we may assume that  $N$  is a minimal normal subgroup of  $G$ . So  $N$  can be viewed as an irreducible and faithful  $G/N$ -module and we will use the additive notation. Let  $N_0$  be the trivial  $H$ -module. Then by 6),  $N_0$  is a submodule of  $N|_H$ , so that  $\text{Hom}(N_0, N|_H) \neq 0$ . Therefore by Nakayama's reciprocity law [7, p. 50] we get  $\text{Hom}(N_0^L, N) \neq 0$  and so there exists a non-trivial homomorphism from  $N_0^L$  to  $N$ , which is an epimorphism because  $N$  is irreducible.

It follows that  $\dim N \leq 6$ , and Lemma 2.1 implies  $\dim N \in \{3, 6\}$ . If  $\dim N = 6$  then  $N \cong N_0^L$ , so  $C_N(S) \neq 0$  and we have a contradiction. If  $\dim N = 3$ , then  $[N, P]$  decomposes into a direct sum of regular modu-

les for  $\langle \sigma \rangle$  by Proposition 2.2. Hence we have  $\dim [N, P] = 2$ . Since  $C_N(P) \cap C_N(\sigma) \neq 0$ , it follows by 6) that  $\dim C_N(\sigma) = 2$ . But this is against Lemma 2.1. ■

### 3. – Conclusion.

**3.1 PROOF OF THE THEOREM.** Let  $S$  be a Sylow 2-subgroup of  $G$ . Since  $G$  is non soluble, the result of Gorenstein and Walter [3] implies that  $G/O(G) \cong PSL(2, q)$  with  $q \equiv 3, 5 \pmod{8}$ . Let  $N = O(G)$ . By [2, p. 348], we know that  $N'$  is nilpotent. We split the proof into two cases:

$q > 5$ : Let  $t$  be a prime dividing  $|N/N'|$  and let  $t \neq p$ . By Proposition 2.3 we have  $C_{N/N'}(S) \neq 1$ . So  $C_N(S) \neq 1$ , a contradiction. Hence  $N/N'$  is a  $p$ -group. By way of contradiction suppose that  $N'$  is not a  $p$ -group. As  $N'$  is nilpotent, we can choose a chief factor  $N'/K$  of  $G$  which is a  $p'$ -group. But then Proposition 2.6, applied to  $M = N'/K$ , yields  $C_{N'}(S) \neq 1$ , a contradiction.

$q = 5$ : Suppose by way of contradiction that  $N$  is not nilpotent. Then there exists a chief factor  $\bar{N}$  of  $G$  below  $N$ , which is central in  $N'$ , but not in  $N$ . So  $\bar{N}$  is a faithful and irreducible module for  $G/C_G(\bar{N})$  which satisfies the conditions of Proposition 2.7. Then  $C_{\bar{N}}(S) \neq 1$  and so  $C_N(S) \neq 1$ , but this is the final contradiction. ■

If  $G$  is assumed to be soluble in the statement of the theorem, then  $O(G)$  need not be nilpotent. In fact, it is easy to construct examples in which  $G$  is 2-nilpotent with a self-centralizing Sylow 2-subgroup of order 4 such that the normal 2-complement is of Fitting length two. For the convenience of the reader we give an example of a group  $G$  such that  $G/N \cong A_4$  where  $N = O(G)$  is of Fitting length two and the Sylow 2-subgroups of  $G$  are self-centralizing.

**EXAMPLE 3.2.** *The group  $H = A_4$  can act faithfully and irreducibly on a vector space  $V$  of dimension 3 over  $GF(3)$  (see Example 2.4). Let  $V_1$  be a subspace of dimension 1, invariant for  $S \in \text{Syl}_2(H)$ . Let  $V_2$  be an  $S$ -invariant complement of  $V_1$  in  $V$ . Set  $T = [V]S$  and  $G = [V]H$ . There exists an irreducible  $T$ -module  $M_1$  of dimension 2 over  $GF(5)$  with kernel  $V_2$ . Let  $M = M_1^G$  be the induced module. Then it is easy to verify that  $C_M(S) = \{0\}$ . Set  $N = MV$ , then we have  $C_N(S) = \{0\}$  and  $N$  is not nilpotent.*

Finally, for every prime  $p \geq 5$ , we construct a finite group  $G$  with a self-centralizing Sylow 2-subgroup such that  $G/O_p(G) \cong PSL(2, p)$  and  $O_p(G)$  is of prescribed derived length. We are indebted to the referee for greatly improving upon our original example.

EXAMPLE 3.3. Let  $p$  be a prime,  $p \equiv 3, 5 \pmod{8}$ . Let  $\mathbb{Z}_p$  be the ring of  $p$ -adic integers and consider the group  $SL(2, \mathbb{Z}_p)$  and its normal sub-

group  $N$  consisting of all matrices of the form  $\begin{pmatrix} 1 + pa & pb \\ pc & 1 + pd \end{pmatrix}$ . Note

that  $N$  is the group  $\mathcal{M}_{1,1,1}$  of [6, p. 387]. Let  $\Gamma = PSL(2, \mathbb{Z}_p)$ . We identify  $N$  with a subgroup of  $\Gamma$ , so that we have  $\Gamma/N \cong PSL(2, p)$ . Moreover (see [6, p. 387 ff.]) it is known that for every positive integer  $d$ , the factor group  $N/N^{(d)}$  is a  $p$ -group of derived length precisely  $d$ . It is easy to check that  $G = \Gamma/N^{(d)}$  is a finite group with a self-centralizing Sylow 2-subgroup of order 4 in which  $O_p(G)$  is of derived length  $d$ .

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