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Mittag-Leffler modules and semi-hereditary rings

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Mittag-Leffler Modules And Semi-hereditary Rings.

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1. - Introduction.

In [2] it was demonstrated that many properties of torsion-free abelian groups carry over to non-singular right modules over right strongly non-singular, right semi-hereditary rings, where a ring R is called *right strongly non-singular* if the finitely generated non-singular right modules are precisely the finitely generated submodules of free modules. A complete characterization of right strongly non-singular right semi-hereditary rings can be found in [9, Theorem 5.18]. In particular, it was shown that right strongly non-singular, right semi-hereditary rings are left semi-hereditary too, so that we shall call such rings right strongly non-singular semi-hereditary. Examples of this type of rings are the semi-prime semi-hereditary right and left Goldie rings, for instance Prüfer domains, as well as infinite dimensional rings like $\mathbb{Z}^{\omega}$.

Following [10], we call a right R-module A a *Mittag-Leffler module* if the natural map $A \otimes_R \left(\prod_{i \in I} M_i\right) \to \prod_{i \in I} (A \otimes_R M_i)$ is a monomorphism for all families $\{M_i\}_{i \in I}$ of left R-modules. Mittag-Leffler modules can be characterized as those modules M for which every finite subset is contained in a pure-projective pure submodule. Moreover, the Mittag-Leffler torsion-free abelian groups are precisely the $\aleph_1$-free groups [4]. In this note we show that this characterization extends to modules over

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right strongly non-singular semi-hereditary rings. Our results particularly generalize recent work by Rothmaler on flat Mittag-Leffler modules over RD-domains [11]. We show that every RD-Ore-domain is a right strongly non-singular semi-hereditary Goldie ring, and give an example that the converse need not to hold.

2. – Non-singularity and purity.

It is easy to see that (1) non-singular right modules over right strongly non-singular semi-hereditary rings are flat, (2) $\mathcal{S}$-closed submodules of non-singular modules are pure (recall that a submodule U of a module M is said to be $\mathcal{S}$-closed in M if M/U is non-singular), and (3) finitely presented modules over a semi-hereditary ring have projective dimension ≤ 1. Our first result describes the right strongly non-singular semi-hereditary rings R for which these three statements can be inverted.

THEOREM 1. *The following conditions are equivalent for a right strongly non-singular semi-hereditary ring R:*

(*a*) *R has no infinite set of orthogonal idempotents.*

(*b*) *R has finite right Goldie dimension.*

(*c*) *A finitely generated right R-module is finitely presented if and only if it has projective dimension ≤ 1.*

(*d*) *R has no proper right ideals which are essential and pure.*

(*e*) *A right R-module is flat if and only if it is non-singular.*

(*f*) *A submodule of a non-singular right R-module is $\mathcal{S}$-closed if and only if it is pure.*

PROOF. $(a)\Rightarrow(b)$ Suppose that R has infinite right Goldie dimension. Since R is a right non-singular ring, it contains a strictly ascending chain $\{I_n\}_{n<\omega}$ of $\mathcal{S}$-closed right ideals [9, Proposition 2.4 and Theorem 3.14]. For every n the right R-module R/I_n is finitely generated and non-singular, hence projective, so that I_n is a direct summand of R_R. If J_n is a right ideal such that $I_n \oplus J_n = R_R$, then $I_n \oplus (J_n \cap I_{n+1}) = I_{n+1}$, so that $\{J_n \cap I_{n+1} \mid n < \omega\}$ is an independent infinite set of direct summands of R_R. But then R has an infinite set of non-zero orthogonal idempotents.

$(b)\Rightarrow(c)$ We have to show only that if (b) holds, then every finitely generated module of projective dimension 1 is finitely presented. Let $M \cong R^m/U$ be a finitely generated module with U projective. Since R is semi-hereditary, U is a direct sum of finitely generated submodules [1]. But R_R has finite Goldie dimension, and therefore $U \subseteq R^m$ must have finite Goldie dimension. Hence the direct sum has a finite number of summands, that is, U is finitely generated.

$(c) \Rightarrow (b)$ If R has infinite right Goldie dimension, R_R contains an infinite independent family of non-zero principal right ideals $r_\lambda R$, $\lambda \in \Lambda$. Then $\bigoplus_{\lambda \in \Lambda} r_\lambda R$ is a projective right ideal of R, so that $R/\bigoplus_{\lambda \in \Lambda} r_\lambda R$ is a cyclic right R-module of projective dimension $\leqslant 1$, which is not finitely presented because $\bigoplus_{\lambda \in \Lambda} r_\lambda R$ is not finitely generated [9, p. 9].

$(b) \Rightarrow (d)$ Suppose that I is an essential, pure right ideal of R. Since R has finite right Goldie dimension and is right non-singular, its maximal right quotient ring Q is semi-simple Artinian [9, Theorem 3.17]. Furthermore, IQ is an essential right ideal of Q [9, Proposition 2.32]. Since Q is semi-simple Artinian, this is only possible if $IQ = Q$. Hence $(R/I) \otimes_R Q \cong Q/IQ = 0$. But I is pure in R, so that R/I is flat. Therefore we obtain the exact sequence $0 \to (R/I) \otimes_R R \to (R/I) \otimes_R Q = 0$, which gives $I = R$.

$(d) \Rightarrow (e)$ It remains to show that a flat module M is non-singular. Let x be an element of a flat module M. Since R is right semi-hereditary, xR is flat [9, p. 11]. But $xR \cong R/\mathrm{ann}_r(x)$, so that the right ideal $ann_r(x)$ is pure in R. Therefore either $\mathrm{ann}_r(x) = R$ or $\mathrm{ann}_r(x)$ is not essential in R. This shows that $Z(M) = 0$.

$(e) \Rightarrow (f)$ Let U be a pure submodule of the non-singular module M. Since M is flat, we know that M/U is a flat R-module. By (e), M/U is non-singular, i.e. U is $\mathcal{S}$-closed in M.

$(f) \Rightarrow (a)$ Suppose that (f) holds and R contains an infinite family $\{e_n \mid n < \omega\}$ of non-zero orthogonal idempotents. Set $I = \sum_n e_n R = \bigoplus_n e_n R$. The right ideal I is pure in R because it is the union of the direct summands $\bigoplus_{i=0}^{n} e_n R$ of R_R. If (f) holds, then I is $\mathcal{S}$-closed in R, so that the non-singular cyclic right R-module R/I is projective. Then I is a direct summand of R. It follows that R_R is a direct sum of infinitely many non-zero right ideals, which is a contradiction.

EXAMPLE 2. *There exists a right strongly non-singular semi-hereditary ring R that does not satisfy the equivalent conditions of Theorem 1.*

PROOF. Consider the strongly non-singular, semi-hereditary ring $R = \mathbb{Z}^{\omega}$ (see [2]). Obviously R does not have finite Goldie dimension.

COROLLARY 3. *The following conditions are equivalent for a ring R without infinite families of orthogonal idempotents*:

(*a*) *R is right strongly non-singular and semi-hereditary.*

(*b*) *R is left strongly non-singular and semi-hereditary.*

Moreover, if R satisfies these conditions, then R is a right and left Goldie ring.

PROOF. Let R be right strongly non-singular and semi-hereditary. By Theorem 1, R has finite right Goldie dimension. Since the maximal right quotient ring Q of R is flat as a right R-module [9, Theorem 5.18], we obtain that the left and right maximal ring of quotients of R coincide [9, Exercise 3.B.23]. Observe that R is a right p.p. ring without infinite families of orthogonal idempotents. In view of [5, Lemma 8.4], such a ring has to be left p.p. too. But every left p.p. ring is left non-singular. In order to show that R is left strongly non-singular, it therefore remains to show that Q is flat as a left R-module by [9, Theorem 5.18] since the multiplication map $Q \otimes_R Q \to Q$ is an isomorphism. By [9, Theorem 3.10], a sufficient condition for this is that every right ideal of R is essentially finitely generated, i.e., R has finite right Goldie dimension. Thus, R is left strongly non-singular.

It remains to show that R has the a.c.c. for right annihilators. But this follows immediately from Theorem 1 and [5, Lemma 1.14].

In view of Theorem 1 and the left/right symmetry proved in Corollary 3 we shall call the rings characterized in Theorem 1 strongly non-singular semi-hereditary Goldie rings. Note that the left/right symmetry may fail if R has an infinite set of orthogonal idempotents (see [9]).

EXAMPLE 4. *A strongly non-singular semi-hereditary Goldie ring need not be semi-prime.*

PROOF. Let R be the ring of lower triangular 2×2-matrices over a field F, so that R is right and left hereditary and Artinian [3]. It is easy to see that R is essential as a right and as a left submodule of $Q = \mathrm{Mat}_2(F)$. By [9, Proposition 2.11], Q is the maximal right and the maximal left ring of quotients of R. Since R is right Artinian, we have that every right ideal of R is essentially finitely generated. [9, Theorem

3.10] yields that ${}_RQ$ is flat and that the multiplication map $Q \otimes_R Q \to Q$ is an isomorphism. Thus, R is right and left strongly non-singular, but is not semi-prime.

3. – Mittag-Leffler modules.

We now turn to the discussion of Mittag-Leffler modules over strongly non-singular semi-hereditary rings. In order to adapt the notion of an $\aleph_1$-free module to modules over strongly non-singular semi-hereditary Goldie rings, a reformulation of the definition used in abelian groups becomes necessary. Otherwise it may happen that R itself may be not $\aleph_1$-free unless R is right hereditary. We say that a non-singular right module M over a right strongly non-singular Goldie ring R is *$\aleph_1$-projective* if the $\mathcal{S}$-closure of every countably generated submodule of M is projective. From the next result it follows immediately that every projective module over a strongly non-singular semi-hereditary Goldie ring is $\aleph_1$-projective.

THEOREM 5. *The following three conditions are equivalent for a right strongly non-singular right Goldie ring R:*

(*a*) *R is semi-hereditary.*

(*b*) *A right R-module M is pure-projective if and only if $M/Z(M)$ is projective and $Z(M)$ is a direct summand of a module of the form $\bigoplus_{i \in I} N_i$ where each N_i is a finitely generated singular module of projective dimension* 1.

(*c*) *The following conditions are equivalent for a right R-module M:*

(i) *M is a non-singular Mittag-Leffler module.*

(ii) *M is $\aleph_1$-projective.*

(iii) *Every finite subset of M is contained in a $\mathcal{S}$-closed projective submodule of M.*

PROOF. $(a) \Rightarrow (b)$ Let M be a pure-projective module. We know that M is a direct summand of a direct sum of finitely presented modules, say $M \oplus N \cong \bigoplus_{i \in I} V_i$ for some R-module N where each V_i is finitely presented. Since R is strongly non-singular, $V_i/Z(V_i)$ is projective, say $V_i = P_i \oplus Z(V_i)$. Then $[M/Z(M)] \oplus [N/Z(N)] \cong (M \oplus N)/Z(M \oplus N) \cong$ $\cong \bigoplus_{i \in I} P_i$ yields that $M/Z(M)$ is projective. Moreover, $Z(M) \oplus Z(N) \cong$

$\cong \bigoplus_{i \in I} Z(V_i)$ where each $Z(V_i)$ is finitely presented as a direct summand of a finitely presented module. We write $Z(V_i) \cong R^{n_i}/U_i$ for some $n_i < \omega$ and finitely generated submodule U_i of R^{n_i}. Since R is a non-singular semi-hereditary ring, U_i is projective, and $Z(V_i)$ has projective dimension 1.

The converse holds by Theorem 1.

$(b) \Rightarrow (a)$ Let I be a finitely generated right ideal of R. Since R/I is finitely presented, it is the direct sum of a projective module and a module of projective dimension at most 1 by (b). Hence I has to be projective.

$(a) \Rightarrow (c)$: (i) $\Rightarrow$ (ii) Let U be a countably generated submodule of a non-singular Mittag-Leffler module M. By [10] there is a pure-projective, countably generated pure submodule V of M that contains U. By Theorem 1 and the already proved implication $(a) \Rightarrow (b)$ of this theorem, V is an $\mathcal{S}$-closed projective submodule of M. In particular, V contains the $\mathcal{S}$-closure U_* of U. By [2, Proposition 2.2] the module V/U_* has projective dimension at most 1. Since V is projective, this yields that U_* has to be projective too.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) By [10] it is enough to show that every finite subset of M is contained in a pure-projective pure submodule of M. But $\mathcal{S}$-closed submodules are pure by Theorem 1.

$(c) \Rightarrow (a)$ Let I be a finitely generated right ideal of R. Consider an exact sequence $0 \to U \to R^n \to I \to 0$ of right R-modules where $n < \omega$. Since R has finite right Goldie-dimension, U contains a finitely generated essential submodule V. Furthermore, R^n is a non-singular Mittag-Leffler module. By (c), the $\mathcal{S}$-closure W of V in R^n is projective [5, Proposition 8.24] yields that W is finitely generated. Since U is $\mathcal{S}$-closed in R^n and V is essential in U, it follows that $U = W$. Thus I is finitely presented, and in particular, a Mittag-Leffler module. By (c), finitely generated non-singular Mittag-Leffler modules are projective.

Since every ideal of a Noetherian integral domain is a Mittag-Leffler module, the ring $\mathbb{Z}[x]$ is an example of a domain over which there exist torsion-free Mittag-Leffler modules which are not $\aleph_1$-projective.

In [11, Section 6.3] Rothmaler studies the structure of flat Mittag-Leffler modules over a right hereditary RD-Ore-domain, i.e., a right hereditary right and left Ore-domain for which purity and relative divisibility coincide. An RD-Ore-domain is right and left semi-hereditary, hence it is a strongly non-singular semi-hereditary Goldie ring. From Example 4 we thus have

EXAMPLE 6. *Every RD-Ore-domain is a strongly non-singular semi-hereditary Goldie ring, but the converse is not true in general.*

We can use Theorem 5 to determine the projective dimension of Mittag-Leffler modules:

COROLLARY 7. *Let R be a ring.*

(a) R is right semi-hereditary if and only if for every Mittag-Leffler right R-module M and every integer $n \geq 0$, if M can be generated by $\leq \aleph_n$ elements then $\text{proj.dim.} M \leq n + 1$.

(b) If R is a strongly non-singular semi-hereditary Goldie ring and M is a non-singular Mittag-Leffler module generated by $\leq \aleph_n$ elements, then $\text{proj.dim.} M \leq n$.

PROOF. If every countably generated Mittag-Leffler right R-module M has projective dimension ≤ 1, then $\text{proj.dim.} R/I \leq 1$ for every finitely generated right ideal I of R, so that R is right semi-hereditary.

Conversely, suppose that R is right semi-hereditary and argue by induction on $n \geq 0$. If $n = 0$, a Mittag-Leffler right R-module generated by $\leq \aleph_0$ elements is pure-projective, and therefore it has projective dimension ≤ 1 because every finitely presented right R-module over a right semi-hereditary ring has projective dimension ≤ 1. And if $n = 0$ and M is a non-singular Mittag-Leffler module over a strongly non-singular Goldie ring generated by $\leq \aleph_0$ elements, then M is projective by Theorem 5.

Suppose $n > 0$. Let M be a Mittag-Leffler right R-module generated by a set $\{x_\nu \mid \nu < \omega_n\} \subseteq M$. For every finite subset X of M fix a pure, countably generated, pure-projective submodule V_X of M containing X. Define a submodule W_α of M generated by $\leq \aleph_{n-1}$ elements by transfinite induction on $\alpha \in \omega_n \times \omega_0$, where $\omega_n \times \omega_0$ denotes the lexicographic product of ω_n and ω_0, in the following way. Set $W_0 = 0$. If $\alpha \in \omega_n \times \omega_0$ is a limit ordinal, set $W_\alpha = \bigcup_{\beta < \alpha} W_\beta$. If $\alpha \in \omega_n \times \omega_0$ is not a limit ordinal, then $\alpha = (\nu, r + 1)$ for some $\nu < \omega_n$ and some $r < \omega_0$. If ν is a limit ordinal, set $W_\alpha = W_{(\nu, r)}$. If ν is not a limit ordinal, then $\alpha = (\mu + 1, r + 1)$. In this case let $X_{(\mu+1, r)}$ be a set of generators of $W_{(\mu+1, r)}$ of cardinality $\leq \aleph_{n-1}$ and set $W_\alpha = \sum\{ V_{X \cup \{x_\mu\}} \mid X \subseteq X_{(\mu+1, r)}, X \text{ finite}\}$. Note that W_α has a set of generators of cardinality $\leq \aleph_{n-1}$.

It is clear that $W_0 \subsetneq W_1 \subsetneq \ldots \subsetneq W_\alpha \subsetneq \ldots$, $\alpha \in \omega_n \times \omega_0$, is an ascending chain of submodules of M. We claim that $W_{(\nu, 0)}$ is pure in M for every ordinal $\nu < \omega_n$. In order to prove the claim, let A be a $k \times m$ matrix over R, Z a $1 \times k$ matrix over M and $Y = (y_1, \ldots, y_m)$ a $1 \times m$ matrix over $W_{(\nu, 0)}$ such that $ZA = Y$. We must show that there exists a $1 \times k$ matrix Z' over $W_{(\nu, 0)}$ such that $Z'A = Y$. Since $y_1, \ldots, y_m \in W_{(\nu, 0)}$ and $(\nu, 0) \in \omega_n \times \omega_0$ is a limit ordinal, there exists $\beta < (\nu, 0)$ such that $y_1, \ldots, y_m \in W_\beta$. Let $\overline{\beta} \leqslant \beta$ be the least ordinal such that $y_1, \ldots, y_m \in$ $\in W_{\overline{\beta}}$. Then $\overline{\beta}$ is not a limit ordinal, and $\overline{\beta}$ must be of the form $(\omega + 1, r + 1)$. Let X be a finite subset of the set $X_{(\omega + 1, r + 1)}$ of generators of $W_{(\omega + 1, r + 1)}$ such that $y_1, \ldots, y_m$ belong to the submodule XR of M generated by X. The pure submodule $V_{X \cup \{x_\omega\}}$ of M is contained in $W_{(\omega + 1, r + 2)}$ and contains $y_1, \ldots, y_m$. Therefore there exists a $1 \times k$ matrix Z' over $V_{X \cup \{x_\omega\}}$ such that $Z'A = Y$. This concludes the proof of the claim, because $V_{X \cup \{x_\omega\}} \subsetneq W_{(\omega + 1, r + 2)} = W_{\overline{\beta} + 1} \subsetneq W_{(\nu, 0)}$.

Since the modules $W_{(\nu, 0)}$ are generated by $\leqslant \aleph_{n-1}$ elements and pure submodules of Mittag-Leffler modules are Mittag-Leffler modules, it follows that the inductive hypothesis can be applied, so that proj. dim. $W_{(\nu, 0)} \leqslant n$ (and proj. dim. $W_{(\nu, 0)} \leqslant n - 1$ if R is a strongly non-singular semi-hereditary Goldie ring and M is non-singular) for every $\nu < \omega_n$. By Auslander's Theorem, the projective dimension of M cannot exceed $n + 1$ (or n if R is a strongly non-singular semi-hereditary Goldie ring and M is non-singular).

If we restrict our discussion to semi-prime rings, the equivalences in Part (*c*) of Theorem 5 can be further improved. Observe that the semi-prime strongly non-singular semi-hereditary rings without infinite sets of orthogonal idempotents are precisely the semi-prime right and left semi-hereditary Goldie rings. Moreover, if R is a semi-prime right Goldie ring, then a right ideal of R is essential if and only if it contains a regular element [5, Lemma 1.11 and Cor. 1.20], so that $Z(M) =$ $= \{x \in M \mid xc = 0$ for some regular element $c \in R\}$ for any right R-module M. In particular if N is a submodule of a non-singular right module M over a semi-prime semi-hereditary Goldie ring, then N is pure in M if and only if $Mc \cap N = Nc$ for every regular element $c \in R$.

Corollary 8. *Let R be a semi-prime, right and left semi-hereditary Goldie ring. The following conditions are equivalent for an R-module M:*

(*a*) *M is a Mittag-Leffler module.*

(*b*) *$Z(M)$ is a Mittag-Leffler module, and $M/Z(M)$ is $\aleph_1$-projective.*

PROOF. Since the class of Mittag-Leffler modules is closed with respect to pure submodules and pure extensions, Theorems 1 and 5 reduce the problem to showing that $M/Z(M)$ is Mittag-Leffler whenever M is Mittag-Leffler. For this, let U be a finitely generated submodule of M, and choose a pure-projective pure submodule V of M which contains U. By Theorem 5, $V = P \oplus Z(V)$ for some projective submodule P of M. Since $[U + Z(M)]/Z(M) \subseteq [P \oplus Z(M)]/Z(M)$, the corollary will follow once we have shown that $P \oplus Z(M)$ is $\mathcal{S}$-closed in M. Suppose that $x \in M$ satisfies $xc \in [P \oplus Z(M)]$ for some regular element $c \in R$. We can find $y \in P$ and a regular $d \in R$ such that $xcd - pd = 0$. But P is pure in V and V is pure in M, so that P is pure in M. Thus $xcd = pd \in P \cap \cap Mcd = Pcd$, and $x \in P \oplus Z(M)$.

The rest of this Section is devoted to completely recover Lemmas 6.10, 6.11, 6.12, Theorem 6.13 and Corollary 6.14 of [11] for the more general class of rings discussed in this paper.

PROPOSITION 9. *Let R be a strongly non-singular semi-hereditary Goldie ring and M an R-module with the property that every countably generated submodule of M is projective. Then M is a non-singular Mittag-Leffler R-module and every finite subset of M is contained in a finitely generated projective pure submodule of M.*

PROOF. Let M be a module satisfying the hypotheses of the statement. It is obvious that M is non-singular.

We claim that if X is a finitely generated submodule of M, then the $\mathcal{S}$-closure C of X in M is finitely generated. In order to prove the claim it is sufficient to show that each countably generated submodule N of C containing X is finitely generated. Any such N is projective, hence $N = = \bigoplus_{i<\omega} N_i$, where the N_i are isomorphic to finitely generated right ideals of R [1]. So it is enough to show that $N = \bigoplus_{i=0}^{n} N_i$ for some $n < \omega$. Choose $n < \omega$ such that $X \subseteq \bigoplus_{i=0}^{n} N_i$ and set $N' = \bigoplus_{i=0}^{n} N_i$. We have $X \subseteq N' \subseteq N \subseteq C$. Since C modulo the submodule generated by X is singular, $N/N' \cong \cong \bigoplus_{i>n} N_i$ also is singular. But the N_i's are isomorphic to right ideals of R, and therefore $N/N' \cong \bigoplus_{i>n} N_i$ is non-singular. Therefore $N' = N$, and N is finitely generated. This proves our claim.

Since every finitely generated submodule of M is projective, it is now clear that the $\mathcal{S}$-closure of every finitely generated submodule of M is a finitely generated projective pure submodule of M. In particular M is a Mittag-Leffler module (Theorem 5).

LEMMA 10. *Let R be a right strongly non-singular right Goldie ring, C a non-singular right R-module and P a finitely generated submodule of C. If C/P is singular, then C has finite Goldie dimension.*

PROOF. Since P is a finitely generated non-singular module over a right strongly non-singular ring, P is a submodule of a finitely generated free module. In particular, P has finite Goldie dimension. Since C is non-singular and C/P is singular, P is an essential submodule of C. This shows $\dim C = \dim P < \infty$.

THEOREM 11. *The following four conditions are equivalent for a right strongly non-singular right Goldie ring R:*

(*a*) *R is a right hereditary ring.*

(*b*) *R is a right noetherian, right hereditary ring.*

(*c*) *R is a right semi-hereditary ring and all submodules of non-singular Mittag-Leffler right R-modules are Mittag-Leffler modules.*

(*d*) *The following conditions are equivalent for a right R-module M:*

(i) *M is a non-singular Mittag-Leffler module.*

(ii) *Every countably generated submodule of M is projective.*

(iii) *Every finite subset of M is contained in a finitely generated projective pure submodule of M.*

(iv) *M is non-singular and every finite subset of M is contained in a finitely presented pure submodule of M.*

(v) *M is non-singular and every submodule of M of finite Goldie dimension is a finitely generated projective module.*

PROOF. $(a) \Rightarrow (d)$ Suppose that R is right hereditary.

(i)$\Rightarrow$(ii) is proved in [11, Cor. 6.3].

(ii)$\Rightarrow$(iii) is proved in Proposition 9.

(iii)$\Rightarrow$(iv) If every element of M is contained in a projective module, M must be non-singular. Moreover, every finitely generated projective submodule is finitely presented.

(iv)$\Rightarrow$(v) Suppose that (iv) holds and let N be a submodule of M of finite Goldie dimension. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite subset of N such that $\sum_{i=1}^{n} x_i R = \bigoplus_{i=1}^{n} x_i R$ is an essential submodule of N. Then $N/\sum_{i=1}^{n} x_i R$ is a singular submodule of $M/\sum_{i=1}^{n} x_i R$, so that if C denotes the $\mathcal{S}$-closure of $\sum_{i=1}^{n} x_i R$ in M, then $N \subseteq C$. By (iv) the subset X is contained in a finitely presented pure submodule D of M. Since M is non-singular, D also is non-singular, hence flat, hence projective. Thus X is contained in the finitely generated projective pure submodule D of M. Therefore C is a submodule of D. Hence N is contained in the projective module D, and N is projective because R is right hereditary. By [1] N is isomorphic to a direct sum of finitely generated right ideals. But N has finite Goldie dimension, and therefore N itself is a finitely generated projective module.

(v) $\Rightarrow$ (i) Suppose that (v) holds. In order to prove that M is a Mittag-Leffler module it is sufficient to show that every finite subset X of M is contained in a pure-projective pure submodule of M [4, Th. 6]. Let P be the submodule of M generated by a finite subset X of M and C be the $\mathcal{S}$-closure of P in M, so that C is pure in M. By Lemma 10 the module C has finite Goldie dimension. By Hypothesis (v) C is a finitely generated projective module.

$(d) \Rightarrow (c)$ Assume that R has the property that the five conditions are equivalent for every right R-module M. Let us show that R is right hereditary. If I is a right ideal of R, then I is a submodule of the non-singular Mittag-Leffler module R_R, which is of finite Goldie dimension. By (d) I is a finitely generated projective module.

Since M is a non-singular Mittag-Leffler module if and only if every countably generated submodule of M is projective, every submodule of a non-singular Mittag-Leffler module is a non-singular Mittag-Leffler module.

$(c) \Rightarrow (b)$ In order to show that R is right noetherian, it is sufficient to show that if I is a countably generated right ideal of R, then I is finitely generated. Since R_R is a non-singular Mittag-Leffler module, every right ideal of R is a non-singular Mittag-Leffler module. Hence every countably generated right ideal I of R is a non-singular pure-projective module, that is, it is projective. Then R/I is a finitely generated module of projective dimension ≤ 1, and therefore it is finitely presented (Theorem 1). Hence I is finitely generated.

$(b) \Rightarrow (a)$ is obvious.

4. – Prüfer rings and indecomposable Mittag-Leffler modules.

Recall that a commutative integral domain is semi-hereditary if and only if it is a *Prüfer* ring, that is, all its localizations at maximal ideals are valuation domains. If R is an integral domain, for every R-module M the submodule $Z(M)$ is exactly the torsion submodule $t(M)$ of M, so that a module is non-singular if and only if it is torsion-free. Hence Theorem 5 gives a complete description of torsion-free Mittag-Leffler modules over Prüfer domains: a torsion-free module over a Prüfer domain is a Mittag-Leffler module if and only if it is $\aleph_1$-projective. More generally, a module M over a Prüfer domain is a Mittag-Leffler module if and only if $M/t(M)$ is $\aleph_1$-projective and $t(M)$ is a torsion Mittag-Leffler module. The structure of torsion Mittag-Leffler modules over a Prüfer domain R depends heavily on the properties of R. For instance, in [4, Prop. 7] it is shown that a torsion abelian group G is a Mittag-Leffler $\mathbb{Z}$-module if and only if $\bigcap_{n>0} nG = 0$. In the next Proposition we describe torsion Mittag-Leffler modules over almost maximal valuation domains and arbitrary Mittag-Leffler modules over maximal valuation rings. Recall that an R-module is *cyclically presented* if it is isomorphic to R/aR for some $a \in R$ [8].

PROPOSITION 12. *Let M be a torsion module over an almost maximal valuation domain R or an arbitrary module over a maximal valuation ring R. The following conditions are equivalent*:

(*a*) *M is a Mittag-Leffler R-module.*

(*b*) *Every finite subset of M is contained in a direct summand of M that is a direct sum of cyclically presented modules.*

(*c*) *Every element of M is contained in a direct summand of M that is a direct sum of cyclically presented modules.*

PROOF. $(a) \Rightarrow (b)$ Let X be a finite subset of M. Then X is contained in a pure-projective pure submodule P of M [4, Th. 6]. The pure-projective module P is a direct sum of cyclically presented modules [8, Th. II.3.4 and Prop. II.4.3]. Hence P decomposes as $P = P' \oplus P''$, where $X \subseteq P'$ and P' is a finite direct sum of cyclically presented modules. By [8, Th. XI.4.2] P' is pure-injective. Since P' is pure in M, P' must be a direct summand of M.

$(b) \Rightarrow (c)$ is obvious.

$(c) \Rightarrow (a)$ Let X be a finite subset of M. By [4, Th. 6] it is sufficient to prove that X is contained in a pure-projective pure submodule of M. By [8, Prop. XIII.2.4] the module M is *separable*, that is, every finite set of elements of M can be embedded in a direct summand which is a direct sum of uniserial modules. Hence it is enough to prove that every uniserial direct summand U of M is cyclically presented. Let x be a nonzero element of a uniserial direct summand U of M. By (c) there exists a direct summand P of M such that P is a finite direct sum of cyclically presented modules and $x \in P$. Let W and Q be direct complements of U and P, so that $M = U \oplus W = P \oplus Q$. Since P has the exchange property

([12] and [8, Cor. VII.2.7]), there are submodules $U' \leqslant U$ and $W' \leqslant W$ such that $M = P \oplus U' \oplus W'$. Since $0 \neq x \in P \cap U$, $P \cap U$ is an essential submodule of the uniserial module U. But $(P \cap U) \cap U' = 0$, so that $U' = 0$ and $M = P \oplus W'$. Then U is a direct summand of $U \oplus \oplus (W/W') \cong (U \oplus W)/W' = M/W' \cong P$. In particular U is pure-projective, that is, U is a direct sum of cyclically presented modules. Hence the uniserial module U must be cyclically presented.

Therefore over an almost maximal valuation domain R the indecomposable torsion Mittag-Leffler modules are only the cyclically presented modules R/aR's, $a \neq 0$, and over a maximal valuation ring R the indecomposable Mittag-Leffler modules are only the cyclically presented modules R/aR's, $a \in R$. The last result of this paper addresses the question whether there exist arbitrarily large indecomposable non-singular Mittag-Leffler modules.

EXAMPLE 13. *Let R be a strongly non-singular semi-hereditary Goldie ring whose additive group is cotorsion-free. Then there exists a proper class of pairwise non-isomorphic, indecomposable, non-singular Mittag-Leffler R-modules.*

PROOF. Let κ be an infinite cardinal. Since R has a cotorsion-free additive group, there exists an $\aleph_1$-projective left R-module M of cardinality at least κ such that $\mathrm{End}_{\mathbb{Z}}(M) \cong R^{\mathrm{op}}$ by [6]. By Theorem 5, M is a non-singular Mittag-Leffler module whose R-endomorphism ring is Center(R). Since R does not contain any infinite family of orthogonal idempotents, the same holds for Center(R). We write $1 = e_1 + \ldots + e_n$ where $\{e_1, \ldots, e_n\}$ is a family of orthogonal, primitive idempotents of Center(R). Then $M_i = e_i(M)$ is an indecomposable Mittag-Leffler module. Since $|M| \geqslant \kappa$ and $M = \bigoplus_{i=1}^{n} M_i$, at least one of the M_i's has cardinality at least κ.

The ring of algebraic integers is an example for a ring as in Example 13.

REFERENCES

[1] F. ALBRECHT, *On projective modules over semi-hereditary rings*, Proc. Amer. Math. Soc., **12** (1961), pp. 638-639.

[2] U. ALBRECHT, *Locally A-projective abelian groups and generalizations*, Pacific J. Math., **141** (1990), pp. 209-227.

[3] F. W. ANDERSON - K. R. FULLER, *Rings and Categories of Modules*, Springer-Verlag (1992).

[4] G. AZUMAYA - A. FACCHINI, *Rings of pure global dimension zero and Mittag-Leffler modules*, J. Pure Appl. Algebra, **62** (1989), pp. 109-122.

[5] A. CHATTERS - C. HAJARNAVIS, *Rings with Chain Conditions*, Pitman Advanced Publishing Program, **44**, Boston, London, Melbourne (1980).

[6] M. DUGAS - R. GÖBEL, *Every cotorsion-free ring is an endomorphism ring*, Proc. London Math. Soc., **45** (1982), pp. 319-336.

[7] A. FACCHINI, *Mittag-Leffler modules, reduced products, and direct products*, Rend. Sem. Mat. Univ. Padova, **85** (1991), pp. 119-132.

[8] L. FUCHS - L. SALCE, *Modules Over Valuation Domains*, Marcel Dekker, New York and Basel (1985).

[9] K. R. GOODEARL, *Ring Theory*, Marcel Dekker, New York and Basel (1976).

[10] M. RAYNAUD - L. GRUSON, *Critères de platitude et de projectivité*, Invent. Math., **13** (1971), pp. 1-89.

[11] PH. ROTHMALER, *Mittag-Leffler Modules and Positive Atomicity*, preprint (1994).

[12] R. B. WARFIELD, *Exchange Rings and Decompositions of Modules*, Math. Ann., **199** (1972), pp. 31-36.

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