Mittag-Leffler Modules And Semi-hereditary Rings.

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1. - Introduction.

In [2] it was demonstrated that many properties of torsion-free abelian groups carry over to non-singular right modules over right strongly non-singular, right semi-hereditary rings, where a ring $R$ is called right strongly non-singular if the finitely generated non-singular right modules are precisely the finitely generated submodules of free modules. A complete characterization of right strongly non-singular right semi-hereditary rings can be found in [9, Theorem 5.18]. In particular, it was shown that right strongly non-singular, right semi-hereditary rings are left semi-hereditary too, so that we shall call such rings right strongly non-singular semi-hereditary. Examples of this type of rings are the semi-prime semi-hereditary right and left Goldie rings, for instance Prüfer domains, as well as infinite dimensional rings like $\mathbb{Z}^\omega$.

Following [10], we call a right $R$-module $A$ a Mittag-Leffler module if the natural map $A \otimes_R \left( \prod_{i \in I} M_i \right) \rightarrow \prod_{i \in I} (A \otimes_R M_i)$ is a monomorphism for all families $\{M_i\}_{i \in I}$ of left $R$-modules. Mittag-Leffler modules can be characterized as those modules $M$ for which every finite subset is contained in a pure-projective pure submodule. Moreover, the Mittag-Leffler torsion-free abelian groups are precisely the $\aleph_0$-free groups [4]. In this note we show that this characterization extends to modules over

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right strongly non-singular semi-hereditary rings. Our results particularly generalize recent work by Rothmaler on flat Mittag-Leffler modules over RD-domains [11]. We show that every RD-Ore-domain is a right strongly non-singular semi-hereditary Goldie ring, and give an example that the converse need not to hold.

2. – Non-singularity and purity.

It is easy to see that (1) non-singular right modules over right strongly non-singular semi-hereditary rings are flat, (2) S-closed sub-modules of non-singular modules are pure (recall that a submodule $U$ of a module $M$ is said to be S-closed in $M$ if $M/U$ is non-singular), and (3) finitely presented modules over a semi-hereditary ring have projective dimension $\leq 1$. Our first result describes the right strongly non-singular semi-hereditary rings $R$ for which these three statements can be inverted.

**Theorem 1.** The following conditions are equivalent for a right strongly non-singular semi-hereditary ring $R$:

(a) $R$ has no infinite set of orthogonal idempotents.
(b) $R$ has finite right Goldie dimension.
(c) A finitely generated right $R$-module is finitely presented if and only if it has projective dimension $\leq 1$.
(d) $R$ has no proper right ideals which are essential and pure.
(e) A right $R$-module is flat if and only if it is non-singular.
(f) A submodule of a non-singular right $R$-module is $S$-closed if and only if it is pure.

**Proof.** $(a) \Rightarrow (b)$ Suppose that $R$ has infinite right Goldie dimension. Since $R$ is a right non-singular ring, it contains a strictly ascending chain $\{I_n\}_{n < \omega}$ of $S$-closed right ideals [9, Proposition 2.4 and Theorem 3.14]. For every $n$ the right $R$-module $R/I_n$ is finitely generated and non-singular, hence projective, so that $I_n$ is a direct summand of $R_R$. If $J_n$ is a right ideal such that $I_n \oplus J_n = R_R$, then $I_n \oplus (J_n \cap I_{n+1}) = I_{n+1}$, so that $\{J_n \cap I_{n+1} \mid n < \omega\}$ is an independent infinite set of direct summands of $R_R$. But then $R$ has an infinite set of non-zero orthogonal idempotents.

$(b) \Rightarrow (c)$ We have to show only that if $(b)$ holds, then every finite-
ly generated module of projective dimension 1 is finitely presented. Let 
\( M = R^m / U \) be a finitely generated module with \( U \) projective. Since \( R \) is 
semi-hereditary, \( U \) is a direct sum of finitely generated submodules [1]. 
But \( R_R \) has finite Goldie dimension, and therefore \( U \subseteq R^m \) must have 
finitive Goldie dimension. Hence the direct sum has a finite number of 
summands, that is, \( U \) is finitely generated.

(c) \( \Rightarrow \) (b) If \( R \) has infinite right Goldie dimension, \( R_R \) contains an 
infinite independent family of non-zero principal right ideals \( r_\lambda R, \lambda \in \).
Then \( \bigoplus_{\lambda \in \Lambda} r_\lambda R \) is a projective right ideal of \( R \), so that \( R/ \bigoplus_{\lambda \in \Lambda} r_\lambda R \) is a 
cyclic right \( R \)-module of projective dimension \( \leq 1 \), which is not finitely 
presented because \( \bigoplus_{\lambda \in \Lambda} r_\lambda R \) is not finitely generated [9, p. 9].

(b) \( \Rightarrow \) (d) Suppose that \( I \) is an essential, pure right ideal of \( R \).
Since \( R \) has finite Goldie dimension and is right non-singular, its 
maximal right quotient ring \( Q \) is semi-simple Artinian [9, Theorem 
3.17]. Furthermore, \( IQ \) is an essential right ideal of \( Q \) [9, Proposition 
2.32]. Since \( Q \) is semi-simple Artinian, this is only possible if \( IQ = Q \). 
Hence \( (R/I) \otimes_R Q \equiv Q/IQ = 0 \). But \( I \) is pure in \( R \), so that \( R/I \) is flat.
Therefore we obtain the exact sequence 0 \( \rightarrow (R/I) \otimes_R R \rightarrow (R/I) \otimes 
Q = 0 \), which gives \( I = R \).

(d) \( \Rightarrow \) (e) It remains to show that a flat module \( M \) is non-singular.
Let \( x \) be an element of a flat module \( M \). Since \( R \) is right semi-hereditary, \( xR \) is flat [9, p. 11]. But \( xR \equiv R/\text{ann}_R(x) \), so that the right ideal 
\( \text{ann}_R(x) \) is pure in \( R \). Therefore either \( \text{ann}_R(x) = R \) or \( \text{ann}_R(x) \) is not 
essential in \( R \). This shows that \( Z(M) = 0 \).

(e) \( \Rightarrow \) (f) Let \( U \) be a pure submodule of the non-singular module 
\( M \). Since \( M \) is flat, we know that \( M/U \) is a flat \( R \)-module. By (e), \( M/U \) is 
non-singular, i.e. \( U \) is \( s \)-closed in \( M \).

(f) \( \Rightarrow \) (a) Suppose that (f) holds and \( R \) contains an infinite family 
\( \{ e_n | n < \omega \} \) of non-zero orthogonal idempotents. Set 
\( I = \sum_n e_n R = \bigoplus_n e_n R \). The right ideal \( I \) is pure in \( R \) because it is the union of the di-
rect summands \( \bigoplus_{i=0}^n e_n R \) of \( R_R \). If (f) holds, then \( I \) is \( s \)-closed in \( R \), so 
that the non-singular cyclic right \( R \)-module \( R/I \) is projective. Then \( I \) is 
a direct summand of \( R \). It follows that \( R_R \) is a direct sum of infinitely 
many non-zero right ideals, which is a contradiction.

**Example 2.** There exists a right strongly non-singular semi-
hereditary ring \( R \) that does not satisfy the equivalent conditions of 
Theorem 1.
PROOF. Consider the strongly non-singular, semi-hereditary ring $R = \mathbb{Z}_\omega$ (see [2]). Obviously $R$ does not have finite Goldie dimension.

COROLLARY 3. The following conditions are equivalent for a ring $R$ without infinite families of orthogonal idempotents:

(a) $R$ is right strongly non-singular and semi-hereditary.

(b) $R$ is left strongly non-singular and semi-hereditary.

Moreover, if $R$ satisfies these conditions, then $R$ is a right and left Goldie ring.

PROOF. Let $R$ be right strongly non-singular and semi-hereditary. By Theorem 1, $R$ has finite right Goldie dimension. Since the maximal right quotient ring $Q$ of $R$ is flat as a right $R$-module [9, Theorem 5.18], we obtain that the left and right maximal ring of quotients of $R$ coincide [9, Exercise 3.B.23]. Observe that $R$ is a right p.p. ring without infinite families of orthogonal idempotents. In view of [5, Lemma 8.4], such a ring has to be left p.p. too. But every left p.p. ring is left non-singular. In order to show that $R$ is left strongly non-singular, it therefore remains to show that $Q$ is flat as a left $R$-module by [9, Theorem 5.18] since the multiplication map $Q \otimes_R Q \to Q$ is an isomorphism. By [9, Theorem 3.10], a sufficient condition for this is that every right ideal of $R$ is essentially finitely generated, i.e., $R$ has finite right Goldie dimension. Thus, $R$ is left strongly non-singular.

It remains to show that $R$ has the a.c.c. for right annihilators. But this follows immediately from Theorem 1 and [5, Lemma 1.14].

In view of Theorem 1 and the left/right symmetry proved in Corollary 3 we shall call the rings characterized in Theorem 1 strongly non-singular semi-hereditary Goldie rings. Note that the left/right symmetry may fail if $R$ has an infinite set of orthogonal idempotents (see [9]).

EXAMPLE 4. A strongly non-singular semi-hereditary Goldie ring need not be semi-prime.

PROOF. Let $R$ be the ring of lower triangular $2 \times 2$-matrices over a field $F$, so that $R$ is right and left hereditary and Artinian [3]. It is easy to see that $R$ is essential as a right and as a left submodule of $Q = \text{Mat}_2(F)$. By [9, Proposition 2.11], $Q$ is the maximal right and the maximal left ring of quotients of $R$. Since $R$ is right Artinian, we have that every right ideal of $R$ is essentially finitely generated. [9, Theorem
yields that $\mathbb{R}Q$ is flat and that the multiplication map $Q \otimes_{\mathbb{R}} Q \rightarrow Q$ is an isomorphism. Thus, $\mathbb{R}$ is right and left strongly non-singular, but is not semi-prime.

3. – Mittag-Leffler modules.

We now turn to the discussion of Mittag-Leffler modules over strongly non-singular semi-hereditary rings. In order to adapt the notion of an $\aleph_1$-free module to modules over strongly non-singular semi-hereditary Goldie rings, a reformulation of the definition used in abelian groups becomes necessary. Otherwise it may happen that $\mathbb{R}$ itself may be not $\aleph_1$-free unless $\mathbb{R}$ is right hereditary. We say that a non-singular right module $M$ over a right strongly non-singular Goldie ring $\mathbb{R}$ is $\aleph_1$-projective if the $S$-closure of every countably generated submodule of $M$ is projective. From the next result it follows immediately that every projective module over a strongly non-singular semi-hereditary Goldie ring is $\aleph_1$-projective.

**Theorem 5.** The following three conditions are equivalent for a right strongly non-singular right Goldie ring $\mathbb{R}$:

(a) $\mathbb{R}$ is semi-hereditary.

(b) A right $\mathbb{R}$-module $M$ is pure-projective if and only if $M/Z(M)$ is projective and $Z(M)$ is a direct summand of a module of the form $\bigoplus_{i \in I} N_i$ where each $N_i$ is a finitely generated singular module of projective dimension 1.

(c) The following conditions are equivalent for a right $\mathbb{R}$-module $M$:

(i) $M$ is a non-singular Mittag-Leffler module.

(ii) $M$ is $\aleph_1$-projective.

(iii) Every finite subset of $M$ is contained in a $S$-closed projective submodule of $M$.

**Proof.** (a) $\Rightarrow$ (b) Let $M$ be a pure-projective module. We know that $M$ is a direct summand of a direct sum of finitely presented modules, say $M \oplus N \cong \bigoplus_{i \in I} V_i$ for some $\mathbb{R}$-module $N$ where each $V_i$ is finitely presented. Since $\mathbb{R}$ is strongly non-singular, $V_i/Z(V_i)$ is projective, say $V_i = P_i \oplus Z(V_i)$. Then $[M/Z(M)] \oplus [N/Z(N)] \cong (M \oplus N)/Z(M \oplus N) \cong \bigoplus_{i \in I} P_i$ yields that $M/Z(M)$ is projective. Moreover, $Z(M) \oplus Z(N) \cong$
\[ \bigoplus_{i \in I} Z(V_i) \text{ where each } Z(V_i) \text{ is finitely presented as a direct summand of a finitely presented module. We write } Z(V_i) \cong R^{n_i}/U_i \text{ for some } n_i < \omega \text{ and finitely generated submodule } U_i \text{ of } R^{n_i}. \text{ Since } R \text{ is a non-singular semi-hereditary ring, } U_i \text{ is projective, and } Z(V_i) \text{ has projective dimension } 1. \]

The converse holds by Theorem 1.

(b) \Rightarrow (a) Let \( I \) be a finitely generated right ideal of \( R \). Since \( R/I \) is finitely presented, it is the direct sum of a projective module and a module of projective dimension at most 1 by (b). Hence \( I \) has to be projective.

(a) \Rightarrow (c): (i) \Rightarrow (ii) Let \( U \) be a countably generated submodule of a non-singular Mittag-Leffler module \( M \). By [10] there is a pure-projective, countably generated pure submodule \( V \) of \( M \) that contains \( U \). By Theorem 1 and the already proved implication (a) \Rightarrow (b) of this theorem, \( V \) is an \( S \)-closed projective submodule of \( M \). In particular, \( V \) contains the \( S \)-closure \( U_* \) of \( U \). By [2, Proposition 2.2] the module \( V/U_* \) has projective dimension at most 1. Since \( V \) is projective, this yields that \( U_* \) has to be projective too.

(ii) \Rightarrow (iii) is obvious.

(iii) \Rightarrow (i) By [10] it is enough to show that every finite subset of \( M \) is contained in a pure-projective pure submodule of \( M \). But \( S \)-closed submodules are pure by Theorem 1.

(c) \Rightarrow (a) Let \( I \) be a finitely generated right ideal of \( R \). Consider an exact sequence \( 0 \to U \to R^n \to I \to 0 \) of right \( R \)-modules where \( n < \omega \). Since \( R \) has finite right Goldie-dimension, \( U \) contains a finitely generated essential submodule \( V \). Furthermore, \( R^n \) is a non-singular Mittag-Leffler module. By (c), the \( S \)-closure \( W \) of \( V \) in \( R^n \) is projective [5, Proposition 8.24] yields that \( W \) is finitely generated. Since \( U \) is \( S \)-closed in \( R^n \) and \( V \) is essential in \( U \), it follows that \( U = W \). Thus \( I \) is finitely presented, and in particular, a Mittag-Leffler module. By (c), finitely generated non-singular Mittag-Leffler modules are projective.

Since every ideal of a Noetherian integral domain is a Mittag-Leffler module, the ring \( \mathbb{Z}[x] \) is an example of a domain over which there exist torsion-free Mittag-Leffler modules which are not \( \aleph_1 \)-projective.

In [11, Section 6.3] Rothmaler studies the structure of flat Mittag-Leffler modules over a right hereditary \( RD \)-Ore-domain, i.e., a right hereditary right and left Ore-domain for which purity and relative divisibility coincide. An \( RD \)-Ore-domain is right and left
semi-hereditary, hence it is a strongly non-singular semi-hereditary Goldie ring. From Example 4 we thus have

**Example 6.** Every RD-Ore-domain is a strongly non-singular semi-hereditary Goldie ring, but the converse is not true in general.

We can use Theorem 5 to determine the projective dimension of Mittag-Leffler modules:

**Corollary 7.** Let $R$ be a ring.

(a) $R$ is right semi-hereditary if and only if for every Mittag-Leffler right $R$-module $M$ and every integer $n \geq 0$, if $M$ can be generated by $\leq \aleph_n$ elements then $\operatorname{proj. dim.} M \leq n + 1$.

(b) If $R$ is a strongly non-singular semi-hereditary Goldie ring and $M$ is a non-singular Mittag-Leffler module generated by $\leq \aleph_n$ elements, then $\operatorname{proj. dim.} M \leq n$.

**Proof.** If every countably generated Mittag-Leffler right $R$-module $M$ has projective dimension $\leq 1$, then $\operatorname{proj. dim.} R/I \leq 1$ for every finitely generated right ideal $I$ of $R$, so that $R$ is right semi-hereditary.

Conversely, suppose that $R$ is right semi-hereditary and argue by induction on $n \geq 0$. If $n = 0$, a Mittag-Leffler right $R$-module generated by $\leq \aleph_0$ elements is pure-projective, and therefore it has projective dimension $\leq 1$ because every finitely presented right $R$-module over a right semi-hereditary ring has projective dimension $\leq 1$. And if $n = 0$ and $M$ is a non-singular Mittag-Leffler module over a strongly non-singular Goldie ring generated by $\leq \aleph_0$ elements, then $M$ is projective by Theorem 5.

Suppose $n > 0$. Let $M$ be a Mittag-Leffler right $R$-module generated by a set $\{x_\nu | \nu < \omega_n\} \subseteq M$. For every finite subset $X$ of $M$ fix a pure, countably generated, pure-projective submodule $V_X$ of $M$ containing $X$. Define a submodule $W_\alpha$ of $M$ generated by $\leq \aleph_{n-1}$ elements by transfinite induction on $\alpha \in \omega_n \times \omega_0$, where $\omega_n \times \omega_0$ denotes the lexicographic product of $\omega_n$ and $\omega_0$, in the following way. Set $W_0 = 0$. If $\alpha \in \omega_n \times \omega_0$ is a limit ordinal, set $W_\alpha = \bigcup_{\beta < \alpha} W_\beta$. If $\alpha \in \omega_n \times \omega_0$ is not a limit ordinal, then $\alpha = (\nu, r + 1)$ for some $\nu < \omega_n$ and some $r < \omega_0$. If $\nu$ is a limit ordinal, set $W_\alpha = W_{(\nu, r)}$. If $\nu$ is not a limit ordinal, then $\alpha = (\mu + 1, r + 1)$. In this case let $X_\mu$ be a set of generators of $W_{(\mu + 1, r)}$ of cardinality $\leq \aleph_{n-1}$ and set $W_\alpha = \sum \{ V_X \cup \{x_\nu\} \mid X \subseteq X_\mu, X \text{ finite} \}$. Note that $W_\alpha$ has a set of generators of cardinality $\leq \aleph_{n-1}$. 


It is clear that $W_0 \subseteq W_1 \subseteq \ldots \subseteq W_n \subseteq \ldots$, $\alpha \in \omega_n \times \omega_0$, is an ascending chain of submodules of $M$. We claim that $W_{(v, 0)}$ is pure in $M$ for every ordinal $v < \omega_n$. In order to prove the claim, let $A$ be a $k \times m$ matrix over $R$, $Z$ a $1 \times k$ matrix over $M$ and $Y = (y_1, \ldots, y_m)$ a $1 \times m$ matrix over $W_{(v, 0)}$ such that $ZA = Y$. We must show that there exists a $1 \times k$ matrix $Z'$ over $W_{(v, 0)}$ such that $Z'A = Y$. Since $y_1, \ldots, y_m \in W_{(v, 0)}$ and $(v, 0) \in \omega_n \times \omega_0$ is a limit ordinal, there exists $\beta < (v, 0)$ such that $y_1, \ldots, y_m \in W_{\beta}$. Let $\beta \leq \beta$ be the least ordinal such that $y_1, \ldots, y_m \in W_{\beta}$. Then $\beta$ is not a limit ordinal, and $\beta$ must be of the form $(\omega + 1, r + 1)$. Let $X$ be a finite subset of the set $X_{(\omega + 1, r + 1)}$ of generators of $W_{(\omega + 1, r + 1)}$ such that $y_1, \ldots, y_m$ belong to the submodule $XR$ of $M$ generated by $X$. The pure submodule $V_{X \cup \{x_0\}}$ of $M$ is contained in $W_{(\omega + 1, r + 2)}$ and contains $y_1, \ldots, y_m$. Therefore there exists a $1 \times k$ matrix $Z'$ over $V_{X \cup \{x_0\}}$ such that $Z'A = Y$. This concludes the proof of the claim, because $V_{X \cup \{x_0\}} \subseteq W_{(\omega + 1, r + 2)} = W_{\beta + 1} \subseteq W_{(v, 0)}$.

Since the modules $W_{(v, 0)}$ are generated by $\leq \kappa_{n-1}$ elements and pure submodules of Mittag-Leffler modules are Mittag-Leffler modules, it follows that the inductive hypothesis can be applied, so that proj. dim. $W_{(v, 0)} \leq n$ (and proj. dim. $W_{(v, 0)} \leq n - 1$ if $R$ is a strongly non-singular semi-hereditary Goldie ring and $M$ is non-singular) for every $v < \omega_n$. By Auslander's Theorem, the projective dimension of $M$ cannot exceed $n + 1$ (or $n$ if $R$ is a strongly non-singular semi-hereditary Goldie ring and $M$ is non-singular).

If we restrict our discussion to semi-prime rings, the equivalences in Part (c) of Theorem 5 can be further improved. Observe that the semi-prime strongly non-singular semi-hereditary rings without infinite sets of orthogonal idempotents are precisely the semi-prime right and left semi-hereditary Goldie rings. Moreover, if $R$ is a semi-prime right Goldie ring, then a right ideal of $R$ is essential if and only if it contains a regular element [5, Lemma 1.11 and Cor. 1.20], so that $Z(M) = \{ x \in M | xc = 0 \text{ for some regular element } c \in R \}$ for any right $R$-module $M$. In particular if $N$ is a submodule of a non-singular right module $M$ over a semi-prime semi-hereditary Goldie ring, then $N$ is pure in $M$ if and only if $Mc \cap N = Nc$ for every regular element $c \in R$.

**Corollary 8.** Let $R$ be a semi-prime, right and left semi-hereditary Goldie ring. The following conditions are equivalent for an $R$-module $M$:

(a) $M$ is a Mittag-Leffler module.

(b) $Z(M)$ is a Mittag-Leffler module, and $M/Z(M)$ is $\kappa_1$-projective.
PROOF. Since the class of Mittag-Leffler modules is closed with respect to pure submodules and pure extensions, Theorems 1 and 5 reduce the problem to showing that $M/Z(M)$ is Mittag-Leffler whenever $M$ is Mittag-Leffler. For this, let $U$ be a finitely generated submodule of $M$, and choose a pure-projective pure submodule $V$ of $M$ which contains $U$. By Theorem 5, $V = P \oplus Z(V)$ for some projective submodule $P$ of $M$. Since $[U + Z(M)]/Z(M) \subseteq [P \oplus Z(M)]/Z(M)$, the corollary will follow once we have shown that $P \oplus Z(M)$ is $S$-closed in $M$. Suppose that $x \in M$ satisfies $x c \in [P \oplus Z(M)]$ for some regular element $c \in R$. We can find $y \in P$ and a regular $d \in R$ such that $x c d - p d = 0$. But $P$ is pure in $V$ and $V$ is pure in $M$, so that $P$ is pure in $M$. Thus $x c d = p d \in P \cap \cap M c d = P c d$, and $x \in P \oplus Z(M)$.

The rest of this Section is devoted to completely recover Lemmas 6.10, 6.11, 6.12, Theorem 6.13 and Corollary 6.14 of [11] for the more general class of rings discussed in this paper.

PROPOSITION 9. Let $R$ be a strongly non-singular semi-hereditary Goldie ring and $M$ an $R$-module with the property that every countably generated submodule of $M$ is projective. Then $M$ is a non-singular Mittag-Leffler $R$-module and every finite subset of $M$ is contained in a finitely generated projective pure submodule of $M$.

PROOF. Let $M$ be a module satisfying the hypotheses of the statement. It is obvious that $M$ is non-singular.

We claim that if $X$ is a finitely generated submodule of $M$, then the $S$-closure $C$ of $X$ in $M$ is finitely generated. In order to prove the claim it is sufficient to show that each countably generated submodule $N$ of $C$ containing $X$ is finitely generated. Any such $N$ is projective, hence $N = \bigoplus_{i < \omega} N_i$, where the $N_i$ are isomorphic to finitely generated right ideals of $R[1]$. So it is enough to show that $N = \bigoplus_{i = 0}^{n} N_i$ for some $n < \omega$. Choose $n < \omega$ such that $X \subseteq \bigoplus_{i = 0}^{n} N_i$ and set $N' = \bigoplus_{i = 0}^{n} N_i$. We have $X \subseteq N' \subseteq N \subseteq C$.

Since $C$ modulo the submodule generated by $X$ is singular, $N/N' \cong \bigoplus_{i > n} N_i$ also is singular. But the $N_i$'s are isomorphic to right ideals of $R$, and therefore $N/N' \cong \bigoplus_{i > n} N_i$ is non-singular. Therefore $N' = N$, and $N$ is finitely generated. This proves our claim.

Since every finitely generated submodule of $M$ is projective, it is now clear that the $S$-closure of every finitely generated submodule of $M$ is a finitely generated projective pure submodule of $M$. In particular $M$ is a Mittag-Leffler module (Theorem 5).
LEMMA 10. Let $R$ be a right strongly non-singular right Goldie ring, $C$ a non-singular right $R$-module and $P$ a finitely generated submodule of $C$. If $C/P$ is singular, then $C$ has finite Goldie dimension.

PROOF. Since $P$ is a finitely generated non-singular module over a right strongly non-singular ring, $P$ is a submodule of a finitely generated free module. In particular, $P$ has finite Goldie dimension. Since $C$ is non-singular and $C/P$ is singular, $P$ is an essential submodule of $C$. This shows $\text{dim } C = \text{dim } P < \infty$.

THEOREM 11. The following four conditions are equivalent for a right strongly non-singular right Goldie ring $R$:

(a) $R$ is a right hereditary ring.
(b) $R$ is a right noetherian, right hereditary ring.
(c) $R$ is a right semi-hereditary ring and all submodules of non-singular Mittag-Leffler right $R$-modules are Mittag-Leffler modules.
(d) The following conditions are equivalent for a right $R$-module $M$:
   (i) $M$ is a non-singular Mittag-Leffler module.
   (ii) Every countably generated submodule of $M$ is projective.
   (iii) Every finite subset of $M$ is contained in a finitely generated projective pure submodule of $M$.
   (iv) $M$ is non-singular and every finite subset of $M$ is contained in a finitely presented pure submodule of $M$.
   (v) $M$ is non-singular and every submodule of $M$ of finite Goldie dimension is a finitely generated projective module.

PROOF. (a) $\Rightarrow$ (d) Suppose that $R$ is right hereditary.

(i) $\Rightarrow$ (ii) is proved in [11, Cor. 6.3].

(ii) $\Rightarrow$ (iii) is proved in Proposition 9.

(iii) $\Rightarrow$ (iv) If every element of $M$ is contained in a projective module, $M$ must be non-singular. Moreover, every finitely generated projective submodule is finitely presented.

(iv) $\Rightarrow$ (v) Suppose that (iv) holds and let $N$ be a submodule of $M$ of finite Goldie dimension. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite subset of $N$ such that $\sum_{i=1}^{n} x_i R = \bigoplus_{i=1}^{n} x_i R$ is an essential submodule of $N$. Then $N/\sum_{i=1}^{n} x_i R$ is a singular submodule of $M/\sum_{i=1}^{n} x_i R$, so that if $C$ denotes the $\mathcal{S}$-closure of $\sum_{i=1}^{n} x_i R$ in $M$, then $N \subseteq C$. By (iv) the subset $X$ is con-
tained in a finitely presented pure submodule $D$ of $M$. Since $M$ is non-singular, $D$ also is non-singular, hence flat, hence projective. Thus $X$ is contained in the finitely generated projective pure submodule $D$ of $M$. Therefore $C$ is a submodule of $D$. Hence $N$ is contained in the projective module $D$, and $N$ is projective because $R$ is right hereditary. By [1] $N$ is isomorphic to a direct sum of finitely generated right ideals. But $N$ has finite Goldie dimension, and therefore $N$ itself is a finitely generated projective module.

(v) $\Rightarrow$ (i) Suppose that (v) holds. In order to prove that $M$ is a Mittag-Leffler module it is sufficient to show that every finite subset $X$ of $M$ is contained in a pure-projective pure submodule of $M$ [4, Th. 6]. Let $P$ be the submodule of $M$ generated by a finite subset $X$ of $M$ and $C$ be the $S$-closure of $P$ in $M$, so that $C$ is pure in $M$. By Lemma 10 the module $C$ has finite Goldie dimension. By Hypothesis (v) $C$ is a finitely generated projective module.

(d) $\Rightarrow$ (c) Assume that $R$ has the property that the five conditions are equivalent for every right $R$-module $M$. Let us show that $R$ is right hereditary. If $I$ is a right ideal of $R$, then $I$ is a submodule of the non-singular Mittag-Leffler module $R_R$, which is of finite Goldie dimension. By (d) $I$ is a finitely generated projective module.

Since $M$ is a non-singular Mittag-Leffler module if and only if every countably generated submodule of $M$ is projective, every submodule of a non-singular Mittag-Leffler module is a non-singular Mittag-Leffler module.

(c) $\Rightarrow$ (b) In order to show that $R$ is right noetherian, it is sufficient to show that if $I$ is a countably generated right ideal of $R$, then $I$ is finitely generated. Since $R_R$ is a non-singular Mittag-Leffler module, every right ideal of $R$ is a non-singular Mittag-Leffler module. Hence every countably generated right ideal $I$ of $R$ is a non-singular pure-projective module, that is, it is projective. Then $R/I$ is a finitely generated module of projective dimension $\leq 1$, and therefore it is finitely presented (Theorem 1). Hence $I$ is finitely generated.

(b) $\Rightarrow$ (a) is obvious.

4. – Prüfer rings and indecomposable Mittag-Leffler modules.

Recall that a commutative integral domain is semi-hereditary if and only if it is a Prüfer ring, that is, all its localizations at maximal ideals are valuation domains. If $R$ is an integral domain, for every $R$-module $M$ the submodule $Z(M)$ is exactly the torsion submodule $t(M)$ of $M$, so that a module is non-singular if and only if it is torsion-free. Hence The-
orem 5 gives a complete description of torsion-free Mittag-Leffler modules over Prüfer domains: a torsion-free module over a Prüfer domain is a Mittag-Leffler module if and only if it is \( \kappa_1 \)-projective. More generally, a module \( M \) over a Prüfer domain is a Mittag-Leffler module if and only if \( M/t(M) \) is \( \kappa_1 \)-projective and \( t(M) \) is a torsion Mittag-Leffler module. The structure of torsion Mittag-Leffler modules over a Prüfer domain \( R \) depends heavily on the properties of \( R \). For instance, in [4, Prop. 7] it is shown that a torsion abelian group \( G \) is a Mittag-Leffler \( \mathbb{Z} \)-module if and only if \( \bigcap_{n>0} nG = 0 \). In the next Proposition we describe torsion Mittag-Leffler modules over almost maximal valuation domains and arbitrary Mittag-Leffler modules over maximal valuation rings. Recall that an \( R \)-module is cyclically presented if it is isomorphic to \( R/aR \) for some \( a \in R \).

**Proposition 12.** Let \( M \) be a torsion module over an almost maximal valuation domain \( R \) or an arbitrary module over a maximal valuation ring \( R \). The following conditions are equivalent:

(a) \( M \) is a Mittag-Leffler \( R \)-module.

(b) Every finite subset of \( M \) is contained in a direct summand of \( M \) that is a direct sum of cyclically presented modules.

(c) Every element of \( M \) is contained in a direct summand of \( M \) that is a direct sum of cyclically presented modules.

**Proof.** (a) \( \Rightarrow \) (b) Let \( X \) be a finite subset of \( M \). Then \( X \) is contained in a pure-projective pure submodule \( P \) of \( M \) [4, Th. 6]. The pure-projective module \( P \) is a direct sum of cyclically presented modules [8, Th. II.3.4 and Prop. II.4.3]. Hence \( P \) decomposes as \( P = P' \oplus P'' \), where \( X \subseteq P' \) and \( P' \) is a finite direct sum of cyclically presented modules. By [8, Th. XI.4.2] \( P' \) is pure-injective. Since \( P' \) is pure in \( M \), \( P' \) must be a direct summand of \( M \).

(b) \( \Rightarrow \) (c) is obvious.

(c) \( \Rightarrow \) (a) Let \( X \) be a finite subset of \( M \). By [4, Th. 6] it is sufficient to prove that \( X \) is contained in a pure-projective pure submodule of \( M \). By [8, Prop. XIII.2.4] the module \( M \) is separable, that is, every finite set of elements of \( M \) can be embedded in a direct summand which is a direct sum of uniserial modules. Hence it is enough to prove that every uniserial direct summand \( U \) of \( M \) is cyclically presented. Let \( x \) be a non-zero element of a uniserial direct summand \( U \) of \( M \). By (c) there exists a direct summand \( P \) of \( M \) such that \( P \) is a finite direct sum of cyclically presented modules and \( x \in P \). Let \( W \) and \( Q \) be direct complements of \( U \) and \( P \), so that \( M = U \oplus W = P \oplus Q \). Since \( P \) has the exchange property...
There are submodules $U' \leq U$ and $W' \leq W$ such that $M = P \oplus U' \oplus W'$. Since $0 \neq x \in P \cap U$, $P \cap U$ is an essential submodule of the uniserial module $U$. But $(P \cap U) \cap U' = 0$, so that $U' = 0$ and $M = P \oplus W'$. Then $U$ is a direct summand of $U \oplus \oplus (W/W') \cong (U \oplus W)/W' = M/W' \cong P$. In particular $U$ is pure-projective, that is, $U$ is a direct sum of cyclically presented modules. Hence the uniserial module $U$ must be cyclically presented.

Therefore over an almost maximal valuation domain $R$ the indecomposable torsion Mittag-Leffler modules are only the cyclically presented modules $R/aR'$'s, $a \neq 0$, and over a maximal valuation ring $R$ the indecomposable Mittag-Leffler modules are only the cyclically presented modules $R/aR$'s, $a \in R$. The last result of this paper addresses the question whether there exist arbitrarily large indecomposable non-singular Mittag-Leffler modules.

**Example 13.** Let $R$ be a strongly non-singular semi-hereditary Goldie ring whose additive group is cotorsion-free. Then there exists a proper class of pairwise non-isomorphic, indecomposable, non-singular Mittag-Leffler $R$-modules.

**Proof.** Let $\kappa$ be an infinite cardinal. Since $R$ has a cotorsion-free additive group, there exists an $\aleph_1$-projective left $R$-module $M$ of cardinality at least $\kappa$ such that $\text{End}_R(M) \cong \text{R}^{op}$ by [6]. By Theorem 5, $M$ is a non-singular Mittag-Leffler module whose $R$-endomorphism ring is $\text{Center}(R)$. Since $R$ does not contain any infinite family of orthogonal idempotents, the same holds for $\text{Center}(R)$. We write $1 = e_1 + \ldots + e_n$ where $\{e_1, \ldots, e_n\}$ is a family of orthogonal, primitive idempotents of $\text{Center}(R)$. Then $M_i = e_i(M)$ is an indecomposable Mittag-Leffler module. Since $|M| > \kappa$ and $M = \bigoplus_{i=1}^{n} M_i$, at least one of the $M_i$'s has cardinality at least $\kappa$.

The ring of algebraic integers is an example for a ring as in Example 13.

**REFERENCES**


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