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A Universal Property of the Cayley-Chow Space of Algebraic Cycles.

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For every projective scheme X and projective embedding $e: X \to \mathbb{P}$, there is a reduced scheme $C_n(X, e)$ parametrizing all cycles of pure dimension n with support in X. The construction is carried over in terms of the given embedding, so that in particular the space of cycles $C_n(X, e)$ comes out equipped with a partition into disjoint subschemes $C_{n,k}(X, e)$, each endowed with a natural projective embedding, for all degrees k of cycles in $\mathbb{P}$. The idea goes back to Cayley [4], but the effective construction is due to Chow [5]. In recent times new important applications appeared [3], [6], [7] which motivate the present contribution to a long standing problem of setting suitable foundations.

Over the complex field, there is a theorem of Barlet [2] saying that different projective embeddings e do always give rise to isomorphic schemes $C_n(X, e)$. This is not true in positive characteristics as shown by Nagata [13]. We therefore confine ourselves to complex schemes.

The question arises how to describe morphisms $T \to C_n(X, e)$ intrinsically by means of incidence correspondences (cycles) Z in $T \times X$, without reference to the embedding. This would be an «explicit» universal property for $C_n(X, e)$, for the essential independence of e is already an «implicit» one.

In general, a (relative) incidence cycle Z only induces a regular map on the smooth locus $T_{sm} \to C_n(X, e)$. The geometrically relevant point is therefore to determine when the induced map on T_{sm} admits a continuous extension $f_Z: T \to C_n(X, e)$, what we call a «semi-regular» (rational) map. This requires a careful study of the limit behaviour of fibre

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cycles, when approaching a singular point moving along the smooth locus in the parameter variety T. We prove that there is a unique «limit cycle» along any path contained within a given branch of T at the singular point. The «regular» cycles Z, those which correspond to semi-regular maps into $C_n(X, e)$, are therefore characterized by the property that the limit cycle is the same one along every branch. This is the main result of the present paper, which is missing in the existing literature.

More precisely, we prove that the space of cycles $C_n(X, e)$ represents a certain functor of regular relative cycles on the category of reduced schemes and semi-regular maps. This implies the earlier result of Andreotti-Norguet [1], that the semi-normalization of $C_n(X, e)$ is essentially independent of the embedding. Finally, once a (regular) cycle Z is known to determine a semi-regular map f_Z, then the property of f_Z being everywhere regular is essentially independent of the embedding e, according to the theorem of Barlet.

Up to date. While the present paper was in the publication procedure, the author had the opportunity of looking into the first chapter of a forthcoming book (J. Kollár, *Rational Curves on Algebraic Varieties*, preliminary version (1994)), containing a treatment of the same subject. However, there are relevant differences both in the methods and in the final results.

1. – The space of cycles.

An introduction to the space of cycles.

Let V be a complex vector space of dimension ≥ 1, and let $\mathbb{P} = \mathbb{P}(V)$ be the associated complex projective space. By the symbol x we will denote both a point in $\mathbb{P}$ and any vector in V representing that point. Likewise, by the symbol u we will denote both a linear form belonging to the dual space $V^\vee$ and the corresponding point in $\mathbb{P}^\vee = \mathbb{P}(V^\vee)$.

Let Z be any subvariety of dimension n and degree k in $\mathbb{P}$. There is an irreducible polynomial $F_Z(u_0, \ldots, u_n)$, homogeneous with respect to each u_i, such that $F_Z(u_0, \ldots, u_n) = 0$ if and only if there is some point $x \in Z$ which satisfies $u_0(x) = \ldots = u_n(x) = 0$, i.e. if and only if the linear space L defined by the linear forms $u_0, \ldots, u_n$ meets Z. This polynomial F_Z, which is unique up to proportionality, is called the *Cayley-Chow form*, or simply the *associated form*, of the variety Z, and its coefficients are called the coordinates of Z. The associated form of a positive cycle $Z = \sum a_i Z_i$ of pure dimension n in $\mathbb{P}$ is defined as $F_Z = \prod F_{Z_i}^{a_i}$. If F is an associated form, we denote in turn by $Z(F)$ the unique cycle whose associated form is F.

PROPOSITION 1.1. *The polynomial F_Z has degree $k = \deg(Z)$ with respect to each u_i* [10; p. 41].

For any square matrix A of order $n+1$, the substitution $u = (u_0, \ldots, u_n) \mapsto Au = (u_0', \ldots, u_n')$, where $u_i' = \sum_j a_{ij} u_j$, replaces the linear forms u by linear forms u' belonging to the subspace generated by u. The substitution $F(u) \mapsto F(Au)$ defines an action of the general linear group $GL(n+1)$ on the space of polynomials in $u_0, \ldots, u_n$, which is identified with the symmetric algebra of the vector space $\bigoplus^{n+1}(V^\vee)$.

PROPOSITION 1.2. *The polynomial F_Z satisfies the identity*:

$$F_Z(Au) = \det(A)^k F_Z(u),$$

where $k = \deg(Z)$ [1; p. 43].

A polynomial $F(u)$ satisfying $F(Au) = \det(A)^k F(u)$ for some integer $k \geqslant 0$ is called a *relative invariant* of $GL(n+1)$, or else an *absolute invariant* of the special linear group $SL(n+1)$. There is a canonical mapping

$$j \colon S(\bigwedge^{n+1}(V^\vee)) \to S(\bigoplus^{n+1}(V^\vee))$$

of the symmetric algebra of the vector space $\bigwedge^{n+1}(V^\vee)$ onto a subalgebra of the ring of polynomials in $u_0, \ldots, u_n$, sending an element F of the former into that polynomial F' such that $F'(u_0, \ldots, u_n) = F(u_0 \wedge \ldots \wedge u_n)$. From invariant theory we recall the following:

THEOREM 1.3 (first main theorem on covariant invariants of $SL(n+1)$). *The subring of $S(\bigoplus^{n+1}(V^\vee))$ consisting of the invariants of $SL(n+1)$ is the image through j of $S(\bigwedge^{n+1}(V^\vee))$* [15; p. 45, Thm. 2.6.A].

In other words, the polynomials in $u_0, \ldots, u_n$ which are invariant under the action of $SL(n+1)$ are those which can be written as polynomials in the wedge product $u_0 \wedge \ldots \wedge u_n$, though not in a unique way.

In the projective space $\mathbb{P}(\bigwedge^{n+1}(V^\vee))$ the classes of the *decomposable* $(n+1)$-vectors $u_0 \wedge \ldots \wedge u_n$ form the support of the Grassmann variety $\mathbb{G} = \mathbb{G}^{n+1}$ of linear spaces of codimension $n+1$ in $\mathbb{P}$. The ideal of $\mathbb{G}$ is the kernel of j, hence $j(S(\bigwedge^{n+1}(V^\vee)))$ is the coordinate ring of $\mathbb{G}$.

THEOREM 1.4. *The coordinate ring* $j(S(\bigwedge^{n+1}(V^\vee)))$ *is a unique factorization domain* [14; p. 38, Prop. 8.5].

Every homogeneous invariant polynomial $F \in j(S(\bigwedge^{n+1}(V^\vee)))$, in particular an associated form F_Z, defines an hypersurface section of $\mathbb{G}$ and, because of the theorem, every effective Cartier divisor in $\mathbb{G}$ is an hypersurface section. Note here that the factorizations of F in the coordinate ring of $\mathbb{G}$ and in $S(\bigoplus^{n+1}(V^\vee))$ are the same one [9; 1.5].

For given integers $n \geq 0$, $k \geq 1$, let $\mathbb{F}_{n,k}$ be the projective space associated to the vector space of polynomials $F(u_0, \ldots, u_n)$ which are homogeneous of degree k with respect to each u_i and invariant under $SL(n+1)$, i.e. the k-th graded piece of $j(S(\bigwedge^{n+1}(V^\vee)))$. As usual, we denote by F both such a polynomial and the corresponding point in $\mathbb{F}_{n,k}$. Let us denote by $\mathbb{C}_{n,k} = C_{n,k}(\mathbb{P})$ the set of associated points $F_Z \in \in \mathbb{F}_{n,k}$ of cycles Z of pure dimension n and degree k in $\mathbb{P}$.

THEOREM 1.5. $\mathbb{C}_{n,k}$ *is an algebraic subset of* $\mathbb{F}_{n,k}$. Proofs are found in [5; satz 2], [10; p. 57, Thm. II].

For any (closed) projective embedding $e: X \to \mathbb{P}$, the set $C_{n,k}(X, e)$ of associated points of cycles of pure dimension n and degree k in $\mathbb{P}$ whose support is contained in $e(X)$ is an algebraic subset of $\mathbb{C}_{n,k} = C_{n,k}(\mathbb{P})$, which will be considered endowed with its reduced scheme structure. Let $C_n(X, e)$ be the disjoint sum over k of all $C_{n,k}(X, e)$.

Furthermore, the set of pairs $(F, x) \in \mathbb{C}_{n,k} \times \mathbb{P}$ such that the point x belongs to the support of the cycle $Z(F)$ is an algebraic subset $\mathbb{U}_{n,k}$, the incidence subset, and there is an analogous incidence subset within $C_{n,k}(X, e) \times X$.

We end with a helpful definition. Any purely dimensional cycle Z of degree k in $\mathbb{P}$ may uniquely be written as a linear combination $Z = a_1 Z_1 + \ldots + a_s Z_s$, with multiplicities $1 \leq a_1 < \ldots < a_s$, such that each Z_i is a simple cycle (no multiple components) and Z_i, Z_j have no common components for each pair $i \neq j$. Let $k_i := \deg(Z_i)$.

DEFINITION 1.6. With the above notations, we call the $2s$-tuple

$$\theta := (a_1, \ldots, a_s; k_1, \ldots, k_s)$$

the type of Z and a *type of cycles* of degree k. Let $\mathbb{C}_{n,\theta}$ denote the set of associated points of cycles Z of type θ.

The collection of all $\mathbb{C}_{n,\theta}$ is a partition of $\mathbb{C}_{n,k}$ into disjoint locally closed subsets [9; § 2]. Therefore, if T is a variety and $f: T \to \mathbb{C}_{n,k}$ is a regular map, there is a unique θ such that $f(T) \subseteq \overline{\mathbb{C}}_{n,\theta}$ and $f(T) \cap \mathbb{C}_{n,\theta} \neq \neq \emptyset$ (hence $f^{-1}(\mathbb{C}_{n,\theta})$ is a dense open subset of T).

DEFINITION 1.7. In the above setting we say that the regular map f is of type θ.

2. – Families of cycles.

Some general setting for families of cycles.

By a scheme we mean any disjoint union of separated schemes of finite type over the complex field, for which the irreducible components still make sense. This is a useful generality for the parameter schemes of families, which in addition will always be reduced. Moreover, a subscheme will always be closed, a variety will be an integral scheme, and a point will be a closed point.

Let T be a parameter variety, $m := \dim(T)$, X a carrier scheme, Z a cycle of pure dimension $m + n$ on $T \times X$, p: $|Z| \to T$ the restriction of pr_T to the support of Z, and write the reduced fibre $p^{-1}(t)_{\text{red}}$ as $t \times \times |Z|_t$. We always assume that p is a proper map, which happens of course if the carrier X is projective. We say that Z is a *relative cycle* of relative dimension n over T if every fibre of p has pure dimension n. So every irreducible component of $|Z|$ projects onto T. If $Z = [U]$ is the cycle of a closed subscheme U of $T \times X$ (of pure dimension $m + n$) as defined in [8; 1.5], then also there is a collection of fibre schemes $U(t)$, closed subschemes of the carrier X, with support $U(t)_{\text{red}} = |Z|_t$.

In this setting, for every smooth point $t \in T_{sm}$ there is a *fibre cycle* $Z(t)$ of pure dimension n with support $|Z|_t$. If $Z = [U]$ then $Z(t)$ gives to every irreducible component Y of $U(t)$ the multiplicity of the primary ideal generated by the maximal ideal of the local ring of T at t inside the local ring of U along $t \times Y$ [8; 10.1.1]. This definition then extends by linearity.

All these cycles $Z(t)$ are algebraically equivalent on X (as T is irreducible). If $X = \mathbb{P}$ they all have one and the same degree. If the carrier X also is smooth, then $Z \cdot (t \times X) = t \times Z(t)$ holds [8; 10.1.2], which classically was a definition.

Consider in particular the case of fibre dimension $n = 0$. Then p is a finite map. Assume that $Z = [U]$ with U reduced. Then the *ramification index* of p at $(t, x) \in |Z|$ is equal to the multiplicity of the fibre cycle $Z(t)$ at x, for every $t \in T_{sm}$ [8; 7.1.15].

Topologically, the ramification index of p at any point (t, x) is the cardinality of $|Z(t')| \cap N$ for all sufficiently small neighborhoods N of x and every $t' \in T_{sm}$ near t (in the complex topology) and out of the branch locus of p [11; 3.12]. If T is locally irreducible (unibranch) at t, then the ramification formula holds [11; 3.25]

Let us write «ramif. p at (t, x)» or else «ramif. $[U]|T$ at (t, x)» for the ramification index, and by linearity define «ramif. $Z|T$ at (t, x)» for every cycle Z of relative dimension 0 over T. This preserves the usual properties of ramification indices, such as the ramification formula, for instance.

A topological description of the ramification index of a cycle Z at a point (t, x) is in terms of cycles of the form $Z(t')|N$, sum of points belonging to $|Z(t')| \cap N$ each counted with the same multiplicity as in $Z(t')$, for every sufficiently small neighborhood N of x and point $t' \in T_{sm}$ sufficiently near t.

LEMMA 2.1. *Assume that T is a locally irreducible variety. There are neighborhood bases in the complex topologies of T and X such that, for every $(t, x) \in |Z|$, and basic neighborhoods T' of $t \in T$, N of $x \in X$,*

$$\text{ramif. } Z|T \text{ at } (t, x) = \deg Z(t')|N$$

holds for $t' \in T_{sm} \cap T'$.

PROOF. By linearity, we may assume $Z = [U]$ with U reduced. A standard elementary argument involving some general topology of locally compact spaces tells us that, for every connected open subset T' of T and every open subset U' of $p^{-1}(T')$ with $\partial U' \cap p^{-1}(T') = \emptyset$, the restriction $p' : U' \to T'$ still is a finite map (of analytic spaces), and the open subsets U' of some $p^{-1}(T')$ as before form a basis in U. Thus a basis in X consists of the open subsets N such that $U' = U \cap (T \times N)$ is in the basis of U.

The basic neighborhoods U' of a given point (t, x), such that $\deg p|U'$ takes the smallest value = ramif. p at (t, x), clearly still form a neighborhood basis at that point. Similarly, a basis of neighborhoods N at a given point x is defined.

With respect to these neighborhood bases, since mult. $Z(t')$ at x = ramif. p at (t', x), one has:

$$\deg Z(t')|N = \sum_x \text{ramif. } p \text{ at } (t', x),$$

sum over $x \in |Z(t')| \cap N$, i.e. over $(t', x) \in p^{-1}(t') \cap U'$. Therefore

$$\deg Z(t')|N = \deg p|U'$$

is just the ramification formula for the finite map $p|U'$ at t', holding because $T_{sm} \cap T'$ is connected (as T is locally irreducible). And ramif. $Z|T$ at (t, x) = ramif. p at $(t, x) = \deg p|U'$ is just by definition. ∎

Taking fibre cycles is a special case of refined pull-back [8; 8.1.2]. Let $\alpha\colon S \to T$ be any morphism of varieties such that $\alpha(S) \cap T_{sm} \neq \emptyset$. *There is in* $S \times X$ *a unique relative cycle* $\alpha^! Z$ *over* S *such that* $(\alpha^! Z)(s) = Z(\alpha s)$ *for every* $s \in S_{sm} \cap \alpha^{-1}(T_{sm})$.

We define it as follows. By linearity we are reduced to the case $Z = [U]$, U a closed subscheme of $T \times X$. Assume first that T is smooth. Every irreducible component Y of the fibre product $U_S := S \times_T U$ is «proper», i.e. of $\dim Y = \dim S + n$ ($\geqslant$ is from [8; Lemma 7.1], $\leqslant$ is from the present setting). This implies that the refined pull-back $\alpha^! Z$, which is in principle only defined as a cycle class on U_S, is indeed a true cycle. It gives to Y as before the multiplicity of the primary ideal generated by the ideal of U_S in $S \times U$ inside the local ring of $S \times U$ along Y [8; 7.1.1]. For general T, using the restriction $\alpha^{-1}(T_{sm}) \to T_{sm}$, take the refined pull-back of the restriction of Z to $p^{-1}(T_{sm})$, it is a cycle on $\alpha^{-1}(T_{sm}) \times_T p^{-1}(T_{sm})$, and take then its closure in $S \times_T U$. If we denote by t again the inclusion mapping of the smooth one-point scheme $t \in T_{sm}$ into T, then $t^! Z = Z(t)$.

3. - Regular families.

Consider now again a relative cycle Z over T in $T \times X$, and any (possibly singular) point $t \in T$. There are germs of maps $\alpha\colon (S, s) \to (T, t)$, $\alpha(s) = t$, where S is a smooth curve and $\alpha(S - s) \subset T_{sm}$. For every such α there is a unique local irreducible component T' of the germ (T, t) such that $\alpha(S) \subset T'$ (inclusion of germs), and every local component T' is run by some α. For every α as before, define $Z(t, \alpha) := \alpha^! Z(s)$. Clearly $Z(t, \alpha) = Z(t)$ if $t \in T_{sm}$. We call the collection of $Z(t, \alpha)$, for every t and α, the *full family of cycles defined by* Z. They all belong to one and the same algebraic equivalence class on X.

LEMMA 3.1. *Let* $\nu\colon \overline{T} \to T$ *be the normalization map and consider the relative cycle* $\nu^! Z$ *over* $\overline{T}$. *There is a bijection between points* $t' \in \nu^{-1}(t)$ *and branches* T' *of the germ* (T, t), *such that the neighborhood of* t' *covers through* ν *the corresponding branch* T'. *Every curve* $\alpha\colon S \to T$, $\alpha(s) = t$, *with* $\alpha(S - s) \subset T_{sm}$ *has a unique lifting* $\overline{\alpha}\colon S \to \overline{T}$, $\overline{\alpha}(s) = t'$, *such that* $\alpha = \nu \circ \overline{\alpha}$. *Then:*

$$Z(t, \alpha) = \nu^! Z(t', \overline{\alpha}).$$

PROOF. Nothing but $\alpha^! Z = \overline{\alpha}^! (\nu^! Z)$. ∎

We therefore *assume from now on that we are given a locally irreducible variety T. Assume moreover that the carrier X is a projective scheme.* We are going to prove that $Z(t, \alpha)$ is then independent of α. By linearity, we may also assume that $Z = [U]$ with $U \subset T \times X$ a reduced subscheme.

Choose now a projective embedding $e\colon X \hookrightarrow \mathbb{P}$, and denote by $e.Z$ the image of Z in $T \times \mathbb{P}$ (under $1_T \times e$). However we will usually omit reference to e in the notation, except sometimes to remark what could in principle depend on it. Using the embedding, we find a «limit cycle» of $Z(t')$, as $t' \in T_{sm}$ tends to t. It could be described as the unique cycle whose section with a general linear space $L(u)$ is the limit of the family of linear sections $Z(t') \cdot L(u)$, as t' tends to t, where the limit in a family of 0-cycles is meant as the limit fibre in a ramified covering, with multiplicities viewed as ramification indices. This is in the spirit of some early treatment of intersection multiplicities.

Let $\mathbb{G}$ denote in this section the grassmannian of linear spaces of codimension n in $\mathbb{P}$, where n is the relative dimension of Z over T, and let $\mathbb{L} \subset \mathbb{P} \times \mathbb{G}$ be the universal linear space over $\mathbb{G}$, whose fibre cycles we denote by $L(u)$, instead of $\mathbb{L}(u)$. The diagram

$$\begin{array}{ccccc} U \times_{\mathbb{P}} \mathbb{L} & \to & \mathbb{L} & \to & \mathbb{G} \\ \downarrow & & \downarrow & & \\ U & \to & \mathbb{P} & & \\ \downarrow & & & & \\ T & & & & \end{array}$$

determines a mapping $\pi\colon U \times_{\mathbb{P}} \mathbb{L} \to T \times \mathbb{G}$, whose reduced fibre over (t', u) is the support of $|Z|_{t'} \cap L(u)$. The set $(T \times \mathbb{G})_0$ of all (t', u) such that $\dim |Z|_{t'} \cap L(u) = 0$ is a dense open subset of $T \times \mathbb{G}$, and so is $\pi^{-1}(T \times \mathbb{G})_0$ too. Similarly define $(T_{sm} \times \mathbb{G})_0$. The restriction $\pi^{-1}(T \times \times \mathbb{G})_0 \to (T \times \mathbb{G})_0$ is a finite map.

Denote by $Z \cdot_{\mathbb{P}} \mathbb{L}$ the cycle on $U \times_{\mathbb{P}} \mathbb{L}$ whose fibre cycle is

$$(Z \cdot_{\mathbb{P}} \mathbb{L})(t', u) = Z(t') \cdot L(u) \quad \text{for } (t', u) \in (T_{sm} \times \mathbb{G})_0.$$

The general construction gives $Z \cdot_{\mathbb{P}} \mathbb{L} = (Z \times \mathbb{G}) \cdot (T \times \mathbb{L})$ in $T \times \mathbb{P} \times \times \mathbb{G}$ [8; Cor. 10.1]. This provides an interpretation of intersection multiplicities as ramification indices:

$$\text{mult. } Z(t') \cdot L(u) \text{ at } x = \text{ ramif. } Z \cdot_{\mathbb{P}} \mathbb{L} | T \times \mathbb{G} \text{ at } (t', x, u)$$

for $(t', x, u) \in \pi^{-1}(T_{sm} \times \mathbb{G})_0$. The set R_m of points $(t', x, u) \in \in \pi^{-1}(T \times \mathbb{G})_0$ where $Z \cdot_{\mathbb{P}} \mathbb{L}$ has ramification index $\geqslant m$ is a closed algebraic set.

DEFINITION 3.2. *In the present setting, for every irreducible component Y of the fibre support $|Z|_t$, define m_Y to be the smallest m such that $(t \times Y \times \mathbb{G}) \cap \pi^{-1}(T \times \mathbb{G})_0$ is contained within R_m. This means that the ramification index of $Z \cdot_{\mathbb{P}} \mathbb{L}$ at (t, x, u) is m_Y for almost all pairs (x, u) with $x \in Y$. Define a «limit cycle» with support $|Z|_t$:*

$$\lim_t e.\ Z := \sum m_Y \cdot Y .$$

If $t \in T_{sm}$ then $\lim_t e.\ Z = Z(t)$. Indeed, for every Y and almost all (x, u) with $x \in Y$ one has: ramif. $Z \cdot_{\mathbb{P}} \mathbb{L}$ at $(t, x, u) =$ mult. $Z(t) \cdot \cdot L(u)$ at $x =$ (mult. Y in $Z(t)$) $\times$ (mult. $Y \cdot L(u)$ at x), together with: (mult. $Y \cdot L(u)$ at x) $= 1$.

PROPOSITION 3.3. *Assume that T is a locally irreducible variety, and let Z in $T \times X$ be a relative cycle over T. Let moreover e: $X \to \mathbb{P}$ be a projective embedding. For every curve α: $S \to T$ with $\alpha(s) = t$ and $\alpha(S - s) \subset T_{sm}$, then:*

$$Z(t, \alpha) = \lim_t e.\ Z .$$

PROOF. Remark that:

$$Z(t, \alpha) := \alpha^! Z(s) = \lim_s e.\ \alpha^! Z ,$$

as s is a smooth point of S. So what we are going to prove is:

$$\lim_s e.\ \alpha^! Z = \lim_t e.\ Z .$$

The following is a pull-back diagram:

$$\begin{array}{ccc} U_S \times_{\mathbb{P}} \mathbb{L} & \longrightarrow & U \times_{\mathbb{P}} \mathbb{L} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ S \times \mathbb{G} & \xrightarrow{\alpha \times 1} & T \times \mathbb{G} . \end{array}$$

Furthermore $(\alpha \times 1)^! (Z \cdot_{\mathbb{P}} \mathbb{L}) = \alpha^! (Z \cdot_{\mathbb{P}} \mathbb{L})$ [8; 6.2(c)], and $\alpha^! (Z \cdot_{\mathbb{P}} \mathbb{L}) = (\alpha^! Z) \cdot_{\mathbb{P}} \mathbb{L}$ in $U_S \times_{\mathbb{P}} \mathbb{L}$, because $\alpha^! (Z \times \mathbb{G} \cdot T \times \mathbb{L}) = \alpha^! (Z \times \mathbb{G}) \cdot \alpha^! (T \times \times \mathbb{L}) = (\alpha^! Z \times \mathbb{G}) \cdot (S \times \mathbb{L})$ in $S \times \mathbb{P} \times \mathbb{G}$. In the present situation, the following holds:

ramif. $\alpha^! Z \cdot_{\mathbb{P}} \mathbb{L} | S \times \mathbb{G}$ at $(s, x, u) =$ ramif. $Z \cdot_{\mathbb{P}} \mathbb{L} | T \times \mathbb{G}$ at (t, x, u),

which gives the desired formula and so completes the proof. This comes from:

LEMMA 3.4. *Assume that T is a locally irreducible variety, and let Z in $T \times X$ be a relative cycle of relative dimension 0 over T. Let α: $S \to T$ be a regular map of varieties such that $\alpha(S) \cap T_{sm} \neq \emptyset$. Then the pull-back cycle $\alpha^! Z$ over S has ramification indices*:

$$\text{ramif. } \alpha^! Z | S \text{ at } (s, x) = \text{ramif. } Z | T \text{ at } (\alpha s, x).$$

PROOF. By linearity assume $Z = [U]$ with U reduced. A neighborhood basis at $(s, x) \in U_S$ is given by the open subsets of the form $U_S \cap \cap (S' \times U')$ where, for some basic neighborhood T' of $\alpha(s) \in T$, $S' \subset \subset \alpha^{-1}(T')$ is a neighborhood of $s \in S$, $U' \subset p^{-1}(T')$ is a basic neighborhood of $(t, x) \in U$ of the form $U' = U \cap T' \times N$ where N is a basic neighborhood of $x \in X$, and the neighborhood bases in T and X are taken as in lemma (2.1). This implies that

$$\text{ramif. } Z | T \text{ at } (\alpha s, x) = \deg Z(t') | N$$

for every $t' \in T_{sm} \cap T'$, in particular for $t' = \alpha(s')$ with $s' \in S_{sm} \cap \cap \alpha^{-1}(T_{sm}) \cap S'$. Likewise for such s'

$$\text{ramif. } \alpha^! Z | S \text{ at } (s, x) = \deg \alpha^! Z(s') | N$$

and clearly $\alpha^! Z(s') | N = Z(\alpha s') | N$. ∎ ∎

THEOREM 3.5. *Let Z in $T \times X$ be a relative cycle over T, and assume that the carrier X is projective. Then the cycle $Z(t, \alpha)$ is constant for α running within a given local component T' of the germ (T, t).*

PROOF. From lemma (3.1) and proposition (3.3) it follows that $Z(t, \alpha) = \nu^! Z(t', \bar{\alpha}) = \lim_{t'} e.\ \nu^! Z$, where $t' \in \nu^{-1}(t)$ corresponds to the branch T', and e: $X \to \mathbb{P}$ is a chosen projective embedding. ∎

Let us say that the relative cycle Z over T is a *regular cycle*, or defines a *regular family* of cycles, iff for every $t \in T$ the cycle $Z(t, \alpha)$ is constant, i.e. also independent of the branch T', and write in this case $Z(t) := Z(t, \alpha)$.

PROPOSITION 3.6. *Let U be a closed subscheme of $T \times X$ which is flat over T. Then the associated cycle $[U]$ defines a regular family of cycles*

$$[U](t) = [U(t)].$$

PROOF. If T is smooth then $[U](t) = [U(t)]$ holds [8; 10.1.2]. If α: $S \to T$ is any map of varieties then $\alpha^![U] = [U_S]$, where $U_S :=$ $:= S \times_T U$ [8; 6.2(*b*)]. Thus, if S is a smooth curve with $\alpha(s) = t$, as usual, it follows that $\alpha^![U](s) = [U_S](s) = [U_S(s)] = [U(t)]$, independent of α. ■

4. – Induced semi-regular maps.

THEOREM 4.1. *Let Z on $T \times \mathbb{P}$ be a relative cycle of relative dimension n over a smooth variety T, and let k be the degree of every $Z(t)$. There is a regular map f_Z: $T \to \mathbb{C}_{n,k}$ sending $t \in T$ into the associated point of $Z(t)$.*

PROOF. By linearity, we may assume $Z = [U]$ with $U \subset T \times \mathbb{P}$ a reduced subscheme. Let $\mathbb{G}$ be the grassmannian of linear spaces of codimension $n+1$ in $\mathbb{P}$, and let $\mathbb{L} \subset \mathbb{P} \times \mathbb{G}$ be the universal linear space over $\mathbb{G}$. Consider the diagram:

$$
\begin{array}{ccc}
U \times_{\mathbb{P}} \mathbb{L} & \to \mathbb{L} & \to \mathbb{G} \\
\downarrow & \downarrow & \\
U & \to \mathbb{P} & \\
\downarrow & & \\
T & &
\end{array}
$$

and the induced map π: $U \times_{\mathbb{P}} \mathbb{L} \to T \times \mathbb{G}$.

Projection λ: $U \times_{\mathbb{P}} \mathbb{L} \to U$ is obtained by base extension from the locally trivial fibration $\mathbb{L} \to \mathbb{P}$, hence $U \times_{\mathbb{P}} \mathbb{L}$ is reduced. The image set $\pi(U \times_{\mathbb{P}} \mathbb{L})$ consists of all $(t, u) \in T \times \mathbb{G}$ such that $L(u)$ meets $|Z(t)|$. It has codimension 1 in $T \times \mathbb{G}$, as π is birational onto its image. Define $I_Z := [U \times_{\mathbb{P}} \mathbb{L}]$, $H_Z := \pi_* I_Z = [\pi(U \times_{\mathbb{P}} \mathbb{L})]$. We claim that the fibre cycle $H_Z(t)$ is the hypersurface section of $\mathbb{G}$ defined by the associated form of $Z(t)$.

By base extension again we obtain fibrations λ_t: $U(t) \times_{\mathbb{P}} \mathbb{L} \to U(t)$, whose pull-backs λ_t^* therefore preserve irreducibility of cycles, and proper projections π_t: $U(t) \times_{\mathbb{P}} \mathbb{L} \to \mathbb{G}$, which are still birational onto their images. Therefore $I_{Z(t)} := \lambda_t^* Z(t)$ is nothing but the «incidence cycle» of $Z(t)$, whose projection $H_{Z(t)} := \pi_*(I_{Z(t)})$ is the hypersurface section of $\mathbb{G}$ defined by the associated form of $Z(t)$. From [8; Prop. 10.1(*b*)] we have $(\lambda^* Z)(t) = \lambda_t^*(Z(t))$, i.e. $I_Z(t) = I_{Z(t)}$. From [8; Prop. 10.1(*a*)] we have $(\pi_* I_Z)(t) = \pi_{t*}(I_Z(t))$. So finally $H_Z(t) = H_{Z(t)}$. In particular, its line bundle is $\mathcal{O}(H_Z(t)) = \mathcal{O}_{\mathbb{G}}(k)$.

Now H_Z is an effective Cartier divisor in $T \times \mathbb{G}$ and its line bundle $\mathcal{O}(H_Z)$ has a unique (up to proportionality) global section Φ with

$\operatorname{div}(\Phi) = H_Z$. Thus the section $\Phi(t) := \Phi | t \times \mathbb{G}$ of $\mathcal{O}(H_Z) | t \times \mathbb{G} = \mathcal{O}(H_Z(t))$ has $\operatorname{div}(\Phi(t)) = H_Z(t)$, so that we can identify $\Phi(t)$ with the associated form of $Z(t)$, up to the identification $\mathcal{O}(H_Z(t)) = \mathcal{O}_{\mathbb{G}}(k)$ and the canonical identification of forms $F(u_0 \wedge \ldots \wedge u_n)$ of degree k in the coordinate ring of $\mathbb{G}$ with global sections of $\mathcal{O}_{\mathbb{G}}(k)$. All these identifications can be made compatible so to define a regular map $T \to \mathbb{P}H^0(\mathcal{O}_{\mathbb{G}}(k)) = \mathbb{F}_{n,k}$ which clearly factors through $\mathbb{C}_{n,k}$. This follows from a general:

LEMMA 4.2. *Let $\mathcal{M}$ be a line bundle on a product variety $V \times W$, such that every $\mathcal{M}_v := \mathcal{M} | v \times W$ belongs to the isomorphism class of some given line bundle $\mathcal{L}$ on W, and assume that W is a complete variety. Then there is an isomorphism $\mathcal{M} \cong pr_V^*(\mathcal{N}) \otimes pr_W^*(\mathcal{L})$ for some line bundle $\mathcal{N}$ on V. If a global section $\Phi \in H^0(V \times W, \mathcal{M})$ is given, define $f(v) \in H^0(W, \mathcal{L})$ to be the global section corresponding to $\Phi | v \times W$ under the induced isomorphism $\mathcal{M}_v \cong \mathcal{L}$. Then f: $V \to H^0(W, \mathcal{L})$ is a regular map.*

PROOF. That $\mathcal{N}$ exists is by the so called zig-zag principle. Since regularity of f is a local question we may assume that V is affine and $\mathcal{N}$ is trivial on V. Now replacing $\mathcal{N}$ with an isomorphic line bundle induces an isomorphism of $\mathcal{M}$ which does not change the map f. So we may also assume that $\mathcal{N} = \mathcal{O}_V$. In this situation, there is an induced isomorphism $H^0(V \times W, \mathcal{M}) \cong H^0(V, \mathcal{O}_V) \otimes H^0(W, \mathcal{L})$. If Φ is sent into $\sum_i f_i \otimes \Phi_i$, then $f(v) = \sum_i f_i(v) \Phi_i$. ■ ■

PROPOSITION 4.3. *Let Z in $T \times \mathbb{P}$ be a relative cycle of relative dimension n over a variety T, and let k be the degree of $Z(t)$ for $t \in T_{sm}$. The closed graph of the induced map $T_{sm} \to \mathbb{C}_{n,k}$ defines a correspondence f from T into $\mathbb{C}_{n,k}$. The image set $f\{t\}$ is the set of associated points of cycles $Z(t, \alpha)$. In particular, it is a finite set.*

PROOF. For every curve α: $S \to T$, $\alpha(s) = t$, $\alpha(S - s) \subset T_{sm}$, the composition $S - s \to T_{sm} \to \mathbb{C}_{n,k}$ extends to φ: $S \to \mathbb{C}_{n,k}$ and clearly $\varphi(s) \in f\{t\}$. This φ agrees with the induced map of the relative cycle $\alpha^! Z$ over S, on $S - s$ hence on the whole of S. Thus $\varphi(s)$ is the associated point of $Z(t, \alpha)$. Clearly, every associated form $F \in f\{t\}$ is of the form $F = \varphi(s)$ for some α. ■

In particular, we see that Z being a regular relative cycle over T is a necessary and sufficient condition for the induced map $T_{sm} \to \mathbb{C}_{n,k}$ to have a continuous extension f_Z: $T \to \mathbb{C}_{n,k}$, what we call a semi-regular (rational) map.

DEFINITION 4.4. If S, T are reduced schemes, we call $f\colon S \to T$ a *semi-regular (rational) map* if the restriction $S_{sm} \to T$ is a regular map and f is continuous, either in the Zariski or in the complex topology, equivalently. It is necessarily a regular map if S is a semi-normal variety.

5. – The universal family.

PROPOSITION 5.1. *For every semi-regular map $f\colon T \to \mathrm{C}_{n,k}$ of a variety T, there is in $T \times \mathrm{P}$ a unique regular cycle Z over T such that $f_Z = f$.*

PROOF. We begin with a useful remark. Suppose we have found some dense open subset T_1 of T_{sm} and in $T_1 \times \mathrm{P}$ a relative cycle Z_1 over T_1 such that $f(t)$ is the associated point of $Z_1(t)$ for $t \in T_1$. We claim that the closure Z of Z_1 in $T \times \mathrm{P}$ satisfies the theorem.

Consider the normalization map $\nu\colon \overline{T} \to T$. We have the relative cycle $\nu^! Z$ over $\overline{T}$ with $\nu^! Z(t') = Z(\nu t')$ if $t' \in \nu^{-1}(T_1)$. Now $\nu^! Z$ defines a regular family and induces a regular map $\bar{f}\colon \overline{T} \to \mathrm{C}_{n,k}$, which clearly agrees with $f \circ \nu$, on $\nu^{-1}(T_1)$ hence on the whole of $\overline{T}$. Let $\alpha\colon S \to T$, $\alpha(s) = t$ be a smooth curve through t with $\alpha(S - s) \subset T_{sm}$, and take the lifting $\bar{\alpha}\colon S \to \overline{T}$, $\bar{\alpha}(s) = \bar{t}'$. We know that $Z(t, \alpha) = \nu^! Z(t')$, and the associated point of $\nu^! Z(t')$ is $\bar{f}(t') = f(\nu t') = f(t)$, hence $Z(t, \alpha)$ is the cycle of the associated point $f(t)$, independent of α. This proves the claim.

We are therefore allowed to assume that T is smooth and f is a regular map. Furthermore, if f is of type θ, we may also assume that $f(T) \subset \mathrm{C}_\theta$.

We first consider the case $\theta = (1; k)$, when the cycle of $f(t)$ is a simple cycle for every t. Using the incidence subscheme $\mathrm{U}_{n,k} \subset \mathrm{C}_{n,k} \times \mathrm{P}$, define $U := (T \times_{\mathrm{C}_{n,k}} \mathrm{U}_{n,k})_{\mathrm{red}}$, a closed subscheme of $T \times \mathrm{P}$, and $p :=$ $:= pr_T | U$. By general dimension and smoothness arguments, we know that there is some dense open subset T_1 of T such that: every irreducible component of $p^{-1}(T_1)$ dominates T_1, $U_{sm} \cap p^{-1}(T_1) \to T_1$ is a smooth map, $U_{sg} \cap p^{-1}(T_1) \to T_1$ has all fibres of dimension $< n$. Let Z_1 be the cycle of the open subscheme $p^{-1}(T_1)$ of U. Clearly Z_1 is a relative cycle over T_1 and the fibre cycle $Z_1(t)$ is a simple cycle and therefore agrees with the cycle with associated point $f(t)$ for $t \in T_1$. So the statement follows from the remark at the beginning.

Next we come to the general case, type $\theta = (a_i; k_i)$. By means of the regular map $\psi\colon \times_i \mathrm{C}_{n,k_i} \to \mathrm{C}_{n,k}$ defined by $\psi(F_1, \ldots, F_s) = \prod_i F_i^{a_i}$, we introduce the space $\widetilde{\mathrm{C}}_{n,\theta} := \psi^{-1}(\mathrm{C}_{n,\theta})$ of all sequences $(F_1, \ldots, F_s)$ of pairwise coprime simple associated forms, and the bijective restriction $\psi\colon \tilde{\mathbb{C}}_{n,\theta} \to \mathbb{C}_{n,\theta}$. Then form the fibre diagram:

$$\begin{array}{ccc} \tilde{T} \to \tilde{\mathbb{C}}_{n,\theta} & \xrightarrow{pr_i} & \mathbb{C}_{n,k_i} \\ \downarrow \quad\ \downarrow \psi & & \\ T \xrightarrow{f} \mathbb{C}_{n,\theta} & & \end{array}$$

where $\tilde{T} := T \times_{\mathbb{C}_{n,\theta}} \tilde{\mathbb{C}}_{n,\theta}$. The composite map $\tilde{T} \to \mathbb{C}_{n,k_i}$ is of type $(1; k_i)$, so by the preceding proof there is a regular cycle $\tilde{Z}_i$ in $\tilde{T} \times \mathbb{P}$ over $\tilde{T}$ whose fibre over $(t, F_1, \ldots, F_s) \in \tilde{T}$ is the cycle of F_i. Define $Z_i := (1 \times \psi)_* (\tilde{Z}_i)$ and then $Z := \sum a_i Z_i$. Clearly Z is a regular cycle over T such that $f(t)$ is the associated point of $Z(t)$ for every $t \in T$. ■

PROPOSITION 5.2. *Let Z in $T \times X$ be a regular cycle over T, and assume that the carrier X is projective. Let $f\colon S \to T$ be any semi-regular map of varieties. There is in $S \times X$ a unique regular cycle $f^! Z$ over S such that $f^! Z(s) = Z(fs)$ for every $s \in S$.*

PROOF. With respect to some projective embedding $e\colon X \to \mathbb{P}$, the regular cycle Z induces a semi-regular map $f_Z\colon T \to \mathbb{C}_{n,k}$ and the composed semi-regular map $f_Z \circ f$ is in turn induced by a unique regular cycle $f^! Z$ over S. ■

It is easy to extend everything more generally to reduced, not necessarily irreducible, parameter spaces T. A cycle Z in $T \times X$ will be a relative cycle over T if the projection $|Z| \to T$ has all fibres of some pure dimension n and if, furthermore, every irreducible component of $|Z|$ projects onto some irreducible component of T. In other words, for every irreducible component T' of T a relative cycle Z' over T' is given, and Z is the sum of all these Z'. In this setting there are limit cycles $Z'(t, \alpha)$ for $t \in T'$, and Z will be said to be regular if $Z'(t, \alpha)$ is independent both of the curve α and of the irreducible component T' containing t. In this case we write $Z(t) := Z'(t, \alpha)$. The cycles $Z(t)$ need not belong to one and the same algebraic equivalence class on X. They do if the parameter space T is connected. With these definitions, all the results in the last two sections hold more generally for any reduced T. A regular cycle Z in $T \times \mathbb{P}$ determines a semi-regular map $T \to \mathbb{C}_n$, and indeed $T \to \mathbb{C}_{n,k}$ if all cycles $Z(t)$ have a common degree k in $\mathbb{P}$. Conversely any semi-regular map $f\colon S \to T$ determines a pull-back $f^! Z$ of regular cycles Z over T, provided that the carrier X is projective.

Therefore, from proposition (5.1) applied to the identity map of

$\mathbb{C}_{n,k}$, we obtain a regular cycle $\mathbb{Z}_{n,k}$ over $\mathbb{C}_{n,k}$, which we call the *universal cycle*. Summing over k, we also have a universal cycle $\mathbb{Z}_n$ over $\mathbb{C}_n$. For every projective embedding $e: X \to \mathbb{P}$, proposition (5.2) applied to the inclusion map $e_{n,k}: C_{n,k}(X, e) \to \mathbb{C}_{n,k}$ gives a universal cycle $Z_{n,k}(X, e) = e^{!}_{n,k} \mathbb{Z}_{n,k}$ in $C_{n,k}(X, e) \times X$.

6. – The semi-universal property.

The results in the last section allow the following:

DEFINITION 6.1. *For every projective carrier scheme X, a set-valued contravariant functor $\mathcal{F}_n(X)$ on the category $\mathcal{R}_0$ of reduced schemes T and semi-regular maps $f: S \to T$ is constructed, by defining $\mathcal{F}_n(X)(T)$ to be the set of all regular cycles Z in $T \times X$ of relative dimension n over T, and $\mathcal{F}_n(X)(f)$ to be the function $Z \mapsto f^! Z$.*

Moreover, for every projective embedding $e: X \to \mathbb{P}$, a natural transformation of functors $\mathcal{F}_n(X)(T) \to \mathrm{Hom}_{\mathcal{R}_0}(T, C_n(X, e))$ is defined by $Z \mapsto f_{e \cdot Z}$ and is in fact an isomorphism. We state this as:

THEOREM 6.2. *The space of cycles $C_n(X, e)$ represents the functor $\mathcal{F}_n(X)$ on the category $\mathcal{R}_0$ of reduced schemes and semi-regular maps.*

This implies the following result of Andreotti-Norguet [1; p. 52].

COROLLARY 6.3. *If e, e' are two projective embeddings of the carrier X, there is a semi-isomorphism $C_n(X, e) \cong C_n(X, e')$, i.e. an homeomorphism which is an isomorphism of the smooth loci, sending the associated point of $e(Z)$ into the associated point of $e'(Z)$ for every cycle Z of pure dimension n in X.*

Actually this is an isomorphism according to [2; p. 115, Cor.].

Concerning the relation to the Hilbert scheme, from proposition (3.6) and theorem (6.2) we have:

COROLLARY 6.4. *Let $e: X \to \mathbb{P}$ be a projective embedding and let $H_n(X)$ be the Hilbert scheme parametrizing closed subschemes U of X of pure dimension n. There is a semi-regular map $H_n(X)_{\mathrm{red}} \to C_n(X, e)$ sending the associated (Hilbert) point of the subscheme U into the associated (Cayley-Chow) point of the cycle $[U]$.*

That this is in fact a regular map is proved in [12; Ch. 5, § 4] and [2; Ch. V].

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