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## A Corollary to the Evans-Griffith Syzygy Theorem.

ANNE-MARIE SIMON (\*)

ABSTRACT - Height two ideals of finite projective dimension in a Cohen-Macaulay or Gorenstein local ring are investigated, providing slight extensions of results of Serre and Evans-Griffith concerning the problem to know when they are two-generated, when the quotient ring is Cohen-Macaulay if they are three-generated.

### Introduction.

This note is concerned with the height two ideals of a Cohen-Macaulay noetherian local ring: when are they two-generated, what can we say about them when they are three-generated?

In the second direction we have an important theorem of Evans-Griffith.

**THEOREM 1.** ([E.G.81] *Theorem 2.1*, or [E.G.85] *Theorem 4.4*). *Let  $A$  be a regular local ring containing a field. If  $I$  is an unmixed three generated ideal of height two, then the ring  $A/I$  is Cohen-Macaulay.*

In the first direction we have first a Serre's theorem, one formulation of it is the following.

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**THEOREM 2.** ([Se, Proposition 5] [B-E]). *Let  $I$  be a height two ideal of a regular local ring  $A$ . If the quotient ring  $A/I$  is Gorenstein, then the ideal  $I$  is two-generated.*

However, another formulation of Serre was a little bit different.

**THEOREM 2'.** [Se, corollaire à la Proposition 2]. *Let  $A$  be a noetherian domain such that all projective  $A$ -modules of rank one or two are free, and let  $I$  be a non-zero ideal of projective dimension less or equal to one. Then the ideal  $I$  is two-generated if and only if the  $A$ -module  $\text{Ext}_A^1(I, A)$  is principal.*

We note that in Theorem 2' only the case where  $I$  is an ideal of projective dimension one and of height two is of some interest, at least to us.

In Theorem 2, the hypothesis on  $I$  imply that the projective dimension of  $I$  is one, they imply also that the  $A$ -module  $\text{Ext}_A^1(I, A)$  is principal, since  $\text{Ext}_A^1(I, A) \simeq \text{Ext}_A^2(A/I, A)$ . Indeed, the last module  $\text{Ext}_A^2(A/I, A)$  is the canonical module of the ring  $A/I$ , hence it is principal since  $A/I$  is Gorenstein (a local ring is Gorenstein if and only if it is Cohen-Macaulay and if its canonical module is principal).

Then we have another Evans-Griffith's theorem.

**THEOREM 3.** ([E.G.81], Theorem 2.2, or [E.G.85], Theorem 4.7). *Let  $A$  be a regular local ring containing a field and let  $I$  be a prime ideal of height two such that the  $A$ -module  $\text{Ext}_A^2(A/I, A)$  is principal. Then  $I$  is two generated.*

There is an evident analogy between Theorem 3 and Theorem 2', in the hypothesis, the conclusions and even the proofs. Both proofs use an extension  $0 \rightarrow A \rightarrow M \rightarrow I \rightarrow 0$  whose class generates the principal  $A$ -module  $\text{Ext}_A^1(I, A)$ , and one observes that  $\text{Ext}_A^1(M, A) = 0$ . From this observation the conclusion of Theorem 2' follows rather quickly, while for theorem 3 one has to use the syzygy theorem; and the hypothesis that  $I$  is a prime ideal of a regular ring is strongly used. Concerning that last Theorem 3 we have also to mention [Br.E.G.].

The aim of this note is to provide slight generalizations of the above-mentioned theorems. The generalizations of Theorem 1 and 2 are straightforward, indeed the proofs are essentially the same. The generalization we give of Theorem 3 requires not only the preceding extensions but also a rather different proof, using the linkage theory as developed in [P.S] or [U] as well as the syzygy theorem.

As a general reference for homological background we quote [E.G.85], [St], [U].

## 1. Preliminaries.

To state the syzygy theorem we must recall the Serre  $k$ -condition.

DEFINITION. *An  $A$ -module  $M$  is said to be  $S_k$  if, for all prime ideals  $p$  of  $A$  one has  $\text{depth } M_p \geq \min\{k, \text{ht } p\}$ .*

So, if an  $A$ -module is  $S_k$  for some  $k > 0$ , then all the associated prime ideals of  $M$  are minimal in  $\text{Spec } A$ .

A key result is the theorem of Auslander-Bridger which shows that a finitely generated  $A$ -module of finite projective dimension is  $S_k$  if and only if it is a  $k^{\text{th}}$  syzygy (see [E.G.85] Theorem 3.8) when the ring itself is  $S_k$ .

THE SYZYGY THEOREM. ([E.G.81], *Theorem 1.1*, or [E.G.85], *Theorem 3.15*, see also [Br] or [Og]). *Let  $A$  be a noetherian local ring containing a field and let  $M$  be a finitely generated  $S_k$ -module over  $A$  of finite projective dimension. Then if  $M$  is not free, it has rank at least  $k$ .*

We note that the rank of a finitely generated  $A$ -module of finite projective dimension is a well-defined natural number: if  $0 \rightarrow A^{n_s} \rightarrow \dots \rightarrow A^{n_0} \rightarrow M \rightarrow 0$  is a free resolution of  $M$ , then  $\text{rank } M = r = \sum_{i=0}^s (-1)^i n_i$ ; and, for all minimal prime ideals  $p$  of  $\text{Spec } A$  one has  $M_p \simeq A_p^r$ .

In the syzygy theorem, the hypothesis that the local ring contains a field is still essential. Indeed, the proof uses a Big Cohen-Macaulay module, only available by now in equal characteristic.

Here is the straightforward generalization of theorem 1.

PROPOSITION 1. *Let  $A$  be a Cohen-Macaulay noetherian local ring containing a field, and let  $I$  be an unmixed three generated ideal of height two of finite projective dimension. Then the ideal  $I$  is perfect, i.e. the quotient ring  $A/I$  is Cohen-Macaulay. Moreover, the module  $\text{Ext}_A^2(A/I, A)$  is also a perfect module, i.e. it is a Cohen-Macaulay module of projective dimension two.*

PROOF We resolve  $A/I$  and have an exact sequence  $0 \rightarrow M \rightarrow A^3 \rightarrow A \rightarrow A/I \rightarrow 0$ , where  $M$  is a second syzygy of  $A/I$ .

We need to show that  $M$  is free. (If  $M$  is free, using the Auslander-Buchsbaum equality we obtain  $\text{depth } A/I = \dim A - \text{pd } A/I = \dim A - 2 = \dim A/I$ ).

We observe that  $M$  is a finitely generated  $A$ -module of finite projective dimension and of rank two.

On the other hand, the  $A$ -module  $M$  is  $S_3$ . Indeed, let  $p$  be a prime ideal of  $A$  at which we localize.

If  $p \nmid I$ , then  $M_p \simeq A_p^2$  and  $\text{depth } M_p = ht\ p \geq \min\{3, ht\ p\}$ .

If  $p \supset I$  and  $ht\ p = 2$ , the above exact sequence localized at  $p$  shows that  $\text{depth } M_p = 2 \geq \min\{3, 2\}$ .

If  $p \supset I$  and  $ht\ p > 2$ , the above exact sequence localized at  $p$  shows that  $\text{depth } M_p \geq 3$ , because  $\text{depth } (A/I)_p \geq 1$ ,  $I$  being unmixed.

The freeness of the module  $M$  follows now from the syzygy theorem and  $M \simeq A^2$ .

For the second assertion, we apply the functor  $\text{Hom}_A(\cdot, A) = (\cdot)^*$  to the exact sequence  $0 \rightarrow A^2 \rightarrow A^3 \rightarrow A \rightarrow A/I \rightarrow 0$ . We obtain a complex  $0 \rightarrow A^* \rightarrow A^{3*} \rightarrow A^{2*} \rightarrow 0$  whose homology is concentrated in degree 2, where it is  $\text{Ext}_A^2(A/I, A)$ .

So  $pd\ \text{Ext}_A^2(A/I, A) = 2$ , and again the conclusion follows from the Auslander-Buchsbaum equality:  $\dim A - 2 = \text{depth } \text{Ext}_A^2(A/I, A) \leq \dim \text{Ext}_A^2(A/I, A) \leq \dim A - 2$ .

We give now the straightforward generalization of Theorem 2, though this has nothing to do with the syzygy theorem.

**PROPOSITION. 2.** *Let  $I$  be a height two ideal of a Gorenstein local ring. If the projective dimension of  $I$  is finite and if the quotient ring  $A/I$  is Gorenstein, then the ideal  $I$  is two-generated*

**PROOF.** The hypotheses imply that the ideal  $I$  is perfect, i.e. the projective dimension of  $A/I$  is two, the height of  $I$ .

A minimal resolution of  $A/I$  has the form

$$0 \rightarrow A^{m-1} \xrightarrow{\alpha} A^m \rightarrow A \rightarrow A/I \rightarrow 0,$$

and  $\text{Ext}_A^2(A/I, A) = \text{coker } \text{Hom}_A(\alpha, A)$ : the sequence  $A^m \xrightarrow{\alpha^t} A^{m-1} \rightarrow \text{Ext}_A^2(A/I, A) \rightarrow 0$  is exact.

As the resolution of  $A/I$  is minimal the entries of the matrix associated to  $\alpha$  and to its transposed  $\alpha^t$  are in the maximal ideal of  $A$ . This shows that the minimal number of generators of  $\text{Ext}_A^2(A/I, A)$  is  $m - 1$ .

On the other hand, this number is one since  $A/I$  is assumed to be Gorenstein, so  $1 = m - 1$ ,  $m = 2$  and the ideal  $I$  is two generated.

## 2. An extension of Theorem 3.

**PROPOSITION 3.** *Let  $A$  be a Gorenstein noetherian local ring containing a field and let  $I$  be an unmixed ideal of finite projective dimen-*

sion and of height two. If the  $A$ -module  $\text{Ext}_A^2(A/I, A)$  is principal, then the ideal  $I$  can be generated by 2 elements.

PROOF. We choose in  $I$  a regular sequence  $x_1, x_2$  and use it to make an algebraic link: if  $J = (x_1, x_2): I$ , then  $I = (x_1, x_2): J$  since  $I$  is unmixed and since the ring  $A$  is Gorenstein [P.S.]; moreover, the ideal  $J$  is also unmixed. The isomorphisms  $\text{Ext}_A^2(A/I, A) \simeq \text{Hom}_A(A/I, A/(x_1, x_2)) \simeq J/(x_1, x_2)$  and the hypothesis on  $I$  imply that the linked ideal  $J$  is a height two ideal three generated:  $J = (x_1, x_2, y)$  for some  $y$  in  $A$ .

As  $I = (x_1, x_2): J$ , we have an exact sequence

$$0 \rightarrow I/(x_1, x_2) \rightarrow A/(x_1, x_2) \xrightarrow{y} J/(x_1, x_2) \rightarrow 0$$

which shows that  $A/I \simeq J/(x_1, x_2)$ . So we have also an exact sequence

$$0 \rightarrow A/I \rightarrow A/(x_1, x_2) \rightarrow A/J \rightarrow 0,$$

this shows that the  $A$ -module  $A/J$  is of finite projective dimension.

By proposition 1, we conclude that the ring  $A/J$  is Cohen-Macaulay; but then  $A/I$  is also Cohen-Macaulay by the linkage theory in a Gorenstein ring  $A$  ([P.S], or [UL]). Consequently the ring  $A/I$  is Gorenstein since it is Cohen-Macaulay and since its canonical module  $\text{Ext}_A^2(A/I, A)$  is principal, and the conclusion follows from proposition 2.

NOTE. the above proposition is to be compared with a geometric result of Fiorentini and Lascu ([Fi.La.], Theorem 2 (iii)).

The hypothesis in Proposition 3 are slightly weaker than in Proposition 2; in Proposition 3, the ring  $A/I$  is not assumed to be Cohen-Macaulay in advance.

### 3. Some Examples.

EXAMPLE 1. To the twisted cubic curve  $(s^3, s^2t, st^2, t^3)$  of the projective space  $\mathbb{P}_K^3$  is associated an ideal  $I$  of the regular local ring  $A = K[[X_0, X_1, X_2, X_3]]$ . This ideal is a height two prime ideal which is three-generated:  $I = (X_0X_3 - X_1X_2, X_1^2 - X_0X_2, X_2^2 - X_1X_3)$ . Hence the ring  $A/I$  is Cohen-Macaulay, not Gorenstein. In fact the ideal  $I$  is the ideal of the  $2 \times 2$  minors of the matrix

$$\phi = \begin{pmatrix} X_1 & X_2 \\ X_2 & X_3 \\ X_0 & X_1 \end{pmatrix}.$$

A minimal projective resolution of the  $A$ -module  $A/I$  is given by

$$0 \rightarrow A^2 \xrightarrow{\phi} A^3 \rightarrow A \rightarrow A/I \rightarrow 0,$$

this shows that the canonical module of  $A/I$ ,  $\text{Ext}_A^2(A/I, A) = \text{coker } \phi^t$  (where  $\phi^t$  is the transposed of  $\phi$ ) is minimally generated by 2 elements.

**EXAMPLE 2.** To the quartic curve  $(s^4, s^3t, st^3, t^4)$  of the projective space  $\mathbb{P}_K^3$  is associated a height two prime ideal  $I$  of the ring  $A = K[X_0, X_1, X_2, X_3]$ , this ideal  $I$  is four-generated:  $I = (X_0X_3 - X_1X_2, X_1^3 - X_0^2X_2, X_2^3 - X_1X_3^2, X_0X_2^2 - X_1^2X_3)$ . The quotient ring  $A/I$  is not Cohen-Macaulay, however it is a Buchsbaum local ring.

**EXAMPLE 3.** Bertini constructed an example of a non Cohen-Macaulay factorial ring  $B$  which is an image of a regular local ring  $A: B = A/I$ , the height  $g$  of  $I$  is greater than 3. Since  $B$  is factorial, the module  $\text{Ext}_A^g(B, A)$  is principal. This illustrates the fact that the hypothesis in Proposition 3 are weaker than those in Proposition 2. On the other hand, Theorem 1 is concerned with unmixed ideals  $I$  of height  $g$  generated by  $g + 1$  element. When  $g = 2$ , when the ring  $A$  is regular, the quotient  $A/I$  is Cohen-Macaulay. When  $g > 2$ , this conclusion is not valid anymore. Indeed, in Bertin's example we can choose in the ideal  $I$  a regular sequence  $x_1, \dots, x_g$  such that  $IA_I = (x_1, \dots, x_g)A_I$  ( $I$  is a prime ideal of the regular ring  $A$ ). This gives us a link (even a geometric link):  $J = (x_1, \dots, x_g):I$ . The ideal  $J$  is an unmixed ideal of height  $g$  of the ring  $A$ , the quotient ring  $A/J$  is not Cohen-Macaulay (since  $A/I$  is not), but  $J$  can be generated by  $g + 1$  elements: the module  $J/(x_1, \dots, x_g) = \text{Hom}_A(A/I, A/x_1, \dots, x_g) = \text{Ext}_A^g(A/I, A)$  is principal.

Schenzel gave other examples of prime ideal of height  $g$  in a regular local ring, minimally generated by  $g + 1$  elements, and such that the quotient ring is not Cohen-Macaulay.

This work was done while the author was visiting the University of Ferrara.

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