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Edge-of-the-Wedge Theorem for Elliptic Systems.

ANDREA D'AGNOLO (*)

ABSTRACT - Let M be a real analytic manifold, X a complexification of M , $N \subset M$ a submanifold, and $Y \subset X$ a complexification of N . One denotes by \mathcal{C}_M the sheaf of real analytic functions on M , and by \mathcal{B}_M the sheaf of Sato hyperfunctions. Let \mathcal{K} be an elliptic system of linear differential operators on M for which Y is non-characteristic. Using the language of the microlocal study of sheaves of [K-S] we give a new proof of a result of Kashiwara-Kawai [K-K] which asserts that

$$(\dagger) \quad H^j \mu_N(\mathcal{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{K}, *)) = 0 \quad \text{for } * = \mathcal{C}_M, \mathcal{B}_M, \quad j < \text{cod}_M N,$$

where μ_N denotes the Sato microlocalization functor. For $\text{cod}_M N = 1$, the previous result reduces to the Holmgren's theorem for hyperfunctions, and of course in this case the ellipticity assumption is not necessary. For $\text{cod}_M N > 1$, this implies that the sheaf of analytic (resp. hyperfunction) solutions to \mathcal{K} satisfies the edge-of-the-wedge theorem for two wedges in M with edge N . Dropping the ellipticity hypothesis in this higher codimensional case, we then show how (\dagger) no longer holds for $* = \mathcal{C}_M$. In the frame of tempered distributions, Liess [L] gives an example of constant coefficient system for which the edge-of-the-wedge theorem is not true. We don't know whether (\dagger) holds or not for $* = \mathcal{B}_M$ in the non-elliptic case.

1. Notations and statement of the result.

1.1. Let X be a real analytic manifold and $N \subset M \subset X$ real analytic submanifolds. One denotes by $\pi: T^*X \rightarrow X$ the cotangent bundle to X , and by T_N^*X the conormal bundle to N in X . The embedding $f: M \rightarrow X$ induces a smooth morphism ${}^t f'_N: T_N^*X \rightarrow T_N^*M$.

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Let $\gamma \subset T_N X$ be an open convex cone of the normal bundle $T_N X$. We denote by γ^a its antipodal, and by $\gamma^0 \subset T_N^* X$ its polar. For a subset $U \subset X$ one denotes by $C_N(U) \subset T_N X$ its normal Whitney cone.

DEFINITION 1.1. An open connected set $U \subset X$ is called a wedge in X with profile γ if $C_N(X \setminus U) \cap \gamma = \emptyset$. The submanifold N is called the edge of U . We denote by \mathfrak{W}_γ the family of wedges with profile γ .

1.2. Let us recall some notions from [K-S]. Let $D^b(X)$ denote the derived category of the category of bounded complexes of sheaves of \mathbb{C} -vector spaces on X . For F an object of $D^b(X)$, one denotes by $SS(F)$ its micro-support, a closed, conic, involutive subset of $T^* X$. One says that M is non-characteristic for F if $SS(F) \cap T_M^* X \subset M \times_X T_X^* X$. Recall that in this case, one has $f^! F \simeq F|_M \otimes or_{M/X}[-\text{cod}_X M]$, where $or_{M/X}$ denotes the relative orientation sheaf of M in X .

Denote by $\mu_N(F)$ the Sato microlocalization of F along N , an object of $D^b(T_N^* X)$.

PROPOSITION 1.2. (cf. [K-S, Theorem 4.3.2])

(i) $R\Gamma_N(\mu_N(F)) \simeq F|_N \otimes or_{N/X}[-\text{cod}_X N]$,

(ii) for $\gamma \subset T_N X$ an open proper convex cone, there is an isomorphism for all $j \in \mathbb{Z}$:

$$H_{\gamma^{0a}}^j(T_N^* X; \mu_N(F) \otimes or_{N/X}) \simeq \varinjlim_{U \in \mathfrak{W}_\gamma} H^{j - \text{cod}_X N}(U; F),$$

The main tool of this paper will be the following result on commutation for microlocalization and inverse image due to Kashiwara-Schapira.

THEOREM 1.3. (cf. [K-S, Corollary 6.7.3]) *Assume that M is non-characteristic for F . Then the natural morphism:*

$$\mu_N(f^! F) \rightarrow R^! f'_* \mu_N(F)$$

is an isomorphism.

1.3. We will consider the following geometrical frame.

Let M be a real analytic manifold of dimension n , and let $N \subset M$ be a real analytic submanifold of codimension d . Let X be a complexification

of M , $Y \subset X$ a complexification of N , and consider the embeddings.

$$\begin{array}{ccc} M & \xrightarrow{f} & X \\ j \uparrow & & \uparrow g \\ N & \xrightarrow{i} & Y \end{array}$$

One denotes by \mathcal{O}_X the sheaf of germs of holomorphic functions on X , and by \mathcal{D}_X the sheaf of rings of linear holomorphic differential operators on X . The sheaf $\mathcal{O}_X|_M$ of real analytic functions is denoted by \mathcal{A}_M . Moreover, one considers the sheaves:

$$\begin{aligned} \mathcal{B}_M &= R\Gamma_M(\mathcal{O}_X) \otimes or_{M/X}[n] = f^! \mathcal{O}_X \otimes or_{M/X}[n], \\ \mathcal{C}_M &= \mu_M(\mathcal{O}_X) \otimes or_{M/X}[n]. \end{aligned}$$

These are the sheaves of Sato’s hyperfunctions and microfunctions respectively.

Let \mathcal{N} be a left coherent \mathcal{D}_X -module. One says that \mathcal{N} is *non-characteristic* for Y if $\text{char}(\mathcal{N}) \cap T_Y^* X \subset Y \times_X T^* X$ (here $\text{char}(\mathcal{N}) \subset T^* X$ denotes the characteristic variety of \mathcal{N}), and one denotes by \mathcal{N}_Y the induced system on Y , a left coherent \mathcal{D}_Y -module. One says that \mathcal{N} is *elliptic* if \mathcal{N} is non-characteristic for M . Recall that in this case, by the fundamental theorem of Sato, one has:

$$(1.1) \quad R \mathcal{H}om_{\mathcal{D}_X}(\mathcal{N}, \mathcal{A}_M) \simeq R \mathcal{H}om_{\mathcal{D}_X}(\mathcal{N}, \mathcal{B}_M).$$

1.4. In the next section we will give a new proof of the following theorem of Kashiwara and Kawai:

THEOREM 1.4. (cf. [K-K]). *Let \mathcal{N} be a left coherent elliptic \mathcal{D}_X -module, non-characteristic for Y . Then*

$$(1.2) \quad H^j \mu_N(R \mathcal{H}om_{\mathcal{D}_X}(\mathcal{N}, *)) = 0 \quad \text{for } * = \mathcal{A}_M, \mathcal{B}_M, \quad j < d.$$

Let us discuss here some corollaries of this result.

1.4.1. Let X be a complex analytic manifold. One denotes by \bar{X} the complex conjugate of X , and by X^R the underlying real analytic manifold to X . Identifying X^R to the diagonal of $X \times \bar{X}$, the complex manifold $X \times \bar{X}$ is a natural complexification of X^R .

Let $S \subset X^R$ be a real analytic submanifold (identified to a subset of X), and let $S^C \subset X \times \bar{X}$ be a complexification of S . Denoting by $\bar{\partial}$ the

Cauchy-Riemann system (i.e. $\bar{\partial} = \mathcal{O}_X \boxtimes \mathcal{O}_{\bar{X}}$), one has an obvious isomorphism

$$(1.3) \quad \mu_S(\mathcal{O}_X) \simeq \mu_S(\mathbb{R} \mathcal{H}om_{\mathcal{O}_X \times \bar{X}}(\bar{\partial}, \mathcal{B}_{X^R})).$$

Assume S is generic, i.e. $TS +_S i^*TS = S \times_X TX$. Then the embedding $g: S^c \rightarrow X \times \bar{X}$ is non-characteristic for $\bar{\partial}$ (which is, of course, elliptic) and hence, combining (1.3) with Theorem 1.4, one recovers the well known result:

COROLLARY 1.5 *Let $S \subset X$ be a generic submanifold with $\text{cod}_X S = d$. Then*

$$H^j \mu_S(\mathcal{O}_X) = 0 \quad \text{for } j < d.$$

1.4.2 Let us go back to the notations of 1.3, and assume that N is a hypersurface of M defined by the equation $\phi(x) = 0$ with $d\phi \neq 0$. Assume that $M \setminus N$ has the two open connected components $M^\pm = \{x; \pm \phi(x) > 0\}$. Let $\mathcal{N} = \mathcal{O}_X / \mathcal{O}_X P$ for an elliptic differential operator P non-characteristic for Y .

By Theorem 1.4 we then recover the classical Holmgren's theorem for hyperfunctions:

COROLLARY 1.6. *Let $u \in \mathcal{B}_M$ be a solution of $Pu = 0$ such that $u|_{M^+} = 0$. Then $u = 0$.*

As it is well known, this result remains true even for non elliptic operators.

1.4.3 Assume now $\text{cod}_M N = d > 1$, and let \mathcal{N} be a left coherent elliptic \mathcal{O}_X -module which is non-characteristic for Y . Let γ be an open convex proper cone of the normal bundle $T_N M$, and let $U \in \mathfrak{W}_\gamma$ be a wedge with profile γ .

By Proposition 1.2 and Theorem 1.4, one has

$$\begin{aligned} (1.4) \quad & \lim_{U \in \mathfrak{W}_\gamma} \Gamma(U; \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{B}_M)) \simeq \\ & \simeq H^d \mathbb{R} \Gamma_{\gamma, \text{oa}}(T_N^* M; \mu_N(\mathbb{R} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{B}_M) \otimes \text{or}_{N/X})) \simeq \\ & \simeq \Gamma_{\gamma, \text{oa}}(T_N^* M; H^d \mu_N(\mathbb{R} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{B}_M) \otimes \text{or}_{N/X})). \end{aligned}$$

The induced morphism

$$b_\gamma: \Gamma(U; \mathfrak{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{B}_M)) \rightarrow \Gamma_{\gamma^{0a}}(T_N^*M; H^d \mu_N(\mathbb{R} \mathfrak{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{B}_M) \otimes or_{N/X}))$$

is called «boundary value morphism» (cf. [S]). Using (1.1), one easily sees that b_γ is injective by analytic continuation.

Let γ_1, γ_2 be open convex proper cones of $T_N M$ and denote by $\langle \gamma_1, \gamma_2 \rangle$ their convex envelope. One deduces the following edge-of-the-wedge theorem.

COROLLARY 1.7. *Let $U_i \in \mathfrak{W}_{\gamma_i}$ (for $i = 1, 2$), and let $u_i \in \Gamma(U_i; \mathfrak{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{B}_M))$ with $b_{\gamma_1}(u_1) = b_{\gamma_2}(u_2)$. Then there exist a wedge $U \in \mathfrak{W}_{\langle \gamma_1, \gamma_2 \rangle}$ with $U \supset U_1 \cup U_2$, and a section $u \in \Gamma(U; \mathfrak{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{B}_M))$ such that $u|_{U_1} = u_1, u|_{U_2} = u_2$.*

PROOF. We will neglect orientation sheaves for simplicity. Notice that $\tilde{u} = b_{\gamma_1}(u_1) = b_{\gamma_2}(u_2)$ is a section of $H^d \mu_N(\mathbb{R} \mathfrak{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{B}_M))$ whose support is contained in $\gamma_1^{0a} \cap \gamma_2^{0a}$. If $\gamma_1^{0a} \cap \gamma_2^{0a} = \{0\}$, the result follows by (i) of Proposition 1.2. If $\gamma_1^{0a} \cap \gamma_2^{0a} \neq \{0\}$, one remarks that $\text{Int}(\gamma_1^{0a} \cap \gamma_2^{0a})^{0a}$ is precisely the convex envelope of γ_1 and γ_2 , and hence by (1.4) there exists a wedge $U' \in \mathfrak{W}_{\langle \gamma_1, \gamma_2 \rangle}$ and a section $u \in \Gamma(U'; \mathfrak{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{B}_M))$ with $b_{\langle \gamma_1, \gamma_2 \rangle}(u) = \tilde{u}$. Again by analytic continuation, one checks that u extends to an open set $U \in \mathfrak{W}_{\langle \gamma_1, \gamma_2 \rangle}$ with $U \supset U_1 \cup U_2$. **Q.E.D.**

Notice that in the case where one replaces N by M, M by X^R, X by $X \times \bar{X}$, and \mathcal{N} by $\bar{\partial}$, the boundary value morphism considered above is the classical:

$$b_\gamma: \Gamma(U; \mathcal{O}_X) \rightarrow \Gamma_{\gamma^{0a}}(T_M^*X; \mathcal{C}_M)$$

2. Proof of Theorem 1.4.

Set $F = \mathbb{R} \mathfrak{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{O}_X)$, the complex of holomorphic solutions to \mathcal{N} , and consider the natural projections

$$T_N^*Y \xrightarrow{t_{g_N}} T_N^*X \xrightarrow{t_{f_N}} T_N^*M.$$

We shall reduce the proof of Theorem 1.4 to the two following isomorphisms:

(2.1) if \mathcal{M} is an elliptic left coherent \mathcal{O}_X -module, one has:

$$\mu_N(\mathbb{R} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{B}_M)) \simeq \mathbb{R} f'_{N*} \mu_N(F) \otimes or_{M/X}[n],$$

(2.2) if \mathcal{M} is a left coherent \mathcal{O}_X -module non-characteristic for Y , one has:

$$\mathbb{R} {}^t g'_{N*} \mu_N(F) \otimes or_{N/X}[n] \simeq \mathbb{R} \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}_Y, \mathcal{C}_N) \otimes or_{Y/X}[-d].$$

In fact, since the restriction of ${}^t g'_N$ to $\text{char}(\mathcal{M}) \cap T_N^* X$ is finite, it follows from (2.2) that $H^j \mu_N(F) = 0$ for $j < n + d$. The conclusion of Theorem 1.4 then follows by formula (2.1).

Let us prove (2.1). By [K-S], Theorem 11.3.3 one has the equality

$$(2.3) \quad \text{SS}(F) = \text{char}(\mathcal{M}).$$

According to (2.3), \mathcal{M} is elliptic if and only if M is non-characteristic for F . One then has the following chain of isomorphisms:

$$\mu_N(\mathbb{R} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{B}_M)) \simeq \mu_N(f^! F) \otimes or_{M/X}[n] \simeq \mathbb{R} {}^t f'_{N*} \mu_N(F) \otimes or_{M/X}[n],$$

where the second isomorphism follows from Theorem 1.3. This proves (2.1).

Let us prove (2.2). According to (2.3), Y is non-characteristic for \mathcal{M} if and only if Y is non-characteristic for F . One then has the following chain of isomorphisms:

$$\begin{aligned} \mathbb{R} {}^t g'_{N*} \mu_N(F) \otimes or_{N/X}[n] &\simeq \mu_N(g^! F) \otimes or_{N/X}[n] \simeq \\ &\simeq \mu_N(F|_Y) \otimes or_{N/Y}[n - 2d] \simeq \\ &\simeq \mu_N(\mathbb{R} \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}_Y, \mathcal{O}_Y)) \otimes or_{N/Y}[n - 2d] \simeq \\ &\simeq \mathbb{R} \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}_Y, \mathcal{C}_N) \otimes or_{Y/X}[-d], \end{aligned}$$

where the second isomorphism follows from Theorem 1.3, and the third from the Cauchy-Kowalevski-Kashiwara theorem which asserts that, \mathcal{M} being non-characteristic for Y , $\mathbb{R} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X)|_Y \simeq \mathbb{R} \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}_Y, \mathcal{O}_Y)$.

3. Remarks for non-elliptic systems.

As already pointed out, for $\text{cod}_M N = 1$ Theorem 1.4 reduces to the Holmgren theorem, and to prove the latter the ellipticity assumption is not necessary.

For $\text{cod}_M N > 1$, one may then wonder whether Theorem 1.4 holds or not if the ellipticity hypothesis is dropped out.

3.1. In the frame of tempered distribution, Liess [L] gives an example of a differential system with constant coefficients for which the corresponding Corollary 1.7 does not hold.

3.2. In order to deal with the real analytic case (i.e. $*$ = \mathcal{A}_M in (1.2)), consider $M = \mathbf{R}^3$ with coordinates (t, x_1, x_2) , let N be defined by $x_1 = x_2 = 0$, and set $X = \mathbf{C}^3, Y = \mathbf{C} \times \{0\}$. Let \mathcal{N} be the (non-elliptic) module associated to the system

$$D_{x_1} + i x_1 D_t, \quad D_{x_2} + i x_2 D_t,$$

which is non-characteristic for Y .

In this case, one has $H^1 \mu_N \mathbf{R} \mathcal{H} \text{Com}_{\mathcal{O}_X}(\mathcal{N}, \mathcal{A}_M) \neq 0$ as implied by the following:

PROPOSITION 3.1. *One has*

$$H^1 \mathbf{R} \Gamma_N \mathbf{R} \mathcal{H} \text{Com}_{\mathcal{O}_X}(\mathcal{N}, \mathcal{A}_M) \neq 0.$$

PROOF. By a change of holomorphic coordinates, \mathcal{N} is associated to a system of constant coefficient differential equations on X , and hence $H^j \mathbf{R} \mathcal{H} \text{Com}_{\mathcal{O}_X}(\mathcal{N}, \mathcal{A}_M) = 0$ for $j \neq 0$. It is then enough to find a solution $f \in \mathcal{A}_M(M \setminus N)$ for \mathcal{N} which does not extend analytically to M . This is the case for

$$(3.1) \quad f = \frac{1}{2t + i(x_1^2 + x_2^2)}. \quad \text{Q.E.D.}$$

Of course, the function f in (3.1) extends to M as a hyperfunction, since its domain of holomorphy in X contains a wedge with edge N .

3.3. We don't know whether Theorem 1.4 holds or not in the frame of hyperfunctions without the ellipticity assumption. However, note that $H^j \mathbf{R} \Gamma_N \mathbf{R} \mathcal{H} \text{Com}_{\mathcal{O}_X}(\mathcal{N}, \mathcal{B}_M) = 0$ for $j < \text{cod}_M N$, as implied by the following division theorem (cf. [S-K-K]) of which we give here a sheaf theoretical proof.

LEMMA 3.2. *Assume that Y is non-characteristic for \mathcal{N} . Then there is an isomorphism:*

$$\mathbf{R} \mathcal{H} \text{Com}_{\mathcal{O}_X}(\mathcal{N}, \Gamma_N \mathcal{B}_M) \simeq \mathbf{R} \mathcal{H} \text{Com}_{\mathcal{O}_Y}(\mathcal{N}_Y, \mathcal{B}_N) \otimes \text{or}_{N/M}[-d].$$

PROOF. We will neglect orientation sheaves for simplicity. Setting $F = \mathbf{R} \mathcal{H}om_{\omega_X}(\mathcal{N}, \mathcal{O}_X)$, one has the isomorphisms:

$$\mathbf{R} \mathcal{H}om_{\omega_X}(\mathcal{N}, \Gamma_N \mathcal{B}_M) \simeq \mathbf{R} \Gamma_N(F)[n] \simeq i^! g^! F[n] \simeq \mathbf{R} \Gamma_N(F|_Y)[n - 2d].$$

By the Cauchy-Kowalevski-Kashiwara theorem, $F|_Y \simeq \mathbf{R} \mathcal{H}om_{\omega_Y}(\mathcal{N}_Y, \mathcal{O}_Y)$, and one concludes.

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