RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

M. J. FERREIRA
RENATO TRIBUZY

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Rendiconti del Seminario Matematico della Università di Padova, tome 94 (1995), p. 17-23
<http://www.numdam.org/item?id=RSMUP_1995__94__17_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 94 (1995)

On the Type Decomposition of the Second Fundamental Form of a Kähler Submanifold.

M. J. FERREIRA (*) - RENATO TRIBUZY (**)

1. Introduction and statement of results.

Let (M, J) be a connected Kähler manifold of complex dimension m, N be a Riemannian manifold and

$$\varphi: M \to N$$

an isometric immersion.

Let TM denotes the tangent bundle of M and $T^{\mathbb{C}}M = TM \otimes \mathbb{C}$ its complexification. We represent by α either the second fundamental form of φ or its complex bilinear extension.

Decomposition of $T^{\mathbb{C}}M$ according to types

$$T^{\mathbb{C}}M = T'M \oplus T''M, \tag{1}$$

induces a decomposition of α into $(2,0)$, $(0,2)$ and $(1,1)$ parts by restricting to $T'M \otimes T'M$, $T''M \otimes T''M$ and $T'M \otimes T''M \oplus T''M \otimes T'M$ giving rise respectively to the operators $\alpha^{(2,0)}$, $\alpha^{(0,2)}$ and $\alpha^{(1,1)}$.

(*) Permanent address: Departamento de Matemática, Facultade de Ciências, Universidade de Lisboa, Rua Ernesto de Vasconcelos, Bl. C_2, 3°, 1700 Lisboa, Portugal.

(**) Permanent address: Departamento de Matemática - ICE - Universidade do Amazonas, Manaus, AM, Brasil.

We say that φ is (1, 1)-geodesic if $\alpha^{(1,1)} \equiv 0$. The condition $\alpha^{(1,1)} \equiv 0$ is quite interesting. Indeed the vanishing of $\alpha^{(1,1)}$ is equivalent to φ being harmonic when restricted to any complex curve [R]. Owing to this property (1, 1)-geodesic maps are sometimes called pluriharmonic maps. It is easily seen that $\pm$ holomorphic maps between Kähler manifolds are (1, 1)-geodesic, so that (1, 1)-geodesic maps lie between harmonic and $\pm$ holomorphic maps.

(1, 1)-geodesic maps have been studied by several authors, but with the exception of the holomorphic ones very few examples are available. However, if N is flat, Dacjzer and Rodrigues [D-R] showed that the only (1, 1)-geodesic immersions are the minimal immersions.

In a real setting (1, 1)-geodesic maps have also a nice description.

The second fundamental form α in conjunction with the complex structure J give rise to two operators, which we denote respectively by P and Q, defined by

$$P(X, Y) = \frac{1}{2}\{\alpha(X, Y) + \alpha(JX, JY)\},$$

$$Q(X, Y) = \frac{1}{2}\{\alpha(X, Y) - \alpha(JX, JY)\},$$

where $X, Y \in C(TM)$.

We remark that if X' and Y'' are respectively the (1, 0) and (0, 1) components of X and Y with respect to the decomposition (1), we have

$$\alpha^{(1,1)}(X', Y'') = P(X, Y) + iP(X, JY),$$

so that (1, 1)-geodesic maps are also characterized by the vanishing of P.

We say that φ is (2, 0)-geodesic if $\alpha^{(2,0)} \equiv 0$. As above, it can be seen that a map is (2, 0)-geodesic if and only if $Q \equiv 0$. Surprisingly the vanishing of Q is a strong condition. Indeed, when N is a spaceform it can be infered from Codazzi-equations that a (2, 0)-geodesic isometric immersion has parallel second fundamental form. Ferus [F] classified all the (2, 0) geodesic isometric embeddings into $\mathbb{R}^n$. These are, of course, immersions with parallel operator P. When $m = 1$, the isometric immersions with P parallel are precisely those with parallel mean curvature. Isometric immersions with parallel P have been studied by the authors.

In this work we analyze the case of isometric immersions with P totally umbilical, that is, $P = \langle , \rangle H$, where H denotes the mean curvature of φ and $\langle , \rangle$ the metric of M.

We prove that:

THEOREM 1. Let $\varphi: M \to Q^n(c)$ be an isometric immersion into an n-dimensional spaceform with constant sectional curvature c. If P is totally umbilical one has:

(i) if $c = 0$, then either $H \equiv 0$ or $m = 1$;

(ii) if $c > 0$, then $m = 1$;

(iii) if $c < 0$, then either $m = 1$ or φ has constant mean curvature $\|H\| = \sqrt{-c}$.

THEOREM 2. Let $\varphi: M \to G_p(\mathbb{C}^n)$ be an isometric immersion into the Grassmannian of complex p-dimensional subspaces of $\mathbb{C}^n$. If P is totally umbilical, then either $m \leqslant (p-1)(n-p-1)+1$ and φ is $(1,1)$-geodesic or φ is $\pm$ holomorphic.

COROLLARY 1. Let $\varphi: M \to \mathbb{C}P^n$ be an isometric immersion with P totally umbilical. Then either M is a Riemann surface or φ is $\pm$ holomorphic.

THEOREM 3. Let N be a $1/4$-pinched Riemannian manifold and $\varphi: M \to N$ be an isometric immersion. If P is totally umbilical, M is a Riemann surface.

Mapping into Riemannian manifolds with constant sectional curvature c, Dacjzer and Rodrigues [D-R] showed that when $c = 0$ minimality is equivalent to being $(1,1)$ geodesic. Moreover they proved that when $c \neq 0$, the only $(1,1)$-geodesic isometric immersions are the minimal surfaces. These theorems are an easy consequence of the following result:

THEOREM 4. Let $\varphi: M \to Q^n(c)$ be an isometric immersion into an n-dimensional spaceform with sectional curvature c. Therefore:

(i) if $c = 0$, then $\|H\| = \dfrac{1}{\sqrt{2m}}\|P\|$;

(ii) if $c < 0$, then $\|H\| \geqslant \dfrac{1}{\sqrt{2m}}\|P\|$, equality holds if and only if $m = 1$;

(iii) if $c > 0$, then $\|H\| \leqslant \dfrac{1}{\sqrt{2m}}\|P\|$, equality holds if and only if $m = 1$.

THEOREM 5. Let $\varphi: M \to G_p(\mathbb{C}^n)$ be an isometric immersion. Then

$$\|H\| \leq \frac{1}{\sqrt{2}m} \|P\|,$$

when the equality holds either $m \leq (p-1)(n-p-1)+1$ or φ is $\pm$ holomorphic.

REMARK. A similar result with the reversed inequality holds when in Theorem 5 we replace $G_p(\mathbb{C}^n)$ by its dual symmetric space of non-compact type.

Theorem 5 generalizes theorems A and B of [D-T]. Indeed, when $p = 1$, $G_p(\mathbb{C}^{n+1})$ is the n-dimensional complex projective space, so that, if $m > 1$ and φ is (1,1)-geodesic, the equality $\|H\| = \frac{1}{2m}\|P\|$ holds trivially and φ is $\pm$ holomorphic. Theorem 5 also generalizes Theorem 5 of [F-R-T] and Theorem 3.7 of [O-U].

THEOREM 6. Let N be a 1/4-pinched Riemannian manifold and $\varphi: M \to N$ be an isometric immersion. Then

$$\|H\| \leq \frac{1}{\sqrt{2}m} \|P\|,$$

equality holds if and only if M is a Riemann surface.

When M is an s-dimensional (not necessarily complex) pseudoumbilical submani-fold of a spaceform $Q^n(c)$, Chen and Yano [C-Y] showed that the (non-normalized) scalar curvature r of M satisfies

$$r \leq s(s-1)(c + \|H\|^2),$$

and the equality is attained when M is totally umbilical.

When M is a Kähler manifold this inequality can be sharpened without the pseudoumbilicity assumption, as we state in the following result:

THEOREM 7. Let $\varphi: M \to Q^n(c)$ be an isometric immersion into a spaceform with sectional curvature c. Then the scalar curvature of M satisfies

$$r \leq 2m^2(c + \|H\|^2),$$

and the equality is attained when, and only when, φ is (2, 0)-geodesic.

When the target manifold has constant holomorphic sectional curvature c we get:

THEOREM 8. Let $\varphi: M \to H^n(c)$ be an isometric immersion into a Riemannian manifold with constant holomorphic sectional curvature c. If φ is totally real the following inequality holds:

$$r \leqslant 2m^2\left(\frac{1}{4}c + \|H\|^2\right).$$

Moreover, the equality is attained when and only when, φ is (2, 0)-geodesic.

REMARK. When φ is minimal results analogous to those of Theorems 7 and 8 may be found in [D-R] and [D-T].

2. Proof of the statements.

First observe that the isotropy and the parallelism of $T'M$ imply

$$\langle R(X, Y)Z, W\rangle = 0,$$

for every $x \in M$, $X, Y \in T_x^{\mathbf{C}}M$ and $Z, W \in T_x' M$, where R denotes the complex multilinear extension of the curvature tensor R of M.

For each $x \in M$ consider a local orthonormal frame field $\{e_1, \dots, e_m, Je_1, \dots, J_{e_m}\}$ in a neighbourhood of x. we shall use the following notation:

$$\sqrt{2}\,E_j = e_j + iJe_j \in T''$$

and

$$\sqrt{2}\,E_{-j} = \sqrt{2}\,\overline{E}_j \in T' \quad \text{for each } j \in \{1, \dots, m\}.$$

If $\widetilde{R}$ denotes the Riemannian curvature tensor of N, using the complex multilinear extension of the Gauss equation we get

$$0 = \langle \alpha(E_k, \overline{E}_k), \alpha(E_r, \overline{E}_r)\rangle - \langle \alpha(E_k, \overline{E}_r), \alpha(E_r, \overline{E}_k)\rangle + \langle \widetilde{R}(E_k, E_r)\overline{E}_k, \overline{E}_r\rangle. \tag{2}$$

Summing in k and r we obtain

$$0 = m^2\|H\|^2 - \frac{1}{2}\|P\|^2 + \sum_{k,r}\langle \widetilde{R}(E_k, E_r)\overline{E}_k, \overline{E}_r\rangle. \tag{3}$$

When N has constant sectional curvature c

$$\langle \widetilde{R}(E_k, E_r)\overline{E}_k, \overline{E}_r \rangle = c(1 - \delta_{k,r}), \tag{4}$$

hence

$$\sum_{k,r=1}^{m} \langle \widetilde{R}(E_k, E_r)\overline{E}_k, \overline{E}_r \rangle = cm(m-1). \tag{5}$$

From (3) and (5) we get the conclusions of Theorem 4.

Now let $N = G_p(\mathbb{C}^n) \simeq \dfrac{U(n)}{U(p) \times U(n-p)}$. If $\mathcal{U}$ represents the Lie algebra of $U(n)$ and κ the subalgebra corresponding to $U(n) \times U_{(n-p)}$ we can identify $T_{\varphi(x)}N$ with the orthogonal complement $\mathcal{P}$ of k in $\mathcal{U}$ with respect to the Killing-Cartan form of $U(n)$.

Under this identification we know that at x

$$\langle \widetilde{R}(E_k, E_r)\overline{E}_k, \overline{E}_r \rangle = \|[E_k, E_r]\|^2. \tag{6}$$

Using (3) and (6), when P is totally umbilical, we get $\|[E_k, E_r]\|^2 = \|H\|^2 = 0$.

It follows from [F-R-T] that either $m \leqslant (p-1)(n-p-1)+1$ or φ is $\pm$ holomorphic.

When N is a 1/4-pinched Riemannian manifold it is easily seen that

$$\langle \widetilde{R}(E_k, E_r)\overline{E}_k, \overline{E}_r \rangle \geqslant 0 \tag{7}$$

and, as above, we get Theorem 6 from (3) and (7).

Assume now that P is totally umbilical. We get from (2) that

$$(1 - \delta_{k,r})\|H\|^2 + \langle \widetilde{R}(E_k, E_r)\overline{E}_k, \overline{E}_r \rangle = 0. \tag{8}$$

Theorems 1, 2 and 3 are then a straightforward consequence of (4), (6), (7) and (8).

To get Theorems 7 and 8 observe that

$$\sum_{k,r=1} \langle R(E_k, \overline{E}_r)\overline{E}_k, E_r \rangle = \frac{r}{2}.$$

Therefore, from the Gauss equation, we get

$$r = m^2\|H\|^2 - \frac{1}{2}\|Q\|^2 + \sum_{k,r=1}^{m} \langle \widetilde{R}(E_k, \overline{E}_r)\overline{E}_k, E_r \rangle.$$

When

$$N = Q^n(c), \sum_{k, r=1}^{m} \langle \widetilde{R}(E_k, \overline{E}_r)\overline{E}_k, E_r \rangle = m^2 c,$$

hence

$$r \leqslant 2m^2(\|H\|^2 + c),$$

with equality when and only when $Q \equiv 0$.

When N has constant holomorphic sectional curvature c and φ is totally real,

$$\sum_{k, r=1}^{m} \langle \widetilde{R}(E_k, \overline{E}_r)\overline{E}_k, E_r \rangle = \frac{c}{4} m^2,$$

so that

$$r \leqslant 2m^2\left(\|H\|^2 + \frac{1}{4}c\right)$$

with equality when and only when φ is (2, 0)-geodesic.

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Manoscritto pervenuto in redazione il 7 gennaio 1994.