

# RENDICONTI *del* SEMINARIO MATEMATICO *della* UNIVERSITÀ DI PADOVA

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*Rendiconti del Seminario Matematico della Università di Padova,*  
tome 94 (1995), p. 155-163

<[http://www.numdam.org/item?id=RSMUP\\_1995\\_\\_94\\_\\_155\\_0](http://www.numdam.org/item?id=RSMUP_1995__94__155_0)>

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## Minimal Immersions of Surfaces into $n$ -Dimensional Space Forms.

IRWEN VALLE GUADALUPE (\*)

**ABSTRACT** - Using the motion of the ellipse of curvature we study minimal immersions of surfaces into  $n$ -dimensional space forms. In this paper we obtain an extension of Theorem 2 of [9]. Also, we obtain some inequalities relating the integral of the normal curvature with topological invariants.

### 1. Introduction.

Let  $M$  be an oriented surface which is isometrically immersed into an orientable  $n$ -dimensional space form  $Q^n(c)$ ,  $n \geq 4$ , where  $Q^n(c)$  stands for the sphere  $S^n(c)$  of radius  $1/c$ , the Euclidean space  $\mathbb{R}^n$  or the hyperbolic space  $H^n(c)$ , according to  $c$  is positive, zero or negative. If the normal curvature tensor  $R^\perp$  of the immersion is nowhere zero, then exists an orthogonal bundle splitting  $NM = (NM)^* \oplus (NM)^0$  of the normal bundle  $NM$  of the immersion, where  $(NM)^0$  consists of the normal directions that annihilate  $R^\perp$  and  $(NM)^*$  is a 2-plane subbundle of  $NM$ .

Let  $K$  and  $K_N$  be the Gaussian and the normal curvature of  $M$ . Let  $K^*$  be the intrinsic curvature of  $(NM)^*$ .

We shall make use of the *curvature ellipse* of  $x: M \rightarrow Q^n(c)$ , which is, for each  $p$  in  $M$  the subset of  $N_p M$  given by

$$\varepsilon_p = \{B(X, X) \in N_p M; X \in T_p M \text{ and } \|X\| = 1\}$$

where  $B$  is the second fundamental form of the immersion. The first result of this paper is an extension of Theorem 2 of Rodriguez-Guadalupe [9] to the case when  $M$  is not homeomorphic to the sphere  $S^2$ .

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**THEOREM 1.** Let  $x: M \rightarrow S^n(1)$  be a minimal immersion of a complete oriented surface  $M$  into the unit sphere  $S^n(1)$  with  $R^\perp \neq 0$  and  $K \geq 0$ . If  $2K \geq K^*$  at every point, then  $K^*$ , the normal curvature  $K_N$  and the Gaussian curvature  $K$  of  $M$  are constant.

**REMARKS.** (1) If  $K > 0$ , then we obtain a minimal  $S^2$  of constant curvature in  $S^n(1)$ . These were classified by Do Carmo-Wallach [4]. Itoh [6] and Asperti-Ferus-Rodriguez [1] have a similar theorem.

(2) For  $K = 0$  we obtain a «flat» minimal torus. These were studied by Kenmotsu [7], [8].

The second result of this paper is the following.

**THEOREM 2.** Let  $x: M \rightarrow S^n(1)$  be a minimal immersion of a complete oriented surface  $M$  into the unit sphere  $S^n(1)$ . If  $K \geq 0$  at every point, then either  $K \equiv 0$  or the ellipse is a circle.

The following theorem relates an inequality between the integral of the normal curvature with topological invariants.

**THEOREM 3.** Let  $x: M \rightarrow Q^n(c)$  be a minimal immersion of a compact oriented surface  $M$  into an oriented  $n$ -dimensional space form  $Q^n(c)$  of constant curvature  $c$  with  $R^\perp \neq 0$ . Then we have

$$(1.1) \quad \int_M K_N dM \geq 4\pi\chi(M)$$

the equality holds if and only if  $(M \sim S^2)n = 4$ .

**COROLLARY 1.** Let  $x: M \rightarrow S^n(1)$  be a minimal immersion of a compact oriented surface  $M$  into the unit sphere  $S^n(1)$  with  $R^\perp \neq 0$ . Then we have

$$(1.2) \quad \text{Area } (M) \geq 6\pi\chi(M)$$

the equality holds if and only if  $(M \sim S^2)n = 4$ .

**REMARK.** Of course (1.2) has interest only when  $M \sim S^2$ , otherwise  $\chi(M) \leq 0$  and (1.2) becomes trivial.

The proofs of the above results are presented in section 4.

I want to thank Professor Asperti for bringing [2] and [3] for my attention.

## 2. Preliminaries.

Let  $M$  be a surface immersed in a Riemannian manifold  $Q^n$ . For each  $p$  in  $M$ , we use  $T_p M$ ,  $TM$ ,  $N_p M$  and  $NM$  to denote the tangent space of  $M$  at  $p$ , the tangent bundle of  $M$ , the normal space of  $M$  at  $p$  and the normal bundle of  $M$ , respectively. We choose a local field of orthonormal frames  $e_1, e_2, \dots, e_n$  in  $Q^n$  such that restricted to  $M$ , the vectors  $e_1, e_2$  are in  $T_p M$  and  $e_3, \dots, e_n$  are in  $N_p M$ . We shall make use the following convention on the ranges of indices:

$$\begin{aligned} 1 &\leq A, B, C, \dots \leq n, & 1 &\leq i, j, k \leq 2, \\ 3 &\leq \alpha, \beta, \gamma, \dots \leq n \end{aligned}$$

and we shall agree that reapeated indices are summed over the respective ranges. With respect to the frame field of  $Q^n$  chosen above, let  $\omega^1, \omega^2, \dots, \omega^n$  be the field of dual frames. Then the structure equations of  $Q^n$  are given by.

$$(2.1) \quad d\omega_A = - \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$(2.2) \quad d\omega_{AB} = - \sum_C \omega_{AC} \wedge \omega_{CB} + \phi_{AB}, \quad \phi_{AB} = \frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D,$$

$$K_{ABCD} + K_{ABDC} = 0.$$

If we restrict these forms to  $M$ . Then

$$(2.3) \quad \omega_\alpha = 0$$

since  $0 = d\omega_\alpha = - \sum_i \omega_{ai} \wedge \omega_i$ , by Cartan' s lemma we may write

$$(2.4) \quad \omega_{ia} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha,$$

From these formulas, we obtain

$$(2.5) \quad d\omega_i = - \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$(2.6) \quad d\omega_{ij} = - \sum_k \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}, \quad \Omega_{ij} = \frac{1}{2} \sum_l R_{ijkl} \omega_k \wedge \omega_l,$$

$$(2.7) \quad R_{ijkl} = K_{ijkl} + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha),$$

$$(2.8) \quad d\omega_{\alpha\beta} = - \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \Omega_{\alpha\beta}, \quad \Omega_{\alpha\beta} = \frac{1}{2} \sum_l R_{\alpha\beta kl} \omega_k \wedge \omega_l,$$

$$(2.9) \quad R_{\alpha\beta kl} = K_{\alpha\beta kl} + \sum_i (h_{ik}^\alpha h_{il}^\beta - h_{il}^\alpha h_{ik}^\beta).$$

The Riemannian connection of  $M$  is defined by  $(\omega_{ij})$ . The form  $(\omega_{\alpha\beta})$  defines a connection  $\nabla^\perp$  in the normal bundle of  $M$ . We call

$$(2.10) \quad B = \sum_{\alpha, i, j} h_{ij}^\alpha \omega_i \omega_j e_\alpha$$

the *second fundamental form* of  $M$ . The *mean curvature vector* is given by

$$(2.11) \quad H = \sum_\alpha \left( \sum_i h_{ii}^\alpha \right) e_\alpha$$

An immersion is said to be minimal if  $H = 0$ .

Let  $R^\perp$  be the curvature tensor associated with  $\nabla^\perp$ . Let  $\{e_1, e_2\}$  be a tangent frame, if we denote  $B_{ij} = B(e_i, e_j)$ ;  $i, j = 1, 2$  then it is easy to see that

$$(2.12) \quad R^\perp(e_1, e_2) = (B_{11} - B_{22}) \wedge B_{12}.$$

An interesting notion in the study of surfaces in higher codimension is that of the *ellipse of curvature* defined as  $\{B(X, X) \in N_p M : \langle X, X \rangle = 1\}$ . To see that it is an ellipse, we just have to look at the following formula, for

$$(2.13) \quad \begin{cases} X = \cos \theta e_1 + \sin \theta e_2, \\ B(X, X) = H + \cos 2\theta u + \sin 2\theta v, \end{cases}$$

where  $u = (B_{11} - B_{22})/2$ ,  $v = B_{12}$  and  $\{e_1, e_2\}$  is a tangent frame. So we see that, as  $X$  goes once around the unit tangent circle,  $B(X, X)$  goes twice around the ellipse. Of course this ellipse could degenerate into a line segment or a point. Everywhere the ellipse is not a circle we can choose  $\{e_1, e_2\}$  orthonormal such that  $u$  and  $v$  are perpendicular. When this happens they will coincide with the semi-axes of the ellipse.

From (2.12) it follows that if  $R^\perp \neq 0$  then  $u$  and  $v$  are linearly independent and we can define a 2-plane subbundle  $(NM)^*$  of the normal bundle  $NM$ . This plane inherits a Riemannian connection from that of  $NM$ . Let  $R^*$  be its curvature tensor and define its curvature  $K^*$  by

$$(2.14) \quad d\omega_{34} = -K^* \omega_1 \wedge \omega_2$$

if  $\{e_3, e_4\}$  locally generates  $(NM)^*$ .

Now, if  $\xi$  is perpendicular to  $(NM)^*$ , then from (2.12),  $R^\perp(e_1, e_2)\xi = 0$ . Hence, it makes sense to define the normal curvature as

$$(2.15) \quad K_N = \langle R^\perp(e_1, e_2)e_4, e_3 \rangle$$

where  $\{e_1, e_2\}$  and  $\{e_3, e_4\}$  are orthonormal oriented bases of  $T_p M$  and  $N_p M$ , respectively. If  $TM$  and  $(NM)^*$  are oriented, then  $K_N$  is globally defined. In codimension 2,  $NM = (NM)^*$  and  $K_N$  has a sign. In higher codimension, if  $R^\perp \neq 0$ ,  $(NM)^*$  is globally defined and oriented if  $TM$  is. In this case, it is shown in [1] that  $\mathcal{X}(NM)^* = 2\mathcal{X}(M)$ , where  $\mathcal{X}(NM)^*$  denote the Euler characteristic of the plane bundle  $(NM)^*$  and  $\mathcal{X}(M)$  denote the Euler characteristic of the tangle bundle  $TM$ .

### 3. Minimal immersions with $R^\perp \neq 0$ .

In this section we assume that  $M$  has non-zero normal curvature tensor  $R^\perp$ . Also if  $M$  is orientable, then we will always choose orientations in  $TM$  and in  $(NM)^*$  such that  $K_N$  is positive. We have

**PROPOSITION 1.1.** Let  $x: M \rightarrow Q^n(c)$  be a minimal immersion of an oriented surface  $M$  into an orientable  $n$ -dimensional space form  $Q^n(c)$  of constant curvature  $c$ . Then we have

$$(3.1) \quad \Delta(\log |K_N - K + c|) = 2(2K - K^*)$$

if  $(K - c)^2 - K_N^2 > 0$ , and consequently

$$(3.2) \quad \Delta(\log |K_N + K - c|) = 2(2K + K^*).$$

**PROOF.** By Itoh [6] there exists isothermal coordinates  $\{x_1, x_2\}$  such that putting  $X_i = \partial/\partial x_i$ ,  $i = 1, 2$  then  $u = B(X_1, X_1) = -B(X_2, X_2)$  and  $v = B(X_1, X_2)$  are the semi-axes of the ellipse at every point where  $(K - c)^2 - K_N^2 \neq 0$ . Moreover we observe that  $|X_i|^2 = E = ((K - c)^2 - K_N^2)^{-1/4}$ ,  $i = 1, 2$ . If we denote  $\lambda = \langle u, u \rangle^{1/2}$  and  $\mu = \langle v, v \rangle^{1/2}$  and following the same arguments that [10] we have

$$(3.3) \quad \lambda^2 - \mu^2 = 1,$$

$$(3.4) \quad \lambda^2 + \mu^2 = -(K - c)E^2,$$

$$(3.5) \quad 2\lambda\mu = K_N E^2.$$

If  $(K - c)^2 - K_N^2 > 0$  from (3.4) and (3.5) we obtain

$$(3.6) \quad \lambda + \mu = (K_N - K + c/K_N + K - c)^{1/4}.$$

Let  $e_3 = \lambda^{-1}u$  and  $e_4 = \mu^{-1}v$  an oriented frame in  $(NM)^*$ . Now, following the same computations that [10] we get

$$(3.7) \quad \omega_{34}(X_1) = -X_2(f),$$

$$(3.8) \quad \omega_{34}(X_2) = X_1(f),$$

where  $f = \log |\lambda + u|$ .

Hence, we have

$$(3.9) \quad \omega_{34} = -X_2(f)dX_1 + X_1(f)dX_2.$$

Deriving (3.9) and using (2.14) we get

$$\begin{aligned} (3.10) \quad & -K^* \omega_1 \wedge \omega_2 = d\omega_{34} E^{-1} \\ & = (-X_2 X_2(f) dX_2 \wedge dX_1 + X_1 X_1(f) dX_1 \wedge dX_2) E^{-1} = \\ & = (X_1 X_1(f) + X_2 X_2(f)) E^{-1} dX_1 \wedge dX_2 = \tilde{\Delta}(f) E^{-1} \omega_1 \wedge \omega_2 \end{aligned}$$

where  $\tilde{\Delta}$  denotes de Laplacian of the «flat» metric. We know  $\tilde{\Delta}(f) = E\Delta(f)$ , where  $\Delta$  is the Laplacian of the surface. Hence, from (2.18) and (2.22) we get

$$(3.11) \quad \Delta(\log |K_N - K + c/K_N + K - c|) = -4K^*.$$

Using  $E = (K - c)^2 - K_N^2)^{-1/4}$  and the Gaussian curvature  $K$  given by the equation

$$(3.12) \quad K = -\frac{1}{2}E^{-1}\tilde{\Delta}\log E.$$

we obtain

$$(3.13) \quad \Delta(\log |K_N - K + c|) + \Delta(\log |K_N + K - c|) = 8K$$

From (3.11) and (3.13) we get the equations (3.1) and (3.2).

**COROLLARY 1.** Let  $x: M \rightarrow Q^n(c)$  be a minimal immersion with  $K^* > 0$ . Then the ellipse is a circle.

**PROOF.** Suppose that the ellipse is not a circle then from (3.11)  $\Delta(\log |K_N - K + c/K_N + K - c|) < 0$ . So  $K_N > 0$  implies that  $\log \frac{|K_N - K + c|}{|K_N + K - c|}$  is subharmonic and bounded from below. Therefore  $\log \frac{|K_N - K + c|}{|K_N + K - c|}$  is constant and this implies that  $K^* \equiv 0$ . This is a contradiction.

COROLLARY 2. Let  $x: M \rightarrow Q^n(c)$  be a minimal immersion of a compact surface  $M$  with  $2K > K^*$ . Then the ellipse is a circle.

PROOF. Suppose that the ellipse is not a circle then from (3.1)  $\Delta(\log |K_N - K + c|) > 0$ . So we have that  $\log |K_N - K + c|$  is subharmonic and bounded from above and therefore is constant. This implies that  $2K = K^*$ . This is a contradiction.

#### 4. Proof of Theorems.

PROOF OF THEOREM 1. First we consider the case when the ellipse is not a circle, i.e.,  $(K - 1)^2 - K_N^2 > 0$ . Now if  $2K \geq K^*$  then from (3.1) follows that  $\Delta(\log |K_N - K + 1|) \geq 0$ . So we have that  $\log |K_N - K + 1|$  is subharmonic and bounded from above. Then

$$(4.1) \quad K_N - K + 1 = \text{constant}$$

and  $2K = K^*$ . On the other hand from (3.2) we get  $\Delta(\log |K_N + K - 1|) = 2(2K + K^*) = 8K \geq 0$ . Similarly from above we have

$$(4.2) \quad K_N + K - 1 = \text{constant}.$$

From (4.1) and (4.2) follows that  $K^*$ ,  $K_N$  and  $K$  are constant.

In the case that ellipse is a circle the theorem follows by Rodriguez-Guadalupe [9]. This completes the proof of theorem.

PROOF OF THEOREM 2. Suppose that the ellipse is not a circle. From (3.13) we obtain

$$(4.3) \quad \Delta(\log |K_N - K + 1/K_N + K - 1|) = 8K \geq 0.$$

From (3.5), Rodriguez-Guadalupe ([9], p. 9) and  $K \geq 0$  implies  $2\lambda\mu E^{-2} = K_N \leq 1$ . So from (3.4) we have  $(\lambda^2 + \mu^2)E^{-2} = 1$ . Therefore we get

$$\begin{aligned} 0 &\leq (\lambda + \mu)^2 E^{-2} = K_N - K + 1, \\ &= (\lambda^2 + \mu^2)E^{-2} + 2\lambda\mu E^{-2} \leq 2. \end{aligned}$$

This implies that  $|K_N - K + 1|$  is bounded from above. Similarly  $|K_N + K - 1|$  is bounded from above, too. Then  $\log(|K_N - K + 1/K_N + K - 1|)$  is subharmonic and bounded from above and therefore is constant. From (4.3) follows that  $K \equiv 0$ .

PROOF OF THEOREM 3. From Asperti ([2], Prop. 3.6) we have

$$(4.4) \quad K^* = K_N - \frac{\|B^2\|^2}{2K_N}$$

where  $B^2$  is the 3th fundamental form of  $M$ . From (4.4) we obtain

$$(4.5) \quad K_N \geq K^*$$

Integrating (4.5) over  $M$  and applying Ferus-Rodriguez-Asperti ([1], Th. 1) we get

$$(4.6) \quad \int_M K_N dM \geq \int_M K^* dM = 2\pi\chi(NM)^* = 4\pi\chi(M).$$

If  $\int_M K_N dM = 4\pi\chi(M)$  then  $K_N = K^*$  and from (4.4)  $B^2 \equiv 0$ . By Erbacher [5] the codimension is two and  $n = 4$ .

PROOF OF COROLLARY 1. If  $\text{Area}(M) = 6\pi\chi(M)$  then  $\chi(M) > 0$  and, actually  $\text{Area}(M) = 12\pi$ . It follows from Asperti ([3], p. 60) that

$$(4.7) \quad 12\pi \geq 2\pi(s+1)(s+2)$$

where  $s$  is such that  $n = 2 + 2s$ . It is clear now from (4.7) that  $s = 1$  and  $n = 4$ .

On the other hand, if  $n = 4$  and  $x: M \rightarrow S^4(1)$  is a minimal two sphere with  $R^\perp \neq 0$ , then  $K_N = K^*$  and by theorem 3 above and Corollary 1 of Rodriguez-Guadalupe ([9]) we have

$$\text{Area}(M) = 2\pi\chi(M) + 2\pi\chi(NM) = 2\pi\chi(M) + \int_M K_N dM = 6\pi\chi(M).$$

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Manoscritto pervenuto in redazione il 6 giugno 1994.