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An Extension of a Result of H. Hopf to Kähler Submanifolds of $\mathbb{R}^n$.

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In the early fifties H. Hopf, [H], proved that a constant mean curvature surface, homeomorphic to a sphere, immersed in Euclidean 3-space is a standard round sphere.

As Wente has recently shown, [W], the topological assumption on the Euler characteristic is an essential requirement.

Let M be a Kähler manifold of complex dimension m and let $f: M \to \mathbb{R}^n$ be an isometric immersion. Considering the complexified tangent and normal bundles of f we can split the second fundamental tensor α according to type as $\alpha = \alpha^{(2,0)} + \alpha^{(1,1)} + \alpha^{(0,2)}$. We denote with H the mean curvature vector.

It is trivial to see that, when M is a surface, the parallelism of H in the normal bundle can be equivalently expressed by

$$\nabla^{\perp} \alpha^{(1,1)} \equiv 0. \tag{1}$$

It looks thus quite natural to try to generalize the Hopf's result to higher dimensional Kähler immersed submanifolds of $\mathbb{R}^n$, under the assumption (1) (see Corollary below).

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In order to state our theorem we need to recall a further ingredient: the notion of isotropy. This has been introduced (in the real case) by Calabi, [C], and (in the complex case) by Eells and Wood, [EW], in their work on minimal surfaces.

Let ∇ represent the covariant derivative on the pull-back of the trivial $\mathbb{C}^n$-bundle over $\mathbb{R}^n$ and consider its type decomposition $\nabla = \nabla^{(1,0)} + \nabla^{(0,1)} = \nabla' + \nabla''$. Let $\langle , \rangle$ denote the complex bilinear extension of the canonical inner product of $\mathbb{R}^n$.

We say that an isometric immersion $f\colon M \to S^t \subset \mathbb{R}^n$ is second order isotropic if

$$\langle \nabla'^{\alpha} f, \ \nabla'^{\beta} f\rangle \equiv 0 \tag{2}$$

for $\alpha + \beta \geqslant 1$ and $\alpha, \beta \leqslant 2$.

THEOREM. Let M be a compact, connected, simply connected Kähler manifold with positive first Chern class $C_1(M)$. Let $f\colon M \to \mathbb{R}^n$ be an isometric immersion such that $\nabla^{\perp} \alpha^{(1,1)} \equiv 0$. Then M is isometric to a Riemannian product $M_1 \times \ldots \times M_r$ of Kähler manifolds and f splits into a product of immersions

$$f = f_1 \times \ldots \times f_r : M = M_1 \times \ldots \times M_r \to \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_r} = \mathbb{R}^n \tag{3}$$

where, for each $l \in \{1, \ldots r\}$, $f_l : M_l \to \mathbb{R}^{n_l}$ is minimal is some sphere and second order isotropic.

As a consequence we have:

COROLLARY. Under the same assumptions of the theorem one has:

i) if M has codimension 1, then $n = 3$ and $f(M)$ is a round 2-sphere;

ii) if M has codimension 2, then either $f(M)$ is the product of two round 2-spheres in $\mathbb{R}^6$ or $f(M)$ is a round 2-sphere in $\mathbb{R}^4$.

REMARK. If the codimension of M is at least 3 there are other examples, beside round spheres, as we can see considering, for instance, the Veronese surface in $S^4 \subset \mathbb{R}^5$.

PROOF (of the theorem). Consider the (symmetric) 2-form

$$\omega = \langle \alpha^{(2,0)}, H\rangle.$$

We claim that ω is a holomorphic section of $\otimes^2 T^* M^{(1,0)}$. Indeed, let $1 \leqslant i, j, k \leqslant m$ and let $\{z_i\}$ be local holomorphic coordinates on M. We then compute

$$\frac{\partial}{\partial \bar{z}_i}\left\langle \alpha\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right), H\right\rangle =$$

$$= \left\langle \nabla^{\perp}_{\partial/\partial \bar{z}_i} H, \alpha\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right)\right\rangle + \left\langle H, \left(\nabla_{\partial/\partial \bar{z}_i}\alpha\right)\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right)\right\rangle +$$

$$+ \left\langle H, \alpha\left(\nabla_{\partial/\partial \bar{z}_i}\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right)\right\rangle + \left\langle H, \alpha\left(\frac{\partial}{\partial z_j}, \nabla_{\partial/\partial \bar{z}_i}\frac{\partial}{\partial z_k}\right)\right\rangle.$$

We now use (1) and the Codazzi equations

$$(\nabla_{\partial/\partial \bar{z}_i}\alpha)\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) = (\nabla_{\partial/\partial z_j}\alpha)\left(\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial z_k}\right)$$

to see that the first two terms in the above sum are zero. Furthermore, since ∇ preserves type and $\left[\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial z_j}\right] \equiv 0$ we have

$$\nabla_{\partial/\partial \bar{z}_i}\frac{\partial}{\partial z_j} \equiv 0.$$

It follows that $\frac{\partial}{\partial \bar{z}_i}\left\langle \alpha\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right), H\right\rangle \equiv 0$ proving our claim.

On the other hand, since $C_1(M) > 0$, from Yau, [Y], we know the existence of a Kähler metric on M with positive Ricci curvature. Using then a Bochner type technique one proves the non existence of non zero holomorphic sections of $\otimes^2 T^* M^{(1,0)}$ (for instance, as in Kobayashi and WU, [KW]). Hence

(4) $$\omega \equiv 0\ .$$

We now follow [FT]. First of all observe that $f\colon M \to \mathbb{R}^n$ with M compact and the parallelism of H imply that H is never zero. Secondly, (4) and (1) imply that the Weingarten operator A_H defined on TM by

$$\langle A_H X, Y\rangle + \langle \alpha(X, Y), H\rangle$$

is parallel too. Therefore, the pointwise eigenspaces of A_H define parallel distributions $T^1, \ldots, T^r$, orthogonal to each other, such that

$$TM = T^1 \oplus \ldots \oplus T^r.$$

Using de Rham's decomposition theorem we deduce that M can be written as a Riemannian product

$$M = M_1 \times \ldots \times M_r.$$

Furthermore, indicating with J the complex structure of M, (4) implies $J \circ A_H = A_H \circ J$ so that the subbundles T^l are invariant with respect to J. Therefore each factor M_l is Kählerian.

To show that f can be written as a product of immersions as in (3) we adapt a technique of Moore, [M]. For details see [FT].

Minimality of f_l in some sphere follows from the observation that the mean curvature vector H_l is parallel and the immersion is umbilical in the direction of H_l.

In order to check the second order isotropy of f_l the only non trivial point to verify is that

$$\langle \nabla'^2 f_l, \nabla'^2 f_l \rangle \equiv 0.$$

But, if α_l is the second fundamental tensor of f_l, this is equivalent to prove that the form

$$\psi = \langle \alpha_l^{(2,0)}, \alpha_l^{(2,0)} \rangle$$

is identically null.

We proceed as we did for ω to show that ψ is a holomorphic section of $\otimes^4 T^* M^{(1,0)}$ so that the condition $C_1(M) > 0$ implies $\psi \equiv 0$. ∎

PROOF (of the Corollary).

i) Since M has codimension 1 and it is compact then M cannot split into a product. It follows that $f: M \to S^{2m} \subset \mathbb{R}^{2m+1}$ is a minimal isometric immersion. Compactness of M implies that M is diffeomorphic to S^{2m}, but M is Kähler and thus $m = 1$..

ii) Suppose now that M has codimension 2. It follows from $\langle \alpha^{(2,0)}, \alpha^{(2,0)} \rangle \equiv 0$ that the image of $\alpha^{(2,0)}$, Im $\alpha^{(2,0)}$, is orthogonal to Im $\alpha^{(0,2)}$ in the Hermitian inner product. According to the theorem we have two possibilities: either $f = f_1 \times f_2 : M_1 \times M_2 \to \mathbb{R}^6$ with $f_1(M_1)$ and $f_2(M_2)$ round 2-spheres in $\mathbb{R}^3$, or $f: M^{2m} \to \mathbb{R}^{2m+2}$ is minimal in some sphere $S^{2m+1} \subset \mathbb{R}^{2m+2}$. Let us consider this latter case. Observe that $\dim_{\mathbb{R}} T^{\perp} M \otimes \mathbb{C} = 4$. We claim that $\alpha^{(2,0)} \equiv 0$. Indeed

$$\dim_{\mathbb{R}} \text{Im } \alpha^{(2,0)} = \dim_{\mathbb{R}} \text{Im } \alpha^{(0,2)}$$

furtherfore Im $\alpha^{(2,0)}$ and Im $\alpha^{(0,2)}$ are orthogonal and therefore if

$\alpha^{(2,0)} \not\equiv 0$ we would have somewhere:

$$\dim_{\mathbb{R}} \operatorname{Im} \alpha^{(2,0)} + \dim_{\mathbb{R}} \alpha^{(0,2)} + \dim_{\mathbb{R}} T^{\perp} S^{2m+1} \otimes \mathbb{C} \geq 6$$

contradiction.

Now observe that $T^{\perp} M \cap TS^{2m+1}$ is a parallel subbundle of $T^{\perp} M$ because it is orthogonal to the parallel vector field H and M has codimension 2. Let v be a unitary smooth section of $T^{\perp} M \cap TS^{2m+1}$. Then the Weingarten operator A_v is parallel and trace $A_v = 0$ because H is orthogonal to v. Moreover, since A_v is parallel, the same argument used in the proof of the theorem shows that either $A_v \equiv 0$ or f splits into a product of factors. This latter alternative is not possible because of the codimension assumption.

We can thus use a result of Erbacher, [E], on the reduction of codimension, to have that $f(M^{2m})$ is contained in some (affine) $\mathbb{R}^{2m+1}$. Hence $f(M^{2m}) \subseteq S^{2m}$ and from i) it follows that $m = 1$. ■

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