The classification of groups in which every product of four elements can be reordered

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The Classification of Groups in which Every Product of Four Elements can be Reordered.

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Secondly

**Theorem 2.** A finite group $G$ of odd order belongs to $P_4$ if and only if

1. $G$ is abelian, or
2. $|G'| = 3$, or
3. $|G'| = 5$ and $|G/Z(G)| = 25$.

Next it was shown in [1] that if a finite group $G$ belongs to $P_4$, then $G'$ is nilpotent. This was improved in [6] where all $P_4$-groups were shown to be metabelian. Then in [9] a complete description of the non-nilpotent groups in $P_4$ was given:

**Theorem 3.** A group $G$ belongs to $P_4$ if and only if one of the following holds:

1. $G$ has an abelian subgroup of index 2;
2. $G$ is nilpotent of class $\leq 4$ and $G \in P_4$;
3. $G' \cong V_4$;
4. $G = B(a, x)$, where $B \leq Z(G)$, $|a| = 5$ and $a^x = a^2$.

Finally in [7] finite 2-groups of class 2 in $P_4$ were classified by the following:

**Theorem 4.** Let $G$ be a finite 2-group of class 2 with $\exp G' = 2$. Then $G \in P_4$ if and only if

1. $G$ has an abelian subgroup of index 2, or
2. $|G'| \leq 4$, or
3. $|G'| = 8$ and $G/Z(G)$ can be generated by 3 elements, or
4. $|G'| = 8$, $G/Z(G)$ can be generated by 4 elements and $G$ is not diabelian.

Following Philip Hall, a group is said to be *diabelian* if it is a product of 2 abelian subgroups.

These are our starting points in order to characterize all $P_4$-groups.

As one might intuitively expect, the order of the derived subgroup of a $P_4$-group turns out to be «nearly always» bounded. The exceptions are due to the fact that any group $G$ with an abelian subgroup $A$ of index 2 always belongs to $P_4$. This can be seen without difficulty by distinguishing cases according to how many of the 4 factors of a product belong to $A$. In particular the wreath product $G$ of an arbitrary abelian
group $A$ and a group of order 2 always belongs to $P_4$ and $G' \cong A$. Thus there is no restriction (other than commutativity) on the derived subgroup of a $P_4$-group. Then we shall see that a $P_4$-group which does not have an abelian subgroup of index 2 has derived subgroup of order at most 8.

Our classification of $P_4$-group is given in

**Theorem 5.** A group $G$ belongs to $P_4$ if and only if one of the following holds:

(i) $G$ has an abelian subgroup of index 2;
(ii) $|G'| \leq 3$;
(iii) $G' \cong V_4$;
(iv) $G' \cong C_4$ and $G$ has a subgroup $B$ of index 2 with $|B'| = 2$;
(v) $G' \cong C_6$ and $G/Z(G)$ is isomorphic to the holomorph of $C_6$;
(vi) $G' \cong C_5$ and $|G/Z(G)| = 25$;
(vii) $G' \cong C_4 \times C_2$ and, with $C = C_G(G')$, $|G/C| = 2$, $C' = \bigcup_i(G')$, all subgroups of $G'$ are normal in $G$ and $G/Z(G)$ can be generated by 2 elements;
(viii) $G' \cong C_2 \times C_2 \times C_2$, $G$ has class 2 and either $G/Z(G)$ can be generated by 3 elements or $G/Z(G)$ can be generated by 4 elements and $G$ is not diabelian.

Notation is as follows.

- $C_n$ cyclic group of order $n$,
- $V_4$ the 4-group,
- $\Sigma_n$ symmetric group of degree $n$,
- $\Omega_1(G)$ subgroup generated by elements of order $p$ in a $p$-group $G$,
- $\Omega_i(G)$ subgroup generated by the $p^i$-th powers of the elements of a $p$-group $G$,
- $g^x = x^{-1}gx$,
- $[x, y] = x^{-1}y^{-1}xy$,
- $\exp G$ exponent of $G$.

In §2-4 we consider groups with derived subgroup of order 4, 8 and greater than 8, respectively, and in §5 we deal with nilpotent groups. We recall again from [6] that

a $P_4$-group is always metabelian

and this will be assumed from now on without further reference. For convenience we mention some known results (see [7] and [9]) which we shall use.
1.1. Let $G \in P_4$ and $A$ be an abelian subgroup of $G$ containing $G'$. Let $a, b \in A$, $x, y \in G$ and suppose that $[a, x], [a, y]$ and $[a, x^{-1} y]$ are all different from 1 and $[b, y] = 1$.

(i) If $[b, x]$ has order 2 and commutes with $x$, then $[b, x] = [a, x]$, $[a, y]$ or $[a, x][y, a]$.

(ii) If $[a, y]$ has order 2 and commutes with $x$, then $[b, x] = 1$, $[a, x][a, y]$ or $[a, x][a, y]$.

1.2. Let $G$ be a finite 2-group in $P_4$ and $A$ be an abelian subgroup of $G$ containing $G'$. If $G = A(x)$, then one of the following holds:

1. $[A, x^2] = 1$; 2. $G' \cong V_4$; 3. $G' \cong C_4$ and $G' \leq Z(G)$.

1.3. Let $G = A \times B$ be a finite $P_4$-group with $A$ of odd order and $B$ a 2-group. Then either $A$ or $B$ is abelian.

1.4. Let $G$ be a finite nilpotent of class 2 group in $P_4$ with $\exp G' = 4$. Then $G' \cong C_4$.

1.5. Let $G$ be a finite nilpotent of class 2 group with $G' \cong C_4$. Then the following are equivalent:

(i) $G \not\in P_4$;

(ii) there are elements $x_1, x_2, x_3, x_4 \in G$ such that $[x_1, x_2] = [x_2, x_3] = [x_3, x_4]$ of order 4 and $[x_1, x_3] = [x_1, x_4] = [x_2, x_4] = 1$.

1.6. Let $G$ be a finite nilpotent of class 2 group with $G' \cong C_4$. Then $G \in P_4$ if and only if $G$ has a subgroup $B$ of index 2 with $|B'| = 2$.

Sometimes in the following sections we shall only sketch technical proofs, for the sake of brevity. All details can be found in[8].

2. Groups with cyclic derived subgroup of order 4.

If $G' \cong C_4$, then $G$ is nilpotent of class 2 or 3. If $G$ has class 2 and is a finite 2-group, then we know already that $G \in P_4$ if and only if $G$ has a subgroup $B$ of index 2 with $|B'| = 2$ (1.6). It turns out that the hypothesis that $G$ is a finite 2-group here is unnecessary (2.2). Also, rather curiously, we find that if $G$ has class 3, then $G$ always belongs to $P_4$ (2.3). The situation is summed up in 2.4.

First a routine argument (which we omit) establishes
2.1. For any \( n \), an inverse limit of \( P_\infty \)-groups belongs to \( P_n \).

Now we can handle the class 2 groups.

2.2. Let \( G \) be nilpotent of class 2 with \( G' \equiv C_4 \). Then \( G \in P_4 \) if and only if \( G \) has a subgroup \( B \) of index 2 with \(|B'| \leq 2\).

**Proof.** Let \( G \in P_4 \). We proceed in 3 steps.

(i) *Suppose that \( G \) is finite.* Then the 2-complement of \( G \) is abelian and so we may assume that \( G \) is a 2-group. Then \( B \) exists, by 1.6.

(ii) *Suppose that \( G \) is finitely generated.* Then \( G \) is residually finite and there is a subgroup \( N \triangleleft G \) with \( G/N \) finite and \( N \cap G' = 1 \). By (i), there is a subgroup \( B \) of index 2 in \( G \) with \( B \trianglelefteq N \) and \(|(B/N)'| \leq 2\), i.e. \( B'/N/N \equiv B' \) has order at most 2.

(iii) *Suppose that \( G \) is arbitrary.* In this case, the argument proceeds by considering a local system of finitely generated subgroups with derived subgroups equal to \( G' \) and employing the theory of complete projection sets (see [5], volume 2, p. 168).

Conversely, suppose that \( G \) has a subgroup \( B \) of index 2 with \(|B'| \leq 2\). In order to prove that \( G \in P_4 \), we may assume that \( G \) is finitely generated and therefore residually finite. Thus, by 2.1, it suffices to assume that \( G \) is finite. Then \( G \in P_4 \), by 1.6. \( \blacksquare \)

Turning our attention to class 3 groups, we have

2.3. Let \( G \) be nilpotent of class 3 with \( G' \equiv C_4 \). Then \( G \in P_4 \).

**Proof.** Suppose, for a contradiction, that \( G \notin P_4 \) and let \( x_1 x_2 x_3 x_4 \) be a product which cannot be reordered. Let \( G' = \langle a \rangle \). Thus \([x_1, x_2] = a^\alpha, [x_1, x_3] = a^\beta, [x_1, x_4] = a^\gamma, [x_2, x_3] = a^\lambda, [x_2, x_4] = a^\mu, [x_3, x_4] = a^\nu\), where \(-1 \leq \alpha, \beta, \gamma, \lambda, \mu, \nu \leq 2\). Also \( a^{x_i} = a^{\varepsilon_i}, 1 \leq i \leq 4 \). Now \( G/\langle a^2 \rangle \in P_3 \). Let \( y_1 = x_1 x_2 \), \( y_2 = x_3 \), \( y_3 = x_4 \). Then there exists \( \sigma(\neq 1) \in \Sigma_3 \) such that

\[
y_1 y_2 y_3 = y_{\sigma(1)} y_{\sigma(2)} y_{\sigma(3)} a^2.
\]

It follows that \( \alpha \neq 2 \) and so \( \alpha = \pm 1 \). Without loss of generality we may assume that

\[
\begin{align*}
\alpha &= 1; \\
\text{similarly } \lambda &= \pm 1, \nu &= \pm 1.
\end{align*}
\]
Following the idea of the proof of 1.6, we have
\[ x_1 x_2 x_3 x_4 = x_4 x_1 x_2 x_3 t = x_4 x_3 x_2 x_1 u = x_4 x_3 x_1 x_2 v \]
with \( t, u, v \in G' \). Clearly \( G' = \{1, t, u, v\} \). Let \( x_1 x_2 x_3 x_4 = x_4 x_1 x_3 x_2 w \).
Then \( w = u \) or \( v \). Distinguishing these possibilities in the light of (1), we get the required contradiction.

The situation when \( G' \cong C_4 \) can now be summarised in

2.4. Let \( G' \cong C_4 \). Then \( G \in P_4 \) if and only if \( G \) has a subgroup \( B \) of index 2 with \( |B'| \leq 2 \).

PROOF. Let \( G \in P_4 \). If \( G \) has class 2, the result follows by 2.2. If \( G \) has class 3, then \( C = C_G(G') \) has index 2 in \( G \). By the 3-subgroup lemma, \([G', G] = 1\) and so \( C' \leq Z(G) \cap G' \cong C_2 \). Then take \( B = C \).

Conversely, let \( B \leq G \) with \( |G : B| = 2 \) and \( |B'| \leq 2 \). Again 2.2 takes care of the case when \( G \) has class 2 and 2.3 gives the other case.


Since \( P_4 \)-groups are metabelian, we have to consider the case when \( G' \) is isomorphic to \( C_4 \times C_2, C_2 \times C_2 \times C_2 \) or \( C_8 \). These are discussed in §§ 3.1, 3.2, 3.3 respectively. In the second case, we shall see that the \( P_4 \)-groups \( G \) all have class 2. Then local arguments can be applied to the results of §3.2 of [7] in our final classification in §5 below. In the third case, it turns out that the only \( P_4 \)-groups are those with an abelian subgroup of index 2.

3.1. The case \( G' \cong C_4 \times C_2 \).

The automorphism group of \( C_4 \times C_2 \) is dihedral of order 8. Thus if \( G' \cong C_4 \times C_2 \) and \( C = C_G(G') \), then \( G/C \) is trivial, the 4-group, cyclic of order 4 or cyclic or order 2. We distinguish these cases.

3.1.1. Let \( G \) be nilpotent of class 2 with \( G' \) a 2-group of exponent \( \geq 4 \) and order \( \geq 8 \) Then \( G \notin P_4 \).

PROOF. Suppose, for a contradiction, that \( G \in P_4 \). We may assume that \( G \) is finitely generated, so \( G' \) is finite and \( G \) is residually finite. Thus there is \( N < G \) with \( G/N \) finite and \( N \cap G' = 1 \). Now the derived subgroup of \( G/N \) is a 2-group of exponent \( \geq 4 \) and order \( \geq 8 \) and it is the derived subgroup of the Sylow 2-subgroup \( P \) of \( G/N \). By 1.2, \( P' \) has exponent 4 and therefore \( P' \cong C_4 \), by 1.4, a contradiction.
The next two cases are contained in

3.1.2. Let $G' \cong C_4 \times C_2$ and $C = C_G(G')$. If (i) $G/C \cong C_2 \times C_2$ or (ii) $G/C \cong C_4$, then $G \notin P_4$.

**Proof.** (i) We have $G = CX$ where $X = \langle x, y \rangle$. We distinguish the cases $X' = G'$, $X' \cong C_4$ and $\exp X' \leq 2$.

(a) Suppose that $X' = G'$. Then $[x, y] = a$ (say) has order 4 and $\langle a \rangle \not\subseteq X$. Therefore we may assume that $a^x \notin \langle a \rangle$. Routine checking shows that, replacing $y$ by $xy$ if necessary, we may further assume that $a^y = a^{-1}$. Let $z = xy$. It is then an exercise to show that $zxay$ cannot be reordered.

(b) Suppose that $X' \cong C_4$. Then $X' \leq CX = G$. Also

$$[C, X] \notin X'.$$

For, $C' \leq Z(G)$ and if (1) were false, then $G' = C' X'$ implies $G' \leq Z(G)X'$, i.e. $C = C_G(X')$. But then $|G : C| = 2$, a contradiction. Therefore (1) is true. We may assume that there is an element $c \in C$ such that $[c, x] \notin X'$. Let $[x, y] = a$. Then $[c, x] = a^b$, where $|b| = 2$ and $G' = \langle a \rangle \times \langle b \rangle$. Without loss of generality, $a = 0$ or 1.

If $[a, x] = 1$, then $a^y = a^{-1}$ and replacing $y$ by $xy$ if necessary, we may assume that $b^y = b$. Then we obtain $[c, y, x] = a^{2x}$. Suppose that $\alpha = 0$. Then $[c, y] = a^\beta$, $-1 \leq \beta \leq 2$. For $\beta = \pm 1$, take $c, a, y, x$ for $a, b, x, y$ in 1.1 (i) to give $G \notin P_4$. For $\beta = 2$, let $z = xy$. Then routine checking shows that $zcyx$ cannot be reordered. The case $\beta = 0$ is covered replacing $c$ by $ca$ and applying the case $\beta = 2$. Therefore suppose that $\alpha = 1$. Then $[c, y] = a^\beta b$, $-1 \leq \beta \leq 2$ and $\beta = 0$ or 2 is disposed of by taking $c, a, y, x$ again for $a, b, x, y$ in 1.1 (i). If $\beta = -1$, routine checking shows that $zcyx$ cannot be reordered (with $z = xy$ still). And if $\beta = 1$, replace $c$ by $ca$ again and apply the case $\beta = -1$.

Finally suppose that $[a, x] = a^2$. Replacing $y$ by $xy$ if necessary, we may assume that $a^y = a$. If $[c, y] \notin \langle a \rangle$, then we can argue as above with $x$ and $y$ interchanged. On the other hand, if $[c, y] \in \langle a \rangle$, then replacing $x$ by $xy$ if necessary, we may assume $b^y = b$ and another routine check shows that, with $c, a, y, x$ for $a, b, x, y$ in 1.1 (i), $G \notin P_4$.

(c) Suppose that $\exp X' \leq 2$. It is easy to see that $\exp [C, X] = 4$ and so we may assume that $[c, x] = a$ (say) has order 4 for some $c \in C$. Then $[x, yc]$ has order 4 and replacing $y$ by $yc$, cases (a) and (b) apply.

(ii) Here $G/C \cong C_4$ and $G = CX, X = \langle x \rangle$. It is easy to see that there is an element $c \in C$ such that $[c, x] = a$ (say) has order 4. (In fact
\[ [C, X] = G' \] Then \( G' = \langle a \rangle \times \langle b \rangle \) with \( |b| = 2 \) and, replacing \( c \) by \( ca \) if necessary, we have \( a^2 = ab, b^2 = a^2 b \). Taking \( c, b, x, x^2 \) for \( a, b, x, y \) in 1.1 (i) shows that \( G \notin P_4 \).

Now we begin the analysis of the final case, where \( G/C \cong C_2 \).

3.1.3. Let \( G \in P_4, G' \cong C_4 \times C_2, C = C_G(G') \) and \( |G/C| = 2 \). Then either

(i) \( G \) has an abelian subgroup of index 2, or

(ii) \( C' = U_1(G') \), all subgroups of \( G' \) are normal in \( G \) and \( G/Z(G) \) is 2-generator.

**Proof.** Suppose \( C \) is not abelian (otherwise (i) is true). Since \( C' \leq Z(G) \), \( C' \neq G' \). Thus \( C' \cong C_4, V_4 \text{ or } C_2 \) and we distinguish these cases.

(a) Suppose that \( C' \cong C_4 \). Then there are \( c_1, c_2 \in C \) such that \( [c_1, c_2] = a \) (say) has order 4. Let \( G = CX \), where \( X = \langle x \rangle \) is cyclic. If \( [c_1, x] \notin \langle a \rangle \), then with \( G' = \langle a \rangle \times \langle b \rangle \) \((|b| = 2)\), we have \([c_1, x] = a^\lambda b, -1 \leq \lambda \leq 2\). Since \( a \in Z(G), b^x = a^\mu b \). Taking \( c_1, b, x, c_2 \) for \( a, b, x, y \) in 1.1 (i) (and noting that \([c_1, x^{-1}] = a^{-\lambda+\mu} b \) we find \( G \notin P_4 \), a contradiction. Thus \([c_1, a'] = c_2(-1 \leq \lambda \leq 2)\).

Now there must exist \( c_3 \in C \) such that \([c_3, x] = a^\nu b \) \((-1 \leq \nu \leq 2) \).

Now there must exist \( c_3 \in C \) such that \([c_3, x] = a^\nu b \) \((-1 \leq \nu \leq 2) \).

If \( c_2 = b \notin \langle a^2 \rangle \), then \( G' = \langle a \rangle \times \langle b \rangle \). Taking \( c_1, a, x, c_2 \) for \( a, b, x, y \) in 1.1 (i), we find \( G \notin P_4 \), a contradiction. Therefore \([c_1, a] = (a^2) \). Now there are elements \( c_{2}, c_{3} \).
in $C$ such that $[c_2, c_3] = b$ (say) has order 2 and $b \neq a^2$. So $G' = \langle a \rangle \times \langle b \rangle$. By the above argument, $[c_2, x]$ and $[c_3, x]$ lie in $\langle a^2, b \rangle$. But $[c_1, c_2, x] = a[c_2, x] = a_1$ (say) has order 4 and $[c_1, c_2, c_3] = [c_1, c_3]b$ has order 2 and does not belong to $\langle a^2 \rangle = \langle a_1^2 \rangle$. Then the above argument, with $c_1$ replaced by $c_1c_2$ and $c_2$ replaced by $c_3$, gives $G \not\cong P_4$, a contradiction.

(c) Suppose that $C' \cong C_2$. We have to establish (ii). As before let $G = CX$ with $X = \langle x \rangle$ and let $c_1 \in C_1$ such that $[c_1, x] = a$ (say) has order 4. Since $C$ is not abelian, it is not difficult to see that, among such elements $c_1$, there is one with $c_1 \notin Z(C)$. Also the subgroups of order 4 in $G'$ are normal in $G$. Moreover, $C' = \cup_1 (G')(\langle a^2 \rangle)$. For, if not, then $\Omega_1(G') = \cup_1(G') \times C' \leq Z(G)$ and so $a^2 = a^{-1}$. Also there exists $c_2 \in C$ such that $[c_1, c_2] = b$ of order 2 (since $c_1 \notin Z(C)$ and $G' = \langle a \rangle \times \langle b \rangle$). Choosing $c_1, a, x, c_2$ for $a, b, x, y$ in 1.1 (i), we find $G \not\cong P_4$. Thus our claim follows.

Let $G' = \langle a \rangle \times \langle b \rangle$. It is straightforward to show there is $c_2 \in C$ with $[c_2, x] = b$ and $[c_1, c_2] = a^2$. Then

(2) all subgroups of $G'$ are normal in $G$.

For, if not, changing $a$ if necessary, we may assume that $a^2 = a$. But then, with $u_1 = x, u_2 = c_1c_2, u_3 = c_1^{-1}x, u_4 = c_2$, routine checking shows that $u_1u_2u_3u_4$ cannot be reordered. So (2) holds. It remains to show that $G/Z(G)$ can be generated by 2 elements. Let $C_0 = C_C(x)$. Then $C = \langle c_1, c_2 \rangle C_0$ and it suffices to show that $C_0 \leq Z(G)$.

Let $d, e, f, g \in G$ and write $[d, e] = r, [d, f] = s, [d, g] = t, [e, f] = u, [e, g] = v, [f, g] = w$. Suppose that $d, f, g \in C, e \notin C, s = t = 1$. Considering the 23 reorderings of defg, we find that at least one of the following holds:

(3) $w = 1, u = 1, uv \in \langle a^2 \rangle, v = a^2$.

If $\langle c_1, c_2 \rangle, C_0 \neq 1$, then there is $c_3 \in C_0$ such that

$[c_1, c_3] = a^2, [c_2, c_3] = 1$; or $[c_1, c_3] = 1$,

$[c_2, c_3] = a^2$; or $[c_1, c_3] = [c_2, c_3] = a^2$.

In each case, take (respectively) $d = c_2c_3, e = x, f = c_1c_3, g = c_3$; or $d = c_1c_3, e = c_2x, f = c_3, g = c_2$; or $d = c_1c_3, e = x, f = c_2, g = c_3$. In each case, none of the relations (3) holds. Thus $\langle c_1, c_2 \rangle, C_0 \neq 1$ and it suffices to show that $C_0$ is abelian. If not, then there are elements
c₃, c₄ ∈ C₀ such that \([c₃, c₄] = a²\). Take \(d = c₂c₃, e = x, f = c₁c₄, g = c₃\).
Again none of the relations (3) holds. This contradiction completes the proof of 3.1.3. ■

In the opposite direction, we have

3.1.4. Let \(G' \equiv C₄ \times C₂, C = C₆(G'), \ |G/C| = 2, C' = U₁(G'), \) with all subgroups of \(G'\) normal in \(G\) and suppose that \(G/Z(G)\) is 2-generator. Then \(G \in P₄\).

**PROOF.** Suppose, for a contradiction, that \(G \notin P₄\). By hypothesis there are elements \(c₁, c₂ \in C\) such that \(C = \langle c₁, c₂ \rangle Z(G)\). Let \(G = CX, X = \langle x \rangle\). Then \(G' = [C, X]\) and we may assume that \([c₁, x] = a\) (say) has order 4. Thus \(G' = \langle a \rangle \times \langle b \rangle\) with \(|b| = 2\). Replacing \(c₂\) if necessary, we may also assume that \([c₂, x] = b\). Note that \([c₁, c₂] = a²\).

Now there are elements \(d, e, f, g \in G\) such that \(defg\) cannot be reordered. Suppose that \(e \notin C\) while \(d, f, g \in C\). Without loss of generality, \(e = x\). Let \(d = c₁c₂\), \(f = c₁c₂\), \(g = c₁c₂\) and let \(r, ..., w\) be the commutators as in 3.1.3 (c). Since \(fg \neq gf\), \(w = a²\). But \(w = a^{2(αν + βξ)}\) and so

\[
αν + βξ \equiv 1(2).
\]

Now \(r = a^λb^μ\) and since \(de \neq ed\) and \(defg \neq edgf, r \notin \langle a² \rangle\). Thus either \(λ \equiv 1(2)\) or \(μ \equiv 1(2)\). Suppose that \(s = 1\), i.e. \(αμ + βλ \equiv 0(2)\). One checks that \(λ \equiv α(2), μ \equiv β(2)\)

and \(t = a²\), i.e. \(λν + μξ \equiv 1(2)\). Then, since \(def \neq fed\) and \(defg \neq fedg\), \(ru \notin \langle a² \rangle\). But \(ru = a^{λ - α} b^{μ + β}\) and this contradicts (5). Thus \(s = a²\). Similarly if \(t = 1\), we get contradictions to \(defg \neq gedf\) or \(gfed\), and therefore

\[
t = a².
\]

By (4), \(u, v\) and \(uv \notin \langle a² \rangle\) and so \(G' = \langle a², u, v \rangle\). But \(r \notin \langle a² \rangle\) and so \(ru, rv\) or \(ruv \in \langle a² \rangle\). The first possibility is ruled out as above, the second contradicts (6), while \(defg \neq fged\) or \(gfed\) shows that \(ruv \notin \langle a² \rangle\).

Using the facts that \(C \in P₃\) and that \(g^{-1}f^{-1}e^{-1}d^{-1}\) cannot be reordered, there are 7 remaining cases depending on which of \(d, e, f, g\) belong to \(C\) and they are handled in the same fashion. (See [8] for details.) ■
Summarising the results of 3.1, we have proved

3.1.5. Let $G' \cong C_4 \times C_2$. Then $G \in P_4$ if and only if $C = C_G(G')$ has
index 2 in $G$, $C' = U_1(G')$, all subgroups of $G'$ are normal in $G$ and
$G/Z(G)$ is 2-generator.

3.2. The case $G' \cong C_2 \times C_2 \times C_2$.

Suppose that $G$ is nilpotent of class $\geq 3$ with $G' \cong C_2 \times C_2 \times C_2$. We
show that $G \notin P_4$. If $C = C_G(G')$, then $G/C \cong C_2 \times C_2 \times C_4$ or $C_2$. We dis-
tinguish these cases.

3.2.1. Let $G$ be nilpotent with $G' \cong C_2 \times C_2 \times C_2$, $C = C_G(G')$ and
$G/C \cong C_2 \times C_2$. Then $G \notin P_4$.

PROOF. Let $G = CX$, where $X = \langle x, y \rangle$. Then $X' \triangleleft G$ and we consid-
er 4 possibilities for $X'$.

(i) Suppose that $X' = G'$. It is routine to check that there is a
basis $\{a, b, c\}$ of $G'$ such that $a^2 = ab$, $b^2 = b$, $c^2 = c$, $a^y = ac$, $b^y = b$,
$c^y = c$ and $[x, y] = a$. But then $u_1 u_2 u_3 u_4$ cannot be reordered, where
$u_1 = y^2$, $u_2 = x$, $u_3 = y$, $u_4 = x^2$. Therefore $G \notin P_4$.

(ii) Suppose that $X' \cong C_2 \times C_2$. We can take $X' = \langle b, c \rangle$, $[x, y] =
= b$, $b^2 = bc$, $b^y = b$, $c^2 = c$, $c^y = c$. Clearly $G' = C'\langle C, X \rangle X'$ and
since $C' \subseteq Z(G)$, we have $C = C_G([C, X]X')$. Thus, by hypothesis,
$[C, X]X' = G'$ and so there is an element $c_1 \in C$ such that either
$[c_1, x] \notin X'$ or $[c_1, y] \notin X'$. If $[c_1, x] \in X'$, then $[c_1, y] = a \notin X'$ and
we can show that $G' = \langle a \rangle \times (b) \times (c)$ with $a^2 = a$, $a^y = ac$,
$1 \neq [c_1, x] \notin C'$ and $1 \neq [c_1, y] \notin 1 \mod \langle c \rangle$. Then the hypotheses of 1.1 (i) are satis-
fied with $c_1, a, y, x$ for $a, b, x, y$. Since $[a, y] = c$, it follows that
$G \notin P_4$.

Suitable substitutions for $x$ and $y$ reduce the second possibility
$([c_1, x] \notin X')$ to the case just handled.

(iii) Suppose that $X' \cong C_2$. Let $[x, y] = c$. Thus $1 \neq c \in Z(G)$.
Now $C' X' \leq Z(G)$, and so $G' = [C, X]$. Therefore we may assume that
there is an element $c_1 \in C$ such that $[c_1, x] = b \notin \langle c \rangle$. Then $[x, c_1 y] =
= b^y c \notin \langle c \rangle$. But by cases (i) and (ii) we may assume that $\langle x, c_1 \rangle$ has or-
der 2 and so $b^y c \in Z(G)$, i.e. $b \in Z(G)$. Therefore $|Z(G) \cap G'| = 4$.
Choose $c_2 \in X$ and $\varepsilon = 0$ or 1 such that

$[c_2, xy^\varepsilon] = a \notin Z(G)$.
Then

\[ [xy^i, c_2y] = [xy^i, y][xy^i, c_2]y = ca^y \notin Z(G). \]

Therefore \(|\langle xy^i, c_2y \rangle\) \geq 4 and \(G \notin P_4\) by cases (i) and (ii).

(iv) Finally suppose that \(X\) is abelian. Again \(G = [C, X]\) and so there is an element \(c_1 \in C\) such that, without loss of generality, \([c_1, x] = c \neq 1\). Then \([x, c_1y] = c^y \neq 1\) and \(\langle x, c_1y \rangle\) is not abelian. Thus \(G \notin P_4\) by the previous cases.

The next case is much easier.

3.2.2. Let \(G\) be nilpotent with \(G' \cong C_2 \times C_2 \times C_2, C = C_G(G')\) and \(G/C \cong C_4\). Then \(G \notin P_4\).

**Proof.** Let \(G = CX\) with \(X = \langle x \rangle\). Then \(G' = C'[C, X]\) and since \(C' \leq Z(G)\), it follows that \(G' = [C, X]\). One sees easily that \(G'\) is indecomposable as an \(X\)-module. Choose a basis \(\{a, b, c\}\) of \(G'\) such that \([a, x] = b, [b, x] = c, [c, x] = 1\). Also choose \(c_1 \in C\) such that \([c_1, x] = ab^\lambda c^\mu, 0 \leq \lambda, \mu \leq 1\). Then \([a^\lambda b^\mu c_1, x] = a\) and replacing \(c_1\) by \(a^\lambda b^\mu c_1\), we may assume that \([c_1, x] = a\). Taking \(c_1, b, x, x^2\) for \(a, b, x, y\) in 1.1 (i) we find \(G \notin P_4\). ■

The final case is

3.2.3. Let \(G\) be nilpotent with \(G' \cong C_2 \times C_2 \times C_2, C = C_G(G')\) and \(|G/C| = 2\). Then \(G \notin P_4\).

**Proof.** There is a basis \(\{a, b, c\}\) of \(G'\) such that \(a^\lambda = ab, b^\lambda = b, c^\lambda = c\). Also \(G' = C'[C, X]\) and \(C' \leq \langle b, c \rangle\) since \(C' \leq Z(G)\). We may assume that if \(c_1 \in C\) and \([c_1, x] \notin \langle b, c \rangle\), then \([c_1, C] \leq \langle b \rangle\). For, if not, then there exists \(c_2 \in C\) such that \([c_1, c_2] \notin \langle b \rangle\). Then taking \(c_1, a, x, c_2\) for \(a, b, x, y\) in 1.1 (i) we see that \(G \notin P_4\). Now it is not hard to see that \([C, X] \notin \langle a, b \rangle\) and we can find \(c_1, c_3 \in C\) such that \([c_1, x] = ab^\lambda c^\mu, [c_3, x] = b^\mu c\) and \([c_1, c_3] = 1\). Taking \(c_1, c_3, x, x^2\) for \(a, b, x, y\) in 1.1 (i) shows that \(G \notin P_4\). ■

Summarising the previous 3 results, we have

3.2.4. Let \(G\) be nilpotent of class \(\geq 3\) with \(G' \cong C_2 \times C_2 \times C_2\). Then \(G \notin P_4\).
3.3. The case \( G' = C_8 \).

Finally for groups \( G \) with \( |G'| = 8 \) we have

3.3.1. Let \( G' = C_8 \) and suppose that \( G \) does not have an abelian subgroup of index 2. Then \( G \notin P_4 \).

**Proof.** Suppose that \( G \) has class 2. Using 2.1, we may assume that \( G \) is a finite 2-group. But then \( G \notin P_4 \), by 1.2.

Now suppose that class \( G \geq 3 \). Let \( C = C_4(G') \). Then \( G/C \) is either the 4-group or of order 2 and we distinguish these cases.

Case (i). Suppose that \( G/C \cong C_2 \times C_2 \). Let \( G = CX \), where \( X = \langle x, y \rangle \). Thus \( G' = C'[C, X]X' \) and since \( C' \leq Z(G) \), \( C' \triangleleft G' \) and we have \( G' = [C, X]X' \). Let \( G' = \langle a \rangle \). If \( X' \leq \langle a^2 \rangle \), then there are elements \( c \in C, z \in X \setminus C \) such that \( [c, z] = a \). Choose \( t \in X \) such that \( G = C(z, t) \).

Then \( [ct, z] = a^i a^{2i} \) (some integer \( i \)) and so \( [ct, z] \notin \langle a^2 \rangle \). Therefore replacing \( X \) by \( \langle ct, z \rangle \), we may assume that \( X' = G' = \langle a \rangle \). Thus, without loss of generality, \( [x, y] = a, a^x = a^{-1}, a^y = a^3 \). Let \( z = xy \). Then it is routine to check that \( xyz \) cannot be reordered and so \( G \notin P_4 \).

Case (ii). Suppose that \( |G/C| = 2 \). Now \( G = CX \) with \( X = \langle x \rangle \) and \( G' = [C, X] \). So there is an element \( c \in C \) such that \( [c, x] = a \) (say) generates \( G' \). If \( a^x = a^{2i+1} \) (\( i = 1 \) or 2), then \( [a, x] = a^{2i} \) and \( [c, x^2] = a^{2(i+1)} \). Take \( c, a, x, x^2 \) for \( a, b, x, y \) in 1.1. If \( i = 1 \), then \( [c, x^2] = a^4 \) and 1.1 (ii) fails; while if \( i = 2 \), then \( [a, x] = a^4 \) and 1.1 (i) fails. Thus \( G \notin P_4 \).

Therefore we may suppose that \( a^x = a^{-1} \). Since \( C' \leq Z(G), \ C' \leq \langle a^4 \rangle \). By hypothesis \( C' \neq 1 \) and so \( C' = \langle a^4 \rangle \). Then there are elements \( c_1, c_2 \in C \) such that

\[
[c_1, c_2] = a^4, \quad \langle [c_1, x] \rangle = G' = \langle a \rangle.
\]

For, certainly there are elements \( c_1, c_2 \in C \) such that \( [c_1, c_2] = a^4 \). Thus suppose that \( [c_1, x] \in \langle a^2 \rangle \). If \( [c_1, c, c_2] = 1 \), then \( [c, c_2] = a^4 \) and we can take \( c_1 = c \). On the other hand, if \( [c_1 c, c_2] = a^4 \), then since \( [c_1 c, x] \notin \langle a^2 \rangle \), we can take \( c_1 c \) for \( c_1 \). Therefore we have (7) and so we may assume that \( [c_1, x] = a \). Then \( [c_1, x^{-1}] = a \). Take \( c_1, a, x, c_2 \) for \( a, b, x, y \) in 1.1 (ii). We find that \( G \notin P_4 \). □

4. Derived subgroups of order exceeding 8.

We shall prove that if \( G \in P_4 \) and \( G \) does not have an abelian subgroup of index 2, then \( |G'| \leq 8 \). Finite groups are considered in 4.2
and infinite groups in 4.3. In the finite case we argue by induction on order and in anticipation of this we derive some technical results in 4.1.

4.1. Groups with a metabelian subgroup of index 2.

Throughout this subsection we assume that

\[ G \notin P_4, \ G \text{ does not have an abelian subgroup of index } 2, \ N \triangleleft G, \ N \leq G', \ G/N \text{ has an abelian subgroup } B/N \text{ of index } 2 \text{ and } G = \langle B, h \rangle. \]

By rank we always understand Prüfer rank.

4.1.1. \( G'/N \) has rank \( \leq 2 \).

**Proof.** Suppose, for a contradiction, that rank \( G'/N \geq 3 \). Clearly we may assume that \( B' = N \) and it is not difficult to show that there are elements \( w, x, y \in B \) such that

\[ [w, x] \neq 1, \ [w, h] = q \neq N, \ [x, h] = r \notin \langle q, N \rangle, \ [y, h] = s \notin \langle q, r, N \rangle. \]

Then we claim that \( wxhy \) cannot be reordered. For, \( wxhy = wxhys^{-1} \) and modulo \( N \) every reordering of \( wxhy \) is uniquely expressible as

\[ wxhyq^x r^y s^r \]

with \( \alpha, \beta, \gamma \in \{0, -1\} \). Let \( \lambda = \alpha + \beta + \gamma \) and \( a = q^\alpha r^\beta s^\gamma \). Thus with \( \ldots \) representing factors in a reordering (8) of \( wxhy \), \( h \ldots \) has \( \lambda = -3 \), \( h \ldots \) has \( \lambda = -2 \) and \( \ldots h \) had \( \lambda = 0 \). Also \( \ldots hw \) has \( a = q^{-1} \), \( \ldots hx \) has \( a = r^{-1} \) and \( wxhy \neq wxhy \). Thus our claim follows, contradicting \( G \in P_4 \). \( \blacksquare \)

Next we consider some cases when \( G' \) has rank 2.

4.1.2. Let \( G' = \langle a \rangle \times \langle b \rangle \) with \( |a| = 2^m, \ |b| = 2^n, \ m \geq n, \) and let \( N = \Omega_1 \langle a \rangle \triangleleft G \). Then \( m \leq 2 \).

**Proof.** Suppose, for a contradiction, that \( m \geq 3 \). Since \( B \) is not abelian, \( B' = N \). Then \( G' = N[B, h] = [B, h] \). Again it is not difficult to find elements \( x, y \in B \) such that

\[ [x, y] \neq 1, \ [x, h] \notin \langle a^2, b \rangle, \ [y, h] \in \langle a^2, b \rangle. \]

Thus \( [x, h] = a^ib^j \) (i odd) and \( [x^2, h] = a^{2i}b^{2j} \mod N \). Let \( z = x^2 \). Then \( [y, z] = 1 \) and \( yxhz \equiv a^3yha^{-2i}b^{-2j} \mod N \). By considering the position of \( h \) in reorderings of \( yxhz \) (as in 4.1.1 and again working modulo \( N \))
when convenient), we find that \(yxhz\) cannot be reordered, contradicting \(GEP_4\). □

4.1.3. If \(|N| = 2\), then \(G' \neq C_4 \times C_4\).

**Proof.** Suppose, for a contradiction, that \(G' \equiv C_4 \times C_4\) and let \(G' = \langle a \rangle \times \langle b \rangle\) with \(N = \langle a^2 \rangle\). Then \(B' = N\) and \(G' = [B, h]\). As in 4.1.2 there are elements \(x, y \in B\) such that \([x, y] = a^2\), \([x, h] = ab^j\) (replacing \(a\) by \(a^{-1}\) if necessary). We may assume that \(j = 0\) or \(1\). In the former case, we may assume (replacing \(y\) by a suitable element if necessary) that \([y, h] = b\). Thus \([y^2, h] \equiv b^2 \mod N\). Put \(z = y^2\). Then easy checking shows \(yxhz\) cannot be reordered. In the second case, when \([x, h] = ab\), we may assume that \([y, h] \notin \langle a^2, b \rangle\). Thus \([y, h] = a^i b^k\) where either (i) \(i\) is odd and \(k\) is even or (ii) \(i\) is even and \(k\) is odd. If (i) holds, then the previous case applies with \(x\) and \(y\) interchanged. Thus (ii) holds. Replacing \(a\) by \(a^{-1}\) and \(b\) by \(a^2 b\) gives \([y, h] = b^{\pm 1}\). Then the previous case applies with \(xy\) and \(x\) for \(x\) and \(y\). □

4.2. **Bounding \(|G'|\): the finite case.**

Suppose that \(G \in P_4\) and \(G\) does not have an abelian subgroup of index 2. If \(|G'| > 5\), then \(G\) is nilpotent, by Theorem 3. Thus if \(G\) is finite, then \(G = A \times B\) with \(A\) a 2-group and \(|B|\) odd. If \(B' \neq 1\), then 1.3 shows that \(A\) is abelian; and \(|B'| = 3\) or 5, by Theorem 2. Thus \(B\) must be abelian and in order to prove that \(|G'| \leq 8\), we may assume that \(G\) is a 2-group.

4.2.1 Let \(G\) be a finite group in \(P_4\). Then either \(G\) has an abelian subgroup of index 2 or \(|G'| \leq 8\).

**Proof.** Suppose that \(G\) does not have an abelian subgroup of index 2. Thus we may assume that \(G\) is a 2-group (see above).

Assume, for a contradiction, that \(|G'| \geq 16\) and let \(\exp G' = 2^e\).

Choose \(N \triangleleft G\), \(|N| = 2\) and with \(N \leq U_{e-1}(G')\). Then

\[(9)\]

\(G/N\) does not have an abelian subgroup of index 2.

For, suppose that this is not the case. Then, by 4.1.1, \(\exp G'/N \leq 2\) and therefore \(\text{rank } G' \leq 3\). If \(\text{rank } G' = 3\), then \(e \geq 2\) and since \(N \leq \Phi(G')\), \(\text{rank } G'/N = 3\). Thus \(\text{rank } G' \leq 2\) and so, by 4.1.2, \(e \leq 2\). Since \(|G'| \geq 16\), we would have \(G' \equiv C_4 \times C_4\), contradicting 4.1.3. Then (9) follows.

By induction on \(|G|\), we may assume that \(|G'/N| = 8\) and so \(|G'| = 16\). We distinguish the cases in which \(G'\) is isomorphic to
\[ C_8 \times C_2, C_4 \times C_4 \text{ and } G'/N \equiv C_2 \times C_2 \times C_2. \text{ (Recall that } G' \neq C_{16}, \text{ otherwise } G'/N \cong C_8, \text{ contradicting 3.3.1.)} \]

**Case (i).** Suppose that \( G' = \langle a \rangle \times \langle b \rangle, \ |a| = 8, |b| = 2. \) Here \( N = \langle a^4 \rangle \) and \( G'/N \cong C_4 \times C_2. \) Let \( C = C_G(G'/N) \) By 3.1.3

\[ (10) \quad |G : C| = 2, \quad (C/N)' = C' N/N = \langle a^2 \rangle/N \]

and \( N \leq X \leq G' \) implies \( X \triangleleft G. \)

Therefore \( C' = \langle a^2 \rangle. \) Let \( G = \langle C, h \rangle. \) Thus \( G' = C'[C, h] = [C, h] \) since \( a^2 \in \Phi(G'). \) Also \( [a, C] \subseteq N \) and therefore \( [a^2, C] = 1. \) Then we easily find elements \( x, y \in C \) such that \( [x, y] = a^4 \) and \( |[x, h]| = 8. \) Now \( [b, h] \in N \) and \( h \notin C. \) Therefore \( [a, h] \notin N \) and so

\[ (11) \quad [a, h] = a^{z^2}. \]

Let \( H = \langle x, y, h, G' \rangle, \) \( K = \langle x, y, h^2, G' \rangle. \) Then either \( H' = G' \) or \( H' \cong C_8. \) Also \( N \leq K' \) and \( K \subseteq C. \) Thus \( |H : K| = 2 \) and \( K = C \cap H. \) Let \( C^* = C_H(H'/N). \) Then \( K \subseteq C^* \) and so \( C^* = K, \) since \( h \notin C^* \) (by (11)). But \( K \) is not abelian and therefore \( H \) does not have an abelian subgroup of index 2. However, by (10) and (11), \( [C, h^2] \subseteq N \) and thus \( K/N \) is abelian. Then \( H/N \) has an abelian subgroup of index 2, contradicting 4.1.2.

**Case (ii).** Now suppose that \( G' = \langle a \rangle \times \langle b \rangle, \ |a| = |b| = 4. \) We can take \( N = \langle a^2 \rangle. \) As in (i), \( C = C_G(G'/N) \) has index 2 in \( G \) and \( (C/N)' = \langle b^2 \rangle N/N, \) i.e. \( C' \leq \langle a^2, b^2 \rangle. \) All subgroups of \( G'/N \) are normal in \( G/N \) and so \( \langle a \rangle \triangleleft G. \) By assumption, \( C' \neq 1. \) Also \( |C'| \neq 2 \) (by 4.1.3). Thus \( C' = \langle a^2, b^2 \rangle = \Phi(G'). \) Then with \( G = \langle C, h \rangle, \) we have \( G' = [C, h]. \) Since \( \langle a \rangle \triangleleft G, \) \( [b, h] \notin N. \) Also \( \langle b, N \rangle \triangleleft G \) and so \( \langle b^2 \rangle \triangleleft G, \) i.e. \( C' \leq \langle b \rangle. \) By 4.1.3, \( G/(b^2) \) cannot have an abelian subgroup of index 2 and thus, by 3.1.3, \( \langle b \rangle \triangleleft G. \) Similarly \( \langle ab \rangle \triangleleft G. \) Therefore, by conjugation, \( h \) inverts \( a \) and \( b \) and thus every element outside \( C \) does the same. Hence \( C = C_G(G'). \)

In a routine way we find elements \( x, y \in C \) such that (replacing \( a \) and \( b \) by suitable elements if necessary) \( |x, y| \neq 1, [x, h] = a, [y, h] = b. \) Let \( H = \langle x, y, h, G' \rangle, K = \langle x, y, h^2, G' \rangle. \) So \( H' = G' \) and clearly \( |H/K| = 2. \) Since \( [C, h^2] = 1, K/(\langle x, y \rangle) \) is abelian and hence \( H/(\langle x, y \rangle) \) has an abelian subgroup of index 2. If \( H \) has an abelian subgroup \( A \) of index 2, then \( H' \leq A \leq C_H(G') = K \) and so \( A = K, \) contradicting \( K' \neq 1. \) Thus \( H \) does not have an abelian subgroup of index 2, contradicting 4.1.3.

For the remaining case, we need the following two results.
4.2.2. Let $G = \langle x_1, x_2, x_3 \rangle$ be nilpotent of class 2 with $G'$ of exponent 2. Suppose that there is an element $\sigma \in \Sigma_3$, $\sigma \neq 1$, such that $x_1 x_2 x_3 = x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}$. Then $|G'| \leq 4$.

The proof can safely be left as an easy exercise.

4.2.3. Let $G \in P_4$ and $N \leq Z(G)$ with $G/N$ nilpotent of class 2. Let $x_1, x_2, x_3 \in G$ with the property that the only element $\sigma \in \Sigma_3$ such that $x_1 x_2 x_3 = x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} \mod N$ is $\sigma = 1$. Then for all $a \in G'$, $[a, x_1] = 1$ if and only if $[a, x_3] = 1$.

Again the proof is an exercise—assume $[a, x_1] = 1$ and consider a reordering of $x_1 x_2 x_3 a$.

PROOF OF 4.2.1 CONTINUED. Case (iii). Finally suppose that $G'/N \cong C_2 \times C_2 \times C_2$. Let bars denote subgroups and elements of $G$ modulo $N$. By 3.2.4, $G$ has class 2 and by Theorem 4 either (a) $\overline{G}/Z(G)$ is 3-generator or (b) $\overline{G}/Z(G)$ is 4-generator and $\overline{G}$ is not diabelian.

Consider case (a) where $\overline{G} = \langle Z(G), \bar{x}_1, \bar{x}_2, \bar{x}_3 \rangle$. Then $\langle \bar{x}_1, \bar{x}_2, \bar{x}_3 \rangle'$ is elementary of rank 3. Let $a \in G'$. Since $|G: C_G(a, N)| \leq 2$, we may assume that $[a, x_1] = 1$. By 4.2.2, neither $\bar{x}_1 \bar{x}_2 \bar{x}_3$ nor $\bar{x}_2 \bar{x}_1 \bar{x}_3$ can be reordered. Therefore, by 4.2.3, $[a, x_3] = 1$ and hence $[a, x_2] = 1$. Let $\bar{c} \in Z(\overline{C})$. Replacing $x_2$ in the above argument by $\bar{c} x_2$, we obtain similarly $[a, c x_2] = 1$, i.e. $[a, c] = 1$. Thus $a \in Z(G)$ and therefore $G' \leq Z(G)$. But this contradicts Theorem 4.

Now suppose that case (b) holds. By 3.2.6 of [7], $\overline{G} = \langle \bar{A}, \bar{x}, \bar{y} \rangle$ with $\bar{A} \triangleleft \overline{G}$, $\bar{A}$ abelian; and $\overline{G}$ is not diabelian. By choosing $\bar{A}$ as large as possible, we may assume that $N \leq \bar{A} \leq G$ and so $G = \langle A, x, y \rangle$. Again we claim that class $G = 2$. For, let $a \in G'$. As before, we may assume that $[a, x] = 1$. Since $\overline{G}/\bar{A}$ is elementary, it follows from (9) that $\overline{G}/\bar{A} \cong C_2 \times C_2 \times C_2$ and $[\bar{A}, \bar{xy}] \neq 1$. Thus we can find $\bar{c} \in A$ such that $[\bar{c}, \bar{xy}] \neq 1$. Let $z = cx$. Since $\bar{G}$ is not diabelian, it follows easily that $\bar{xy}z$ cannot be reordered. Therefore, by 4.2.3, $[a, z] = 1$ and so $[a, c] = 1$. Since $\bar{A}$ is generated by such elements $\bar{c}$, it follows that $[a, A] = 1$. Also $[\bar{A}, \bar{x}] \neq 1 \neq [\bar{A}, \bar{y} \bar{x}]$ and therefore $\bar{A} \neq C_\bar{A}(\bar{x}) \cup C_\bar{A}(\bar{y} \bar{x})$. Thus there is an element $\bar{d} \in \bar{A}$ such that $[\bar{d}, \bar{x}] \neq 1 \neq [\bar{d}, \bar{y} \bar{x}]$. As before, it follows easily that $\bar{y} \bar{x} \bar{d}$ cannot be reordered. (If $\bar{y} \bar{x} \bar{d} = \bar{d} \bar{x} \bar{y}$, then $[\bar{d}, \bar{y} \bar{x}] = 1$ and $\overline{G} = \bar{A}(\bar{d} \bar{x}, \bar{y} \bar{x})$ would be diabelian.) Since $[a, d] = 1$, 4.2.3 gives $[a, y] = 1$. Thus $a \in Z(G)$ and so $G' \leq Z(G)$, as claimed. As in case (a), this contradicts Theorem 4.

First, a straightforward inverse limit argument (see, for example [5], vol. 2, p. 167) gives

4.3.1. Suppose that every finitely generated subgroup of $G$ is abelian or has an abelian subgroup of index 2. Then either $G$ is abelian or $G'$ has an abelian subgroup of index 2.

Now we can establish

4.3.2. Let $G \in P_4$. Then either $G$ has an abelian subgroup of index 2 or $|G'| \leq 8$.

**Proof.** As we saw at the beginning of 4.2, we may assume that $G$ is nilpotent.

(i) Suppose that $G$ is finitely generated. Then $G'$ is finitely generated and has finite exponent ([3], 2.2). Thus $G'$ is finite. Also $G$ is residually finite and so there is $N < G$ with $G/N$ finite and $G' \cap N = 1$. If $G/N$ has an abelian subgroup $A/N$ of index 2, then $A$ is abelian and has index 2 in $G$. On the other hand, if $G/N$ does not have an abelian subgroup of index 2, then $|G'| = |(G/N)'| \leq 8$, by 4.2.1.

(ii) Now suppose that $G$ is arbitrary. Suppose that $G$ is not abelian and does not have an abelian subgroup of index 2. By 4.3.1, there is a finitely generated subgroup $X$ of $G$ with the same properties. If $|G'| > 8$, then, by enlarging $X$ if necessary, we may assume that $|X'| > 8$. But this would contradict (i). Thus $|G'| \leq 8$.

5. Nilpotent $P_4$-groups.

The classification of the nilpotent $P_4$-groups is as follows.

5.1. Let $G$ be a nilpotent group. Then $G \in P_4$ if and only if one of the following holds:

(i) $G$ has an abelian subgroup of index 2;
(ii) $|G'| \leq 3$;
(iii) $G' \cong V_4$;
(iv) $G' \cong C_4$ and $G$ has a subgroup $B$ of index 2 with $|B'| = 2$;
(v) $G' \cong C_5$ and $|G/Z(G)| = 25$;
(vi) \( G' \cong C_4 \times C_2 \) and, with \( C = C_0(G') \), \( |G/C| = 2 \), \( C' = U_1(G') \), all subgroups of \( G' \) are normal in \( G \) and \( G/Z(G) \) is 2-generator;

(vii) \( G' \cong C_2 \times C_2 \times C_2 \), \( G \) has class 2 and either \( G/Z(G) \) is 3-generator or \( G/Z(G) \) is 4-generator and \( G \) is not diabelian.

**Proof.** Let \( G \in P_4 \) and suppose that (i) does not hold. By 4.3.2, \( |G'| \leq 8 \). Routine arguments allow us to assume that \( G \) is finitely generated. Then, as in 4.3.2 (i), there is a finite quotient \( G/N \) with \( (G/N)' \cong G' \). Therefore, by 1.3 and Theorem 2, either \( |G'| \leq 5 \) or \( |G'| = 8 \). If \( |G'| \leq 4 \), then \( G \) satisfies (ii), (iii) or (iv) (by 2.4). Suppose that \( |G'| = 8 \). By 3.3.1, \( G' \not\cong C_8 \); and if \( G' \cong C_4 \times C_2 \), then \( G \) satisfies (vi), by 3.1.3. Thus we may assume that \( G' \cong C_2 \times C_2 \times C_2 \). By 3.2.4, class \( G = 2 \) and so \( G/Z(G) \) is a finite elementary abelian 2-group. Let \( L = Z(G) \cap N \). Then \( G/L \) is finite and \( Z(G/L) = Z(G)/L \). By Theorem 4, \( G/L \) modulo its centre has rank \( \leq 4 \) and hence rank \( G/Z(G) \leq 4 \). Also if rank \( G/Z(G) = 4 \) and \( G \) is diabelian, then \( G/L \) has structure contradicting Theorem 4. Thus \( G \) satisfies (vii). Finally, if \( |G'| = 5 \), then, by Theorem 2, \( G/Z(G) \cap N \) has order 25 modulo its centre, namely \( Z(G)/Z(G) \cap N \). Thus \( |G/Z(G)| = 25 \) and (v) holds.

Conversely, if \( G \) satisfies (i), then \( G \in P_4 \), by Theorem 3. If \( |G'| = 2 \), then \( G \in P_3 \), by [2]. Suppose that \( |G'| = 3 \). To show that \( G \in P_4 \), we may assume that \( G \) is finitely generated and, by 2.1, even finite and hence a 3-group. Then \( G \in P_4 \), by Theorem 2. If (iii) holds, then \( G \in P_4 \), by Theorem 1. If \( G \) satisfies (iv), then \( G \in P_4 \), by 2.4.

Suppose that (v) holds. To show that \( G \in P_4 \), again we may assume that \( G \) is finite and hence a 5-group and the result follows from Theorem 2. If (vi) holds, 3.1.4 shows that \( G \in P_4 \). Finally, suppose that \( G \) satisfies (vii). We may assume that \( G \) is finitely generated and so \( Z(G) \) is finitely generated. If \( B \) is a complement in \( Z(G) \) of the 2-component of \( Z(G) \), then it suffices to show that \( G/B \in P_4 \), since \( G \) embeds in \( G/B \times G/G' \). But \( G/B \) is a finite 2-group and satisfies (vii) and so \( G/B \in P_4 \), by Theorem 4. □

Taking Theorem 3 and 5.1 together, we have a complete description of \( P_4 \)-groups and their structure is as described in Theorem 5 of §1.

**References**


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